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Theoretical hyperfine structure of muonic helium*†

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The study of the hyperfine structure of the muonic helium atom ($\alpha^{++}\mu^-e^-$) can provide a test of quantum electrodynamics and yield precise values for the magnetic moment and mass of the negative muon. In the lowest-order approximation the hyperfine structure interval $\Delta\nu$ is given by the Fermi formula. The principal corrections, which contribute about 10 MHz to the splitting, arise from the relativistic, radiative, and recoil effects. The theoretical hyperfine structure of the ground state of muonic helium is given as $\Delta\nu = 4465.1 \pm 1.0$ MHz.

I. INTRODUCTION

The need for a theoretical account of the hyperfine structure[1-4] of the recently observed muonic helium atom[5] ($\alpha^{++}\mu^-e^-$) has been accentuated by recently proposed microwave resonance experiments.[6] The combination of the theoretical and experimental values of the hyperfine structure interval $\Delta\nu$ can yield precise values of the magnetic moment and mass of the negative muon, which can be compared with more accurately known values of the positive muon as a test of *CPT* invariance. In addition, a comparison of the hyperfine structure for muonic helium and muonium of sufficient accuracy might detect the effect of axial vector neutral currents predicted in gauge theories.[7]

In a muonic helium atom, the muon is bound closely to the α particle, and the electron wave function is approximately hydrogenic. Therefore the hyperfine splitting of the ground state is given approximately by the Fermi formula[8] and has about the same value as that of muonium,[9] but is inverted because of the different signs of the magnetic moments of μ^+ and μ^-. However, the pseudonucleus $(\alpha\mu)^+$ is an oversized "nucleus" with its charge and magnetic moment spreading over a region having a radius about 10^2 times larger than that of the α particle. Consequently, apart from the usual reduced-mass correction, the principal difference between $\Delta\nu$ for $\alpha^{++}\mu^-e^-$ and for μ^+e^- is due to the penetration of the electron inside the pseudonucleus $(\alpha\mu)^+$. This gives a correction term of relative order m_e/m_μ and is an effect similar to the hyperfine structure anomaly first observed for deuterium,[10] where the electron penetrates inside the distribution of magnetism in the deuteron.

To compute the hyperfine structure of muonic helium, we work first in the Pauli approximation[11] and then consider high-order corrections. The lowest-order hyperfine interaction operator reads,[11] in atomic units,

$$H_{hfs} = \frac{\alpha^2}{4}\left(\frac{m_e}{m_\mu}\right)\left[-\frac{8\pi}{3}\vec{\sigma}_e\cdot\vec{\sigma}_\mu\delta^3(\vec{r}_{e\mu}) + \frac{1}{r_{e\mu}^3}\left(\vec{\sigma}_e\cdot\vec{\sigma}_\mu - \frac{3(\vec{\sigma}_e\cdot\vec{r}_{e\mu})(\vec{\sigma}_\mu\cdot\vec{r}_{e\mu})}{r_{e\mu}^2}\right)'\right], \tag{1.1}$$

where α is the fine-structure constant, m the rest mass, $\vec{\sigma}$ the Pauli spin operator, $r_{e\mu}$ the distance between the electron and the muon, and $\delta^3(\vec{r}_{e\mu})$ the Dirac δ function in three-dimensional space. In (1.1), the prime ( )′ indicates that when ( )′ occurs in any integral over position space, replace ( )′ by zero for $r<\epsilon$, evaluate the integral, and then take the limit of $\epsilon\to 0$. For spherically symmetric states, only the first term of (1.1) survives; then the hyperfine splitting of the ground state of muonic helium is

$$(\Delta\nu)_F = \langle H_{\mathrm{hfs}}\rangle_{\mathrm{singlet}} - \langle H_{\mathrm{hfs}}\rangle_{\mathrm{triplet}} = \tfrac{8}{3}\pi\alpha^2(m_e/m_\mu)\langle\delta^3(r_{e\mu})\rangle , \tag{1.2}$$

where $\langle\delta^3(r_{e\mu})\rangle$ is the expectation value of the spatial part of the wave function. Here we use $(\Delta\nu)_F$ to denote the lowest-order hyperfine splitting.

