Eikonal exchange amplitudes

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We show that the existing and widely used results for an approximation to eikonal exchange amplitudes, which contain an indeterminate phase, are not valid. We derive a new analytic result, containing a well-defined phase, which can be used unambiguously with direct amplitudes to calculate full differential cross sections. The modulus of our result also differs from that of the previous formulations. Comparisons are made with the earlier e^--H approximate exchange calculations and with exact eikonal exchange calculations.

I. INTRODUCTION

Recently, considerable effort has been given to the calculation of the exchange amplitudes for collisions of electrons with atoms or ions by means of the eikonal approximation or some variant thereof.[1-12] These amplitudes are of particular interest because they complement the direct amplitudes obtained by means of often-used Glauber and eikonal approximations. Together with the direct amplitudes they allow for the calculation of self-consistent full amplitudes for indistinguishable particles.

The most widely used method for calculating eikonal exchange amplitudes has been to adapt to the eikonal amplitude a technique used earlier[13,14] for approximating the Born exchange amplitude. The formulation for e^--H collisions was first presented in 1972.[1,2] This approximate eikonal exchange amplitude, originally expressed as a one-dimensional integral, was subsequently reduced to closed form.[3-5]

The techniques and results of these calculations[1-5] have also been used by others for calculating a variety of cross sections.[6-8] All these calculations[1-8] yield approximate eikonal exchange amplitudes which contain an indeterminate phase which leads to serious ambiguities when attempts are made to combine the amplitudes with direct amplitudes.

In Sec. II we present a critique of the formulation that was first presented[1,2] for obtaining this approximate eikonal exchange amplitude and demonstrate that the mathematical treatment leading to the indeterminate phase is not valid. In Sec. III we develop an appropriate mathematical treatment which yields not only a well-defined unambiguous phase, but a modulus for the analytic approximate eikonal exchange amplitude which differs from that of Refs. 2–5. In Sec. IV we compare our results for e^--H elastic scattering with exact eikonal calculations obtained from a numerical evaluation[9] of a two-dimensional integral and, for completeness, with the earlier calculations.[3-5]

II. CRITIQUE OF PREVIOUS FORMULATION

In this section we investigate the formulation of the approximate eikonal exchange amplitude that is currently used in electron-atom or electron-ion collisions. For definiteness we shall consider electron-hydrogen atom (e^--H) collisions and use the notation of Ref. 2.

Let $\vec{r}_1$ and $\vec{r}_2$ be the position vectors of the incident and target electrons, respectively, and let $\vec{r}_j = \vec{z}_j + \vec{b}_j$ be the usual cylindrical coordinate decomposition of $\vec{r}_j$. The "exact" eikonal exchange T matrix for e^--H collisions in which $\vec{k}_i$ and $\vec{k}_f$ are the initial and final momenta of the free electron, respectively, and the target makes a transition from initial state ϕ_i to final state ϕ_f, is given, in the post interaction, by[2]

$$T^{+}_{fi}(\vec{k}_i,\vec{k}_f) = \int d\vec{r}_1 d\vec{r}_2 e^{-i\vec{k}_f\cdot\vec{r}_2}\phi^*_f(\vec{r}_1)\left(\frac{1}{r_{12}} - \frac{1}{r_2}\right) \times e^{i\vec{k}_i\cdot\vec{r}_1}\phi_i(\vec{r}_2)\left(\frac{(r_1+z_1)\vec{b}^2_{12}}{(r_{12}+z_{12})b^2_1}\right)^{i\eta_+}, \quad (1)$$

where $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$, $\vec{b}_{12} = \vec{b}_1 - \vec{b}_2$, $z_{12} = z_1 - z_2$, and $\eta_+ = 1/k_i$. (We use atomic units $m_e = \hbar = e = 1$ throughout.) The factor in large parentheses is the eikonal phase contribution to the scattered wave. We may write the amplitude (1) as a sum of two terms,

$$T^{+}_{fi} = t^+[r_{12}] + t^+[r_2], \quad (2)$$

where $t^+[r_{12}]$ and $t^+[r_2]$ are the terms in Eq. (1) corresponding to the $1/r_{12}$ and $-1/r_2$ parts of the potential, respectively. After a simple change of variables and some straightforward algebra $t^+[r_{12}]$ may be recast in the form

$$t^{+}[r_{12}]=\int d\vec{r}_{12}d\vec{r}_2 e^{i\vec{q}\cdot\vec{r}_2}\phi_f^*(\vec{r}_{12}+\vec{r}_2)\phi_i(\vec{r}_2)\frac{e^{i\vec{k}_i\cdot\vec{r}_{12}}}{r_{12}}\times\left(\frac{r_{12}-z_{12}}{|\vec{r}_{12}+\vec{r}_2|-(z_{12}+z_2)}\right)^{i\eta_+}, \quad (3)$$

where $\vec{q}=\vec{k}_i-\vec{k}_f$ is the momentum transfer.

