Note on the traces of the angular momentum operators

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It is shown that angular momentum operator identities between ladder operators and rotation operators can be used to study the trace problem. Using an explicit representation of the rotation matrices, the author evaluates the trace of the operator $J_-^k\ J_+^k$ in closed form. The identities are found to be particularly useful in providing checks for the sum of the traces.

Angular momentum plays an important role in the study of many branches of physics. Since the time when Rose[1] established the connection between statistical tensors and angular momentum operators, there has been considerable interest in the problem of calculating the traces of the various products of angular momentum operators. In the past these traces have been calculated either by using the direct application of ladder operators[2] or by making use of the Wigner-Eckart theorem.[3] Recently the algebra of angular momentum operators has been extensively used in the study of coherent states[4] and also in the projection of good angular momentum states for a many-body system.[5] In these studies one uses certain angular momentum operator identities[4,5] which connect the products of expoential form of the step-up and step-down operators with that of the rotation operator. The purpose of this note is to look at the trace problem from the new viewpoint of these angular momentum operator identities.

Let us consider the evaluation of the trace of the operator $J_-^k J_+^k$, using the operator identity[4,5]

$$\exp(\rho J_-)\exp(-\omega J_+) =(\exp(-i\alpha J_z)\exp(-i\beta J_y)\exp(-i\gamma J_z)\,, \qquad (1)$$

where the Euler parameters α, β, and γ are related to ρ and ω by the following relations:

$$\exp(\tfrac{1}{2}i\alpha) = [(\rho/\omega)(1-\rho\omega)]^{1/4}\,, \qquad (2a)$$

$$\exp(\tfrac{1}{2}i\gamma) = [(\omega/\rho)(1-\rho\omega)]^{1/4}\,, \qquad (2b)$$

$$\cos\tfrac{1}{2}\beta = (1-\rho\omega)^{1/2}\,. \qquad (2c)$$

Taking the trace of the operator relation (1) in the spherical basis, we get

$$\mathrm{Tr}[\exp\rho J_-\exp(-\omega J_+)] = \sum_m \exp[-i(\alpha+\gamma)m]\,d^j_{mm}(\beta)\,, \qquad (3)$$

where $d^j_{mm}(\beta)$ are the reduced rotation matrices,[1,6] and the sum over m goes from $-j$ to j. Explicitly, $d^j_{mm}(\beta)$ are given by[6]

$$d^j_{mm}(\beta) = \sum_t (-1)^t \frac{(j+m)!\,(j-m)!}{(j+m-t)!\,(j-m-t)!\,(t!)^2} \times (\cos\tfrac{1}{2}\beta)^{2j-2t}(\sin\tfrac{1}{2}\beta)^{2t}\,, \qquad (4)$$

where the sum is taken over all values of t which lead to non-negative factorials.

For the present problem it is sufficient to take $\rho=\omega$. Expressions (2)–(4) then give us

$$\mathrm{Tr}(J_-^k J_+^k) =(k!)^2 \sum_{m,t} \frac{(j+m)!\,(j-m)!}{(j+m-t)!\,(j-m-k)!\,(t!)^2(k-t)!}\,. \qquad (5)$$

We would next like to calculate the sum in expression (5). For this purpose we consider the binomial expansion

$$(1+x)^{j+m}(1+y)^{j-m} = \sum_{t,k} \frac{(j+m)!}{(j+m-t)!\,t!}\,\frac{(j-m)!}{(j-m-k)!\,k!}\,x^t y^k\,. \qquad (6)$$

Summing over m from $-j+t$ to $j-k$, we get

$$(1+x)^j(1+y)^j \frac{\gamma^{-j+t}-\gamma^{j-k+1}}{1-\gamma} = \sum_{m,t,k} \frac{(j+m)!}{(j+m-t)!\,t!}\,\frac{(j-m)!}{(j-m-k)!\,k!}\,x^t y^k\,, \qquad (7)$$

where

$$\gamma = (1+x)/(1+y)\,.$$

Treating x and y as complex variables and using contour integration, we get from expression (7)

$$\frac{1}{(2\pi i)^2}\oint dx\,dy\,x^{-t-1}y^{-k-1} \times (1+x)^j(1+y)^j\frac{\gamma^{-j+t}-\gamma^{j-k+1}}{1-\gamma} = \sum_m \frac{(j+m)!\,(j-m)!}{(j+m-t)!\,(j-m-k)!\,t!\,k!}\,.$$

The contours are unit circles around the origin. Multiplying both sides by $k!/(k-t)!\,t!$ and carrying out summation over t, we finally get

$$\frac{1}{(2\pi i)^2}\oint dx\,dy \frac{x^{-k-1}y^{-k-1}}{y-x} \times [(1+y)^{2j-k+1}(1+2x+xy)^k - (1+y)^k(1+x)^{2j+1}] = \sum_{m,t} \frac{(j+m)!\,(j-m)!}{(j+m-t)!\,(j-m-k)!(t!)^2(k-t)!}. \quad (8)$$

Therefore, we have shown that the summation over m, t in expression (5) is the coefficient of $x^k y^k$ in the expansion of the function

$$(y-x)^{-1}[(1+y)^{2j-k+1}(1+2x+xy)^k - (1+y)^k(1+x)^{2j+1}]. \quad (9)$$

It is now a simple matter to show, by writing $1+2x+xy$ as $(1+x)(1+y)-(y-x)$ in the function given by (9), that the coefficient of $x^k y^k$ is given by

$$\sum_{s=0}^{k} \frac{(2j+1)!}{(2j-k-s)!(k+s+1)!}\,\frac{k!}{(k-s)!\,s!}. \quad (10)$$

Writing the expansion of $(1+x)^{2j+k+1}$ in two different ways, we finally get the following value of the coefficient of $x^k y^k$:

$$(2j+k+1)!/(2k+1)!(2j-k)!.$$

Putting this in expression (5) we find that the trace of the operator $J_-^k J_+^k$ is given by the following closed-form expression:

$$\mathrm{Tr}(J_-^k J_+^k) = (k!)^2(2j+k+1)!/(2k+1)!(2j-k)!. \quad (11)$$

This is the same expression which was first found[3] using the Wigner-Eckart theorem.

A great advantage in the use of operator identities in studying the trace problem is that they provide sum rules which can be used to check the tabulated traces.[2,3] These sum rules can be easily established by noting that the rotation operator on the right-hand side in Eq. (1) can be brought to diagonal form by a similarity transformation. Putting $\rho=\omega=\lambda$ and taking the trace in the diagonal form gives the following relation:

$$\mathrm{Tr}\{[\exp(\lambda J_-)][\exp(-\lambda J_+)]\} = \sin[\tfrac{1}{2}(2j+1)\varphi]/\sin\tfrac{1}{2}\varphi, \quad (12)$$

where φ is given by[4,5]

$$\cos\tfrac{1}{2}\varphi = \tfrac{1}{2}(2-\lambda^2). \quad (13)$$

Putting $\lambda = 1$ and expanding the exponentials in Eq. (12), we establish the following sum rule:

$$\sum_{k=0}^{2j} \frac{(-1)^k}{(k!)^2}\,\mathrm{Tr}(J_-^k J_+^k) = \frac{\sin[(2j+1)\frac{1}{3}\pi]}{\sin\frac{1}{3}\pi}. \quad (14)$$

Lastly I would like to remark that the use of rotation operators provides quite a bit of flexibility in the study of traces of the angular momentum operators, since one could also make use of other representations of reduced rotation matrices, like the one in terms of hypergeometric functions.

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