The lowest-order calculation is based on three types of approximation. First, the muonic helium atom is treated in Pauli nonrelativistic limit. By using a proper relativistic wave function, a relativistic correction term is obtained. Second, the Dirac single-particle theory is used. By considering quantum field theoretical effects, radiative correction terms are obtained. Third, the interaction between the electron and the muon is approximated by the Breit interaction.[12] After including two-photon exchange effect, a recoil (or mass) correction term appears.

In Sec. II, we obtain the lowest-order hyperfine splitting by using correlated wave functions which explicitly contain interparticle coordinates. We then consider the relativistic, radiative, and recoil and other effects in the subsequent sections, III, IV, and V, respectively. In Sec. VI, we summarize the results. Hydrogenic wave functions and their magnetic corrections needed in this work are presented in the Appendix.

II. LOWEST-ORDER HYPERFINE SPLITTING

The calculation of the hyperfine structure of muonic helium requires correlated wave functions. Accurate variational wave functions, which explicitly contain interparticle coordinates, have been obtained[1,13] for the ground as well as for excited states of the muonic helium atom. There is good convergence of matrix elements of radial operators as the number of terms in the wave function is increased.

The correlated wave functions of the ground state have the form[1,13]

$$\psi(\vec{r}_e, \vec{r}_\mu) = \sum^{l+m+n\leq\omega} C_{lmn} U_{lmn}, \tag{2.1}$$

where ω is chosen to have certain selected values, and

$$U_{lmn} = (1/4\pi)\exp(-\tfrac{1}{2}ar_e - \tfrac{1}{2}br_\mu) r_{e\mu}^l r_e^m r_\mu^n. \tag{2.2}$$

Here the parameters a and b and the coefficients C_{lmn} are determined variationally. Hence from (1.2) and (2.1), we obtain the lowest-order hyperfine splitting of the ground state of muonic helium as

$$(\Delta\nu)_F = \frac{8\pi}{3}\alpha^2\left(\frac{m_e}{m_\mu}\right) \sum_{l'=0}^{l'+m'+n'\leq\omega} \sum_{l=0}^{l+m+n\leq\omega} C_{l'm'n'} C_{lmn} \times \frac{(m'+m+n'+n)!}{(a+b)^{m'+m+n'+n}}. \tag{2.3}$$

We have evaluated $(\Delta\nu)_F$ with correlated wave functions of various choices of terms. With $l+m+n\leq\omega$, we obtain $(\Delta\nu)_F$ as presented in Table I and Fig. 1. Even with wave functions up to 455 terms, the lowest-order hyperfine splitting is still converging slowly. To better account for the correlation without increasing the number of terms in the trial wave function, we include more terms with high powers of r_{12} and drop terms with high powers of r_1 and r_2; specifically, we choose $l+m+n\leq\omega$ with $m,n\leq 3$. The convergence of the total energy along this new sequence, about 10^{-3} ppm, is as good as for the wave functions with $l+m+n\leq\omega$, while $(\Delta\nu)_F$ converges faster. Results from this sequence of variational wave functions are presented in Table II and Fig. 2. A least-square hyperbola fit gives the extrapolated value

$$(\Delta\nu)_F = 4455.2 \pm 1.0\ \text{MHz}. \tag{2.4}$$

The error estimate is based on the greatest possible variation of the fitting curve. We note that the difference between the above extrapolated value and our calculated value for the 496-term wave function is about 5 MHz.

The large uncertainty in (2.4) may be understood

TABLE I. Lowest-order hyperfine splitting of the ground state of the muonic helium atom for variational wave functions with $l+m+n\leq\omega$.

ω	Number of terms	$(\Delta\nu)_F$ (MHz)
0	1	4483.9
1	4	4484.4
2	10	4485.1
3	20	4483.2
4	35	4484.0
5	56	4480.0
6	84	4480.2
7	120	4474.7
8	165	4474.7
9	220	4472.5
10	286	4471.0
11	364	4470.0
12	455	4468.9

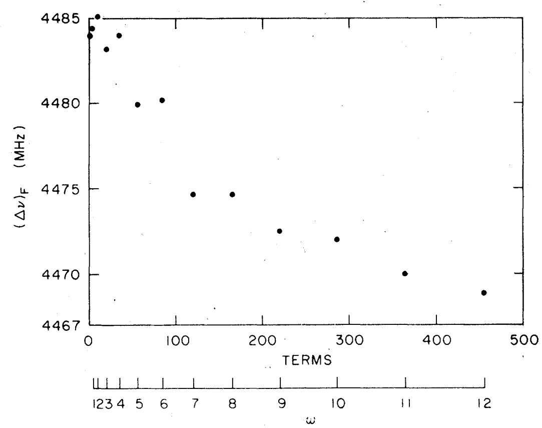

FIG. 1. Lowest-order hyperfine splitting of the ground state of the muonic helium atom for variational wave functions with $l+m+n\leq\omega$.