To obtain the leading term in the expansion of $t^+[r_{12}]$ and of $t^+[r_2]$ in inverse powers of k_i or k_f, the terms $t^+[r_{12}]$ and $t^+[r_2]$ are approximated by[2]

$$t^{+}[r_2]\approx t_A^{+}[r_2]=-(4\pi/k_f^2)\int d\vec{r}_1 e^{i\vec{k}_i\cdot\vec{r}_1}\phi_f^*(\vec{r}_1)\phi_i(r_2=0), \quad (4a)$$

$$t^{+}[r_{12}]\approx t_A^{+}[r_{12}]=\frac{4\pi}{k_i^2}\int d\vec{r}_2 e^{i\vec{q}\cdot\vec{r}_2}\phi_i(\vec{r}_2)\int d\vec{r}_{12}e^{i\vec{k}_i\cdot\vec{r}_{12}}\delta(\vec{r}_{12})\phi_f^*(\vec{r}_{12}+\vec{r}_2)\left(\frac{r_{12}-z_{12}}{|\vec{r}_{12}+\vec{r}_2|-(z_{12}+z_2)}\right)^{i\eta_+}. \quad (4b)$$

In effect, the replacements

$$\frac{e^{-i\vec{k}_f\cdot\vec{r}_2}}{r_2}\to\frac{4\pi}{k_f^2}e^{-i\vec{k}_f\cdot\vec{r}_2}\delta(\vec{r}_2), \quad (5a)$$

$$\frac{e^{i\vec{k}_i\cdot\vec{r}_{12}}}{r_{12}}\to\frac{4\pi}{k_i^2}e^{i\vec{k}_i\cdot\vec{r}_{12}}\delta(\vec{r}_{12}), \quad (5b)$$

have been made for the $\vec{r}_2$ integration in $t^+[r_2]$ and the $\vec{r}_{12}$ integration in $t^+[r_{12}]$, respectively. (These replacements will be discussed in detail in Sec. III.) The second term in the approximate T matrix, $t_A^+[r_2]$, is higher order in inverse powers of k_i, k_f than is the first term $t_A^+[r_{12}]$, and thus is neglected in the approximation.[1,2] We therefore concentrate on $t_A^+[r_{12}]$; it is, in fact, this term which leads to the indeterminate phase. Continuing the reasoning of Ref. 2, we proceed to perform the $\vec{r}_{12}$ integration on the right-hand side of Eq. (4b) which seemingly yields

$$\phi_f^*(\vec{r}_2)(r_2-z_2)^{-i\eta_+}\lim_{r\to 0}e^{i\eta_+\ln(r-z)},$$

the last factor of which is the indeterminate phase usually written as $\lim_{\epsilon\to 0}e^{i\eta_+\ln\epsilon}$. The result thus obtained is[2]

$$t_A^+[r_{12}]=\frac{4\pi}{k_i^2}\lim_{\epsilon\to 0}e^{i\eta_+\ln\epsilon}\int d\vec{r}_2 e^{i\vec{q}\cdot\vec{r}_2}\phi_i(\vec{r}_2)\times\phi_f^*(\vec{r}_2)(r_2-z_2)^{-i\eta_+}. \quad (6)$$

As shown in Refs. 3–5, the integral in Eq. (6) can be performed in closed form for arbitrary bound final states in the hydrogen spectrum, yielding finally a simple analytic approximation for the eikonal exchange amplitude.

The fallacy in the above procedure can be seen as follows. If, apart from the δ function, the integrand in the $\vec{r}_{12}$ integral in Eq. (4b) were well behaved at the origin, there would be no question about evaluating it at the origin. However, the integrand contains the factor $(r_{12}-z_{12})^{i\eta_+}$ which is not well defined at the origin, and it is wholly inappropriate to "evaluate" it at the origin, even as an indeterminate phase. The reason for this is that the δ-function integral

$$f=\int d\vec{r}\,\delta(\vec{r})(r-z)^{i\eta_+} \quad (7)$$

is undefined in a deeper sense, due to the infinite oscillatory nature of $(r-z)^{i\eta_+}$ at the origin. To help us see this let us write out Eq. (7) in rectangular coordinates,

$$f=\int dx\,dy\,dz\,\delta(x)\delta(y)\delta(z)\times[(x^2+y^2+z^2)^{1/2}-z]^{i\eta_+}. \quad (8)$$

Let us now, for sake of argument, integrate over y and z first, yielding

$$f=\int_{-\infty}^{\infty}dx\,\delta(x)(x^2)^{(1/2)i\eta_+}, \quad (9)$$

which may be thought of as a one-dimensional integral exhibiting the nature of the indeterminate three-dimensional integral of Eq. (7). Indeed, following the procedure of Ref. 2, Eq. (9) would yield

$$f=\lim_{\epsilon\to 0}e^{i\eta_+\ln\epsilon}. \quad (10)$$

The fact that Eq. (10) is not a valid result can be shown by replacing $\delta(x)$ in Eq. (9) by a defining sequence of functions and seeing what the formal limit of the thusly defined sequence of integrals yields. This limit, of course, is the precise definition of the δ-function integral. We have carried out this procedure with four well-known sequences for $\delta(x)$:

$$\delta(x)=\lim_{\epsilon\to 0}\frac{\sin(x/\epsilon)}{\pi x}, \quad (11a)$$

$$\delta(x)=\lim_{\epsilon\to 0}\frac{\epsilon[1-\cos(x/\epsilon)]}{\pi x^2}, \quad (11b)$$

$$\delta(x)=\lim_{\epsilon\to 0}\frac{\epsilon}{\pi(\epsilon^2+x^2)}, \quad (11c)$$

$$\delta(x)=\lim_{\epsilon\to 0}\frac{e^{-x^2/\epsilon^2}}{\epsilon\sqrt{\pi}}. \quad (11d)$$

For each of the cases (11a)–(11d), we obtained for the δ-function integral, Eq. (9), an indeterminate phase factor multiplied by a term whose modulus is different from unity. Furthermore, the nonunity modulus terms were different for each case. The results we obtained for f in Eq.

(9), corresponding to Eqs. (11a)–(11d), are

$$f=\lim_{\epsilon\to 0} e^{i\eta_+\ln\epsilon}\Gamma(1+i\eta_+)\frac{\sinh(\frac{1}{2}\pi\eta_+)}{\frac{1}{2}\pi\eta_+}, \qquad (12a)$$

$$f=\lim_{\epsilon\to 0} e^{i\eta_+\ln\epsilon}\frac{\Gamma(1+i\eta_+)}{1-i\eta_+}\frac{\sinh(\frac{1}{2}\pi\eta_+)}{\frac{1}{2}\pi\eta_+}, \qquad (12b)$$

$$f=\lim_{\epsilon\to 0} e^{i\eta_+\ln\epsilon}/\cosh(\tfrac{1}{2}\pi\eta_+), \qquad (12c)$$

$$f=\lim_{\epsilon\to 0} e^{i\eta_+\ln\epsilon}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}i\eta_+)/\sqrt{\pi}. \qquad (12d)$$

Note that in all cases the limit $\eta_+\to 0$ yields the same well-defined value of $f=1$. However, at any finite value of η_+ the results (12a)–(12d) generally all differ in *both* magnitude and phase. Other results can of course be obtained for other representations of the δ function and for other choices of order of integration in Eqs. (7) or (8). The important point is to recognize that it is not sufficient to merely extract $(r-z)^{i\eta_+}$ from the δ-function integral as an indeterminate phase factor. Instead, a careful examination of the origin of the δ function is necessary for a meaningful approximation for the $\vec{r}_{12}$ integral in Eq. (3) for $t^+[r_{12}]$. This shall be made in the next section.