TABLE II. Lowest-order hyperfine splitting of the ground state of the muonic helium atom for variational wave functions with $l+m+n\leq\omega$ and $m,n\leq 3$.

ω with $m,n\leq 3$	Number of terms	$(\Delta\nu)_F$ (MHz)
3	20	4483.2
7	80	4474.7
10	128	4471.2
13	176	4468.5
17	240	4465.5
21	304	4463.4
25	368	4462.0
29	432	4461.1
33	496	4460.4
Extrapolated value		4455.2 ± 1.0

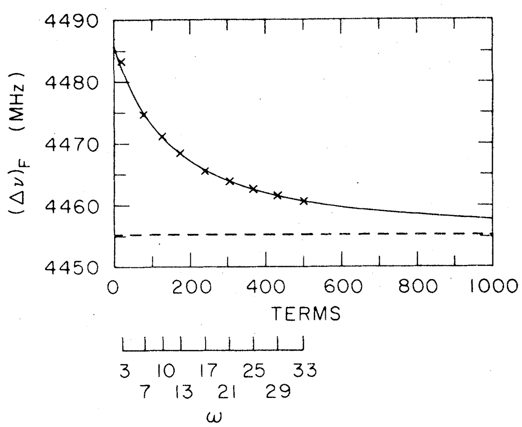

FIG. 2. Lowest-order hyperfine splitting of the ground state of the muonic helium atom for variational wave functions with $l+m+n \le \omega$ and $m, n \le 3$.

in the following way. Because of the singular character of the contact operator $\delta^3(\vec{r}_{e\mu})$, its expectation value is very sensitive to the electron-muon correlation. On the other hand, highly correlated wave functions are very difficult to obtain by the variational method using an energy minimization procedure. The difficulty lies in the fact that a trial wave function in error by $O(\delta)$ can give an energy in error by $O(\delta^2)$. Therefore, to obtain a wave function with 1 ppm accuracy, we would be required to minimize the energy to an accuracy of 10^{-6} ppm. Furthermore, the total energy of the muonic helium atom is comparatively insensitive to the electron-muon correlation because the binding energy of the muon is the dominant part of the total energy. Namely, the electron-muon interaction energy in the muonic helium atom accounts for only 0.25% of the total energy, while in the ordinary helium atom the electron-electron interaction energy accounts for 37% of its total energy.

Had we considered the composite system $\alpha^{++}\mu^- e^-$ as an atom with a point nucleus $(\alpha\mu)^+$, we would have been dealing with a muonium isotope. In fact, it may seem to be a fairly good assumption since in the Bohr model the electron orbit is 400 times larger than the radius of the pseudonucleus $(\alpha\mu)^+$. We would then obtain the lowest-order hyperfine splitting

$$(\Delta\nu)_F = \tfrac{8}{3}\alpha^2(m_e/m_\mu)[1+m_e/(m_\mu+m_\alpha)]^{-3} = 4517.0\ \text{MHz}, \tag{2.5}$$

which differs from the more accurate value 4455.2 MHz by 62 MHz.

III. RELATIVISTIC CORRECTION

Using nonrelativistic wave functions gives an approximation to the hyperfine splitting which is in error of relative order $(Z\alpha)^2$. Thus to compute the hyperfine splitting to this order we have to consider an appropriate relativistic equation.