III. APPROXIMATION FOR EIKONAL EXCHANGE AMPLITUDE

We have seen that the δ-function $\vec{r}_{12}$ integral of Eq. (4b) is not well defined. As remarked earlier, the origin of the δ function is the replacement given by Eq. (5b). Let us investigate the justification of this replacement. Although the formal justification of this replacement is best made via a momentum-space analysis, it suffices for the moment to note that the $\vec{r}_{12}$ coordinate-space integral of Eq. (3) is of the form

$$h=\int \frac{e^{i\vec{k}_i\cdot\vec{r}}}{r}H(\vec{r})d\vec{r}, \qquad (13)$$

where $H(\vec{r})$ is bounded. For k_i very large, so that the exponential term has many oscillations over regions where $H(\vec{r})/r$ varies relatively slowly, the contributions to the integral will be very small except in the neighborhood of the $1/r$ singularity, i.e., at the origin. *If H* were smooth at the origin we would then have, for $\epsilon\ll 1$,

$$h\approx\int_{r<\epsilon} d\vec{r}\,\frac{e^{i\vec{k}_i\cdot\vec{r}}}{r}H(\vec{r})\approx H(0)\int_{r<\epsilon} d\vec{r}\,\frac{e^{i\vec{k}_i\cdot\vec{r}}}{r}$$
$$\approx H(0)\int_{\text{all }r} d\vec{r}\,\frac{e^{i\vec{k}_i\cdot\vec{r}}}{r}=H(0)\,\frac{4\pi}{k_i^2}, \qquad (14)$$

which is of course equivalent to the replacement given by Eq. (5b). In our case, the $H(\vec{r})$ of the $\vec{r}_{12}$ integral in Eq. (3) is *not* smooth at the origin because of the infinite oscillations of $(r-z)^{i\eta_+}$; hence the simple replacement given by Eq. (5b) is not valid. Instead, let us note that for our case we may write

$$H(\vec{r})=G(\vec{r})g(\vec{r}), \qquad (15)$$

where

$$G(\vec{r})=\phi_f^*(\vec{r}+\vec{r}_2)[|\vec{r}+\vec{r}_2|-(z+z_2)]^{-i\eta_+}, \qquad (16a)$$

$$g(\vec{r})=(r-z)^{i\eta_+}. \qquad (16b)$$

Since $G(\vec{r})$ is bounded and is smooth at the origin we may still use the same reasoning for large k_i as before, and again noting that the contributions to the integral will be very small except near the origin we obtain, again for $\epsilon\ll 1$,

$$h=\int d\vec{r}\,\frac{e^{i\vec{k}_i\cdot\vec{r}}}{r}G(\vec{r})g(\vec{r})\approx\int_{r<\epsilon} d\vec{r}\,\frac{e^{i\vec{k}_i\cdot\vec{r}}}{r}G(\vec{r})g(\vec{r})$$
$$\approx G(0)\int_{r<\epsilon} d\vec{r}\,\frac{e^{i\vec{k}_i\cdot\vec{r}}}{r}g(\vec{r})\approx G(0)$$
$$\times\int_{\text{all }r} d\vec{r}\,\frac{e^{i\vec{k}_i\cdot\vec{r}}}{r}(r-z)^{i\eta_+}. \qquad (17)$$

This last integral is completely well defined and can be evaluated, for example, in spherical coordinates to give

$$h\approx G(0)2\pi\int_0^\infty dr\int_{-1}^1 dx\, e^{ik_i rx}r^{1+i\eta_+}(1-x)^{i\eta_+}$$
$$\approx 2\pi G(0)\,\frac{2^{1+i\eta_+}}{1+i\eta_+}\int_0^\infty dr\, r^{1+i\eta_+}e^{-ik_i r}{}_1F_1(1;2+i\eta_+;2ik_i r)$$
$$\approx\frac{4\pi}{k_i^2}G(0)(\tfrac{1}{2}k_i)^{-i\eta_+}\Gamma(1+i\eta_+)e^{\pi\eta_+/2}. \qquad (18)$$

Using this result in Eq. (3) we obtain, instead of Eq. (6),

$$T_{fi}^+\approx t_A^+[r_{12}]=\frac{4\pi}{k_i^2}(\tfrac{1}{2}k_i)^{-i\eta_+}\Gamma(1+i\eta_+)e^{\pi\eta_+/2}$$
$$\times\int d\vec{r}_2 e^{i\vec{q}\cdot\vec{r}_2}\phi_i(\vec{r}_2)\phi_f^*(\vec{r}_2)$$
$$\times(r_2-z_2)^{-i\eta_+}. \qquad (19a)$$

Here the T matrix is completely well defined, with no ambiguous phase. The modulus of this T matrix is that of Eq. (6) multiplied by the factor

$$\{2\pi\eta_+/[1-\exp(-2\pi\eta_+)]\}^{1/2}.$$

This corresponds, for example, to a factor of ~3.3 in the squared modulus at 54 eV.

The same analysis may be made for the prior form of the T matrix. Furthermore, since the integral in Eq. (19) can be performed analytically,[3-5] our final result is also analytic and, for example, for the case in which the initial state is the ground state, may be written as

$$T_{fi}^{\pm} \approx \frac{16\pi^3 \eta_\pm e^{\pi\eta_\pm/2}}{k_\pm^{2+i\eta_\pm} \sinh\pi\eta_\pm} \times D_{\beta\vec{\lambda}} [\beta \mp i(\vec{q}+\vec{\lambda})\cdot\hat{z}]^{-i\eta_\pm} [\beta^2 + (\vec{q}+\vec{\lambda})^2]^{i\eta_\pm - 1} \big|_{\vec{\lambda}=0}, \quad (19b)$$

where $\eta_- = 1/k_f$ and $D_{\beta\vec{\lambda}}$ is a differential operator such that

$$\phi_f^*(\vec{r})\phi_i(\vec{r}) = D_{\beta\vec{\lambda}} r^{-1} e^{-(\beta r + i\vec{\lambda}\cdot\vec{r})}\big|_{\vec{\lambda}=0}. \quad (19c)$$

Here $k_+ = k_i$ and $k_- = k_f$.

It is worth pointing out that for elastic scattering ($k_i = k_f$) from *ns* states there is *no* post-prior "discrepancy" since for the post case the appropriate choice of $\hat{z}$ is $\hat{z} = \hat{k}_i$ and for the prior case the appropriate choice is $\hat{z} = \hat{k}_f$. Thus $\vec{q}\cdot\hat{z}$(post) $= -\vec{q}\cdot\hat{z}$(prior) and therefore our approximations for $T_{ii}^{\pm}$ are identical.[15]

The entire analysis leading to both the indeterminate phase of Ref. 2 and our result can be perhaps even more clearly understood in momentum space. The basic integral to be performed is again, before any approximation, of the form

$$h(\vec{k}_i) = \int d\vec{r}\, \frac{e^{i\vec{k}_i\cdot\vec{r}}}{r} G(\vec{r}) g(\vec{r}). \quad (20)$$

We define the Fourier transforms

$$\overline{G}(\vec{q}) = \int e^{-i\vec{q}\cdot\vec{r}} G(\vec{r})\, d\vec{r}, \quad (21a)$$

$$\overline{g}(\vec{q}) = \int e^{-i\vec{q}\cdot\vec{r}} g(\vec{r})\, d\vec{r}. \quad (21b)$$

Then via the faltung theorem,

$$H(\vec{q}) = (2\pi)^{-3} \int d\vec{k}\, \overline{g}(\vec{k}) \overline{G}(\vec{q}-\vec{k}). \quad (22)$$

If the faltung theorem is used again in Eq. (20) we obtain

$$h(\vec{k}_i) = \frac{4\pi}{(2\pi)^6} \int d\vec{q}\, d\vec{k}\, \frac{\overline{g}(\vec{k})\overline{G}(\vec{q}-\vec{k})}{|\vec{k}_i + \vec{q}|^2} \quad (23a)$$

$$= \frac{4\pi}{(2\pi)^6} \int d\vec{q}\, d\vec{k}\, \frac{\overline{g}(\vec{k})\overline{G}(\vec{q})}{|\vec{k}_i + \vec{q} + \vec{k}|^2}. \quad (23b)$$