A. Relativistic wave functions

Let 1 stand for the electron and 2 for the muon, and we can then write the Breit equation[12] for the muonic helium atom as

$$\{H_{(1)}+H_{(2)}-E\}\psi(\vec{r}_1,\vec{r}_2) = \left\{\frac{Z\alpha}{r_1}+\frac{Z\alpha}{r_2}-\frac{\alpha}{r_{12}}+\frac{\alpha}{2r_{12}}\left(\vec{\alpha}_1\cdot\vec{\alpha}_2+\frac{(\vec{\alpha}_1\cdot\vec{r}_{12})(\vec{\alpha}_2\cdot\vec{r}_{12})}{r_{12}^2}\right)\right\}\psi(\vec{r}_1,\vec{r}_2), \tag{3.1}$$

where $H_{(i)} = c(\vec{\alpha}_i\cdot\vec{p}_i)+\beta_i M_i c^2$ are the Dirac Hamiltonians with reduced masses M_i. By introducing two effective charges Z_1 and Z_2, we can rewrite (3.1) as

$$\{H^0_{(1)}+H^0_{(2)}-E\}\psi(\vec{r}_1,\vec{r}_2) = \left\{\frac{(Z-Z_1)\alpha}{r_1}+\frac{(Z-Z_2)\alpha}{r_2}-\frac{\alpha}{r_{12}}+\frac{\alpha}{2r_{12}}\left(\vec{\alpha}_1\cdot\vec{\alpha}_2+\frac{(\vec{\alpha}_1\cdot\vec{r}_{12})(\vec{\alpha}_2\cdot\vec{r}_{12})}{r_{12}^2}\right)\right\}\psi(\vec{r}_1,\vec{r}_2), \tag{3.2}$$

where $H^0_{(i)} = c(\vec{\alpha}_i\cdot\vec{p}_i)+\beta_i M_i c^2 - Z_i/r_i$. We treat the right-hand side of (3.2) as a small perturbation and solve for the left-hand side.

The unperturbed wave function is $\psi^0(\vec{r}_1,\vec{r}_2)=\psi^0_1\psi^0_2$, where ψ^0_1 and ψ^0_2 are solutions of the Lorentz-invariant Dirac equations

$$\{H^0_{(i)}-E^0_{(i)}\}\psi^0_i = 0,\quad i=1,2. \tag{3.3}$$

The ground-state solutions of (3.3) have the form[14]

$$\left\langle \vec{r}_i \,\middle|\, \begin{matrix} m_i \\ \tfrac{1}{2}\end{matrix}\right\rangle = F(r_i)\begin{pmatrix}1\\0\\C_iY_{10}(\Omega_i)\\-\sqrt{2}C_iY_{11}(\Omega_i)\end{pmatrix},\qquad \left\langle \vec{r}_i \,\middle|\, \begin{matrix} m_i \\ -\tfrac{1}{2}\end{matrix}\right\rangle = F(r_i)\begin{pmatrix}0\\1\\\sqrt{2}C_iY_{1-1}(\Omega_i)\\-C_iY_{10}(\Omega_i)\end{pmatrix}, \tag{3.4}$$

where the radial part is

$$F(r_i) = (M_i Z_i \alpha)^{3/2}\left(\frac{1+\gamma_i}{\Gamma(1+2\gamma_i)\pi}\right)^{1/2} \times (2M_i Z_i \alpha r_i)^{\gamma_i - 1} e^{-M_i Z_i \alpha r_i}, \tag{3.5}$$

and

$$C_i = i(\tfrac{4}{3}\pi)^{1/2}\frac{(1-\gamma_i)}{Z_i\alpha}, \qquad \gamma_i = [1-(Z_i\alpha)^2]^{1/2}. \tag{3.6}$$

Here m_i denotes the magnetic quantum number of the total angular momentum, $Y_{lm}(\Omega_i)$ the spherical harmonics, and $\Gamma(t)$ the gamma function. In split notation,[11] we have

$$\left\langle \vec{r}_i \middle| \begin{matrix} m_i \\ \frac{1}{2} \end{matrix} \right\rangle = F(r_i)\begin{pmatrix} \varphi_i^{(+)} \\ \phi_i^{(+)} \end{pmatrix}, \qquad \left\langle \vec{r}_i \middle| \begin{matrix} m_i \\ -\frac{1}{2} \end{matrix} \right\rangle = F(r_i)\begin{pmatrix} \varphi_i^{(-)} \\ \phi_i^{(-)} \end{pmatrix}, \tag{3.7}$$

where

$$\varphi_i^{(+)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \phi_i^{(+)} = a_i\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_i\begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \varphi_i^{(-)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \phi_i^{(-)} = -b_i^*\begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_i^*\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.8}$$

and

$$a_i = C_i Y_{10}(\Omega_i), \quad b_i = -\sqrt{2}\,C_i Y_{11}(\Omega_i). \tag{3.9}$$