Using the defining equations for g and G, Eqs. (16), we can explicitly evaluate the Fourier transforms $\overline{g}(\vec{k})$ and $\overline{G}(\vec{q})$ since they are of the same general form as the integral in Eq. (19a), and we find

$$\overline{g}(\vec{q}) = 4\pi\Gamma(1+i\eta_+) 2^{i\eta_+} \times \lim_{\gamma\to 0} \left(-\frac{\partial}{\partial\gamma}\right) [(\gamma^2 + q^2)^{-i\eta_+ - 1} (\gamma + i\vec{q}\cdot\hat{z})^{i\eta_+}], \quad (24a)$$

$$\overline{G}(\vec{q}) = e^{i\vec{q}\cdot\vec{r}_2} 4\pi\Gamma(1-i\eta_+) 2^{-i\eta_+} \times \overline{D}_{\beta\vec{\lambda}} [\beta^2 + (\vec{\lambda}-\vec{q})^2]^{i\eta_+ - 1} [\beta - i(\vec{\lambda}-\vec{q})\cdot\hat{z}]^{-i\eta_+} \big|_{\vec{\lambda}=0}, \quad (24b)$$

where $\overline{D}_{\beta\vec{\lambda}}$ is a differential operator such that

$$\phi_f^*(\vec{r}) = \overline{D}_{\beta\vec{\lambda}} r^{-1} e^{-(\beta r + i\vec{\lambda}\cdot\vec{r})}\big|_{\vec{\lambda}=0}. \quad (24c)$$

An examination of Eqs. (24a) and (24b) reveals that $\overline{g}(\vec{q})$ falls off as q^{-3} for $|\vec{q}| \to \infty$ and $\overline{G}(\vec{q})$ falls off at least that rapidly. Thus for large k_i the major contribution to Eq. (23) comes from $q \ll k_i$ and $k \ll k_i$. If we therefore ignore both $\vec{q}$ and $\vec{k}$ in the denominator of Eq. (23b) the resulting integrals can be formally performed to yield

$$h(\vec{k}_i) \approx (4\pi/k_i^2) g(\vec{r}=0) G(\vec{r}=0). \quad (25)$$

The right-hand side, of course, does not exist for our case because $g(\vec{r})$ is undefined at the origin. If we express $g(\vec{r}=0)$ as a limiting expression as $r \to 0$ we in fact reproduce the result Eq. (6), which is that of Ref. 2 with its attendant ambiguities. The problem from the momentum-space point of view is that $\int \overline{g}(\vec{k}) d\vec{k}$ is logarithmically divergent for large k. It therefore is not legitimate to remove the k dependence in the denominator of Eq. (23b) since that dependence is necessary for convergence of the $\vec{k}$ integration. Put another way, while the major contribution to the exact integral comes from $k \ll k_i$, removing $\vec{k}$ from the denominator term causes a slow but steady accumulation of an infinite spurious contribution for large k. On the other hand no such problem arises for the $\vec{q}$ integration. The reason for this is that when $\overline{G}(\vec{q})$ falls off only as q^{-3} for large q, the oscillatory $e^{i\vec{q}\cdot\vec{r}_2}$ term guarantees the convergence of $\int \overline{G}(\vec{q}) d\vec{q}$. Thus, if instead of setting both q and k to zero in the denominator of Eq. (23b) we set only $q = 0$, we obtain the well-defined approximation

$$h(\vec{k}_i) \approx \frac{4\pi}{(2\pi)^3} G(\vec{r}=0) \int d\vec{k}\, \frac{\overline{g}(\vec{k})}{|\vec{k}_i + \vec{k}|^2}$$

$$= G(\vec{r}=0) \int \frac{e^{i\vec{k}_i\cdot\vec{r}}}{r} (r - z)^{i\eta_+} d\vec{r}, \quad (26)$$

which is precisely our present result, Eq. (17).

IV. RESULTS FOR e^--H ELASTIC SCATTERING AND CONCLUSION

From Eq. (19b) the exchange amplitude $g_{ii}^{\pm} \equiv -T_{ii}^{\pm}/2\pi$, for the case of elastic scattering of electrons by hydrogen, is given by

$$g_{ii}^{\pm} = \frac{8\pi\eta e^{\pi\eta/2}}{k^{2+i\eta} \sinh\pi\eta} \frac{(2 - iq^2/2k)^{-i\eta - 1}}{(4+q^2)^{2-i\eta}} \times [-i\eta(4+q^2) + 4(i\eta - 1)(2 - iq^2/2k)], \quad (27)$$

where $k = 1/\eta = k_i$ and θ is the scattering angle. We have evaluated this amplitude and in Tables I, II, and III have compared its squared modulus to the earlier results[2,3] and to the "exact" numeri-

TABLE I. Squared modulus of e^--H elastic scattering exchange amplitude (in units of a_0^2/sr) vs scattering angle θ for 200-eV incident electrons.

θ (deg)	This work	Ref. 3	Exact eikonal post (Ref. 9)
2	3.37×10^{-2}	1.66×10^{-2}	3.71×10^{-2}
3	3.29×10^{-2}	1.62×10^{-2}	3.62×10^{-2}
5	3.05×10^{-2}	1.50×10^{-2}	3.34×10^{-2}
7	2.74×10^{-2}	1.35×10^{-2}	2.97×10^{-2}
10	2.19×10^{-2}	1.08×10^{-2}	2.33×10^{-2}
15	1.32×10^{-2}	6.48×10^{-3}	1.34×10^{-2}
20	7.05×10^{-3}	3.47×10^{-3}	6.68×10^{-3}
40	3.73×10^{-4}	1.83×10^{-4}	1.87×10^{-4}
60	2.31×10^{-5}	1.13×10^{-5}	9.20×10^{-6}
80	2.71×10^{-6}	1.33×10^{-6}	1.27×10^{-5}
100	6.20×10^{-7}	3.05×10^{-7}	1.23×10^{-5}
120	2.30×10^{-7}	1.13×10^{-7}	1.06×10^{-5}

cally obtained eikonal results at the energies (200, 100, and 50 eV) for which they are available.[9] Since the "post" and "prior" results differ in the exact eikonal case we show both results at the two energies (100 and 50 eV) for which they are available. In Fig. 1 we compare $|g_{ii}^{+}|^2$ at $\theta = 2°$ obtained from Eq. (27) to the corresponding result of Ref. 3 over a wide range of incident energies. We also show the exact eikonal results of Ref. 9 at the three energies for which they are available. (As can be seen from the tables, the post and prior exact eikonal results differ negligibly at $\theta = 2°$. Hence we have plotted the points for $|g_{ii}^{+}|^2$ only.) Furthermore, to guide the eye we have connected these three exact points by using a three-term interpolation formula involving inverse powers of k. As can be seen, the good agreement between the exact results and the earlier results of Ref. 3 at 50 eV is wholly accidental, and while the results of the three different calculations converge to each other at extremely high energy, the present result, as expected, is the better high-energy approximation to the eikonal exchange amplitudes.

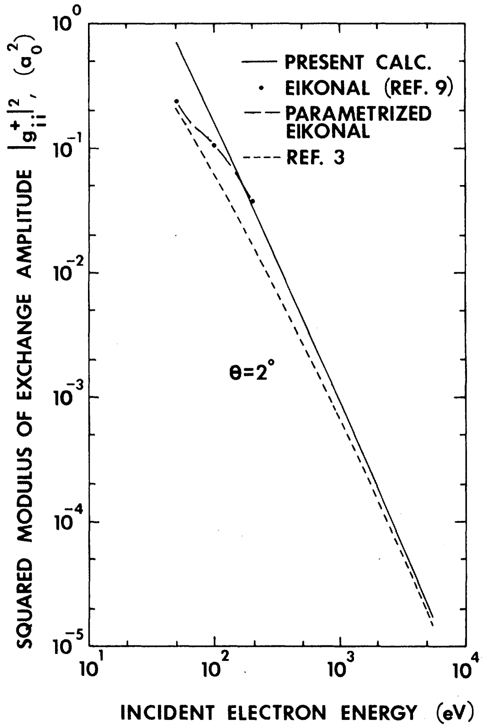

FIG. 1. Squared modulus $|g_{ii}^{+}|^2$, of eikonal exchange amplitude for elastic e^-–H scattering at a scattering angle of 2° as a function of incident electron kinetic energy. The solid curve is the present calculation [Eq. (27)]. The dashed curve is the earlier results of Ref. 3. The three solid dots are the exact eikonal post results of Ref. 9. The long-dashed curve is a simple interpolation of the three exact results.

In conclusion we see that the widely used approximation method for calculating eikonal ex-

TABLE II. Squared modulus of e^--H elastic scattering exchange amplitude (in units of a_0^2/sr) vs scattering angle θ for 100-eV incident electrons.