In the ground state of the muonic helium atom, the total angular momenta j_1 and j_2 of the electron and the muon, respectively, are coupled into singlet and triplet states of the grand total angular momentum J, i.e.,

singlet:

$$\left| \begin{matrix} JM \\ 00 \end{matrix} \right\rangle = \frac{1}{\sqrt{2}}\left\{ \left| \begin{matrix} m_1 & m_2 \\ \frac{1}{2} & -\frac{1}{2} \end{matrix} \right\rangle - \left| \begin{matrix} m_1 & m_2 \\ -\frac{1}{2} & \frac{1}{2} \end{matrix} \right\rangle \right\}; \tag{3.10}$$

triplet:

$$\left| \begin{matrix} J & M \\ 1 & 1 \end{matrix} \right\rangle = \left| \begin{matrix} m_1 & m_2 \\ \frac{1}{2} & \frac{1}{2} \end{matrix} \right\rangle, \quad \left| \begin{matrix} J & M \\ 1 & 0 \end{matrix} \right\rangle = \frac{1}{\sqrt{2}}\left\{ \left| \begin{matrix} m_1 & m_2 \\ \frac{1}{2} & -\frac{1}{2} \end{matrix} \right\rangle + \left| \begin{matrix} m_1 & m_2 \\ -\frac{1}{2} & \frac{1}{2} \end{matrix} \right\rangle \right\}, \quad \left| \begin{matrix} J & M \\ 1 & -1 \end{matrix} \right\rangle = \left| \begin{matrix} m_1 & m_2 \\ -\frac{1}{2} & -\frac{1}{2} \end{matrix} \right\rangle. \tag{3.11}$$

We now reduce the wave functions (3.10) and (3.11) into a form which is convenient for the subsequent calculation. Consider first the singlet wave function. This wave function depends on the coordinates $\vec{r}_1$ and $\vec{r}_2$ and sixteen spinor components. In split notation, we can express it as a four-component wave function:

$$\left\langle \vec{r} \middle| \begin{matrix} J & M \\ 0 & 0 \end{matrix} \right\rangle = \frac{1}{\sqrt{2}}F(r_1)F(r_2)\left[\begin{bmatrix} \varphi_1^{(+)}\varphi_2^{(-)} \\ \varphi_1^{(+)}\phi_2^{(-)} \\ \phi_1^{(+)}\varphi_2^{(-)} \\ \phi_1^{(+)}\phi_2^{(-)} \end{bmatrix} - \begin{bmatrix} \varphi_1^{(-)}\varphi_2^{(+)} \\ \varphi_1^{(-)}\phi_2^{(+)} \\ \phi_1^{(-)}\varphi_2^{(+)} \\ \phi_1^{(-)}\phi_2^{(+)} \end{bmatrix} \right]. \tag{3.12}$$

In terms of the eigenvectors, $|SM_s\rangle$, of the total spin $\vec{S} = \vec{s}_1 + \vec{s}_2$, we can write the singlet wave function in a matrix form

$$\left(\left\langle \vec{r} \middle| \begin{matrix} J & M \\ 0 & 0 \end{matrix} \right\rangle\right) = F(r_1)F(r_2)\begin{bmatrix} 1 & \frac{1}{2}(a_2+a_2^*) & \frac{1}{2}(a_1+a_1^*) & \frac{1}{2}(a_1a_2^*+a_1^*a_2+b_1b_2^*+b_1^*b_2) \\ 0 & -(1/\sqrt{2})b_2^* & (1/\sqrt{2})b_1^* & -(1/\sqrt{2})(a_1b_2^*-b_1^*a_2) \\ 0 & \frac{1}{2}(a_2^*-a_2) & \frac{1}{2}(a_1-a_1^*) & \frac{1}{2}(a_1a_2^*-a_1^*a_2-b_1b_2^*+b_1^*b_2) \\ 0 & -(1/\sqrt{2})b_2 & (1/\sqrt{2})b_1 & (1/\sqrt{2})(b_1a_2^*-a_1^*b_2) \end{bmatrix}, \tag{3.13}$$

where the ith column is the ith component of $\langle \vec{r} | {}^{JM}_{00} \rangle$, expressed in the total spin eigenvectors $|00\rangle$, $|11\rangle$, $|10\rangle$, and $|1-1\rangle$ in the descending order.