θ (deg)	This work	Ref. 3	Exact eikonal post (Ref. 9)	Exact eikonal prior (Ref. 9)
2	1.57×10^{-1}	6.12×10^{-2}	1.05×10^{-1}	1.05×10^{-1}
3	1.55×10^{-1}	6.04×10^{-2}	1.03×10^{-1}	1.03×10^{-1}
5	1.49×10^{-1}	5.81×10^{-2}	9.86×10^{-2}	9.97×10^{-2}
7	1.41×10^{-1}	5.47×10^{-2}	9.24×10^{-2}	9.44×10^{-2}
10	1.24×10^{-1}	4.84×10^{-2}	8.07×10^{-2}	8.43×10^{-2}
15	9.33×10^{-2}	3.63×10^{-2}	5.85×10^{-2}	6.47×10^{-2}
20	6.39×10^{-2}	2.49×10^{-2}	3.81×10^{-2}	4.60×10^{-2}
40	8.35×10^{-3}	3.25×10^{-3}	3.02×10^{-3}	8.47×10^{-3}
60	8.75×10^{-4}	3.40×10^{-4}	2.20×10^{-5}	1.83×10^{-3}
80	1.24×10^{-4}	4.82×10^{-5}	7.69×10^{-5}	4.13×10^{-4}
100	2.93×10^{-5}	1.14×10^{-5}	1.40×10^{-4}	2.55×10^{-4}
120	1.10×10^{-5}	4.28×10^{-6}	1.51×10^{-4}	4.75×10^{-4}

TABLE III. Squared modulus of e^--H elastic scattering exchange amplitude (in units of a_0^2/sr) vs scattering angle θ for 50-eV incident electrons.

θ (deg)	This work	Ref. 3	Exact eikonal post (Ref. 9)	Exact eikonal prior (Ref. 9)
2	7.09×10^{-1}	2.08×10^{-1}	2.39×10^{-1}	2.39×10^{-1}
3	7.04×10^{-1}	2.07×10^{-1}	2.37×10^{-1}	2.38×10^{-1}
5	6.89×10^{-1}	2.02×10^{-1}	2.30×10^{-1}	2.33×10^{-1}
7	6.66×10^{-1}	1.96×10^{-1}	2.21×10^{-1}	2.26×10^{-1}
10	6.22×10^{-1}	1.83×10^{-1}	2.02×10^{-1}	2.11×10^{-1}
15	5.27×10^{-1}	1.55×10^{-1}	1.63×10^{-1}	1.81×10^{-1}
20	4.21×10^{-1}	1.24×10^{-1}	1.21×10^{-1}	1.47×10^{-1}
40	1.11×10^{-1}	3.25×10^{-2}	1.53×10^{-2}	4.55×10^{-2}
60	2.04×10^{-2}	5.99×10^{-3}	3.06×10^{-5}	1.42×10^{-2}
80	3.93×10^{-3}	1.15×10^{-3}	2.40×10^{-3}	3.89×10^{-3}
100	1.04×10^{-3}	3.06×10^{-4}	3.76×10^{-3}	2.28×10^{-3}
120	4.09×10^{-4}	1.20×10^{-4}	3.93×10^{-3}	7.76×10^{-3}

change amplitudes, introduced in Refs. 1–5, which has led to the well-known indeterminate phase has been misformulated. The basic premises leading to that formulation have been recast in correct form in this work, leading to a new approximate eikonal exchange amplitude with a well-defined phase and a different modulus. This new exchange amplitude can now be unambiguously combined with direct eikonal amplitudes to obtain the total differential cross sections for collisions involving indistinguishable particles. Despite our reformulation, however, since the approximation to the exact eikonal exchange amplitude is inherently a very high-energy one, its use is limited by the fact that the contribution from the exchange amplitude is most significant at low energies. Nonetheless the simple analytic form of the approximation is appealing and we anticipate it will be useful in the energy region above ~100 eV for e^--H scattering. Furthermore, the approximation can be extended to more complex atoms and should be of use in corresponding energy regions for such systems. We intend to combine our new exchange amplitudes with corresponding direct amplitudes both in the post and prior cases and for the Glauber case ($\vec{q}\cdot\hat{z}=0$) and to make detailed comparisons with experiment for a number of systems. It is difficult to extend the approximation to lower energies. While the approximation stands on its own merits, it may well not be the first term in a convergent expansion leading to the exact eikonal result.[16] Nevertheless corrections to the approximation to improve its accuracy and extend its validity to lower energies are possible, and we are currently working in that direction.

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