Following the same procedure, we can write the triplet in a similar form. Since any one of the triplet states may be used in the calculation of the hyperfine splitting, we consider explicitly only

$$\left(\left\langle \vec{r} \middle| \begin{matrix} J & M \\ 1 & 1 \end{matrix} \right\rangle\right) = F(r_1)F(r_2) \begin{bmatrix} 0 & (1/\sqrt{2})b_2 & -(1/\sqrt{2})b_1 & (1/\sqrt{2})(a_1b_2 - b_1a_2) \\ 1 & a_2 & a_1 & a_1a_2 \\ 0 & (1/\sqrt{2})b_2 & (1/\sqrt{2})b_1 & (1/\sqrt{2})(a_1b_2 + b_1a_2) \\ 0 & 0 & 0 & b_1b_2 \end{bmatrix}. \tag{3.14}$$

We can easily show that in the nonrelativistic limit the singlet (3.13) and triplet (3.14) reduce to the familiar Pauli-Schrödinger wave functions.

B. Relativistic hyperfine splitting

The residual Coulomb interaction and Breit interaction in (3.2) are considered by a first-order perturbation method. Since the residual Coulomb interaction only contributes to level shift but not to the level splitting, we are left with the Breit interaction, namely,

$$H_B = \frac{\alpha}{2r_{12}}\left(\vec{\alpha}_1 \cdot \vec{\alpha}_2 + \frac{\vec{\alpha}_1 \cdot \vec{r}_{12}\vec{\alpha}_2 \cdot \vec{r}_{12}}{r_{12}^2}\right), \tag{3.15}$$

which gives rise to the hyperfine splitting. The relativistic hyperfine splitting of the ground state of muonic helium can thus be written as

$$\Delta\nu = \left\langle \begin{matrix} J & M \\ 0 & 0 \end{matrix} \middle| H_B \middle| \begin{matrix} J & M \\ 0 & 0 \end{matrix} \right\rangle - \left\langle \begin{matrix} J & M \\ 1 & 1 \end{matrix} \middle| H_B \middle| \begin{matrix} J & M \\ 1 & 1 \end{matrix} \right\rangle. \tag{3.16}$$

To calculate (3.16), we write H_B in split notation,

$$H_B = \frac{\alpha}{2r_{12}} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \left(\vec{\sigma}_1 \cdot \vec{\sigma}_2 + \frac{\vec{\sigma}_1 \cdot \vec{r}_{12}\vec{\sigma}_2 \cdot \vec{r}_{12}}{r_{12}^2}\right), \tag{3.17}$$

where σ's are the Pauli spin matrices. We now take the eigenvectors of the total spin as bases and express everything in a matrix form. We can show that

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.18}$$

and

$$\sigma_{12} \equiv \left(\frac{\vec{\sigma}_1 \cdot \vec{r}_{12}\vec{\sigma}_2 \cdot \vec{r}_{12}}{r_{12}^2}\right) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & t_z^2 & \sqrt{2}t_z(t_x - it_y) & (t_x - it_y)^2 \\ 0 & \sqrt{2}t_z(t_x + it_y) & (t_x^2 + t_y^2 - t_z^2) & -\sqrt{2}t_z(t_x - it_y) \\ 0 & (t_x + it_y)^2 & -\sqrt{2}t_z(t_x + it_y) & t_z^2 \end{bmatrix}, \tag{3.19}$$

where $t_x = x_{12}/r_{12}$, $t_y = y_{12}/r_{12}$, and $t_z = z_{12}/r_{12}$ are direction cosines of the vector $\vec{r}_{12}$.

Therefore for the singlet, we have

$$\left\langle \begin{matrix} J & M \\ 0 & 0 \end{matrix} \middle| H_B \middle| \begin{matrix} J & M \\ 0 & 0 \end{matrix} \right\rangle = \int d^3r_1 d^3r_2 \frac{\alpha}{2r_{12}} \mathrm{Tr}\left\{\left(\left\langle \begin{matrix} J & M \\ 0 & 0 \end{matrix} \middle| \vec{r} \right\rangle \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) (\vec{\sigma}_1 \cdot \vec{\sigma}_2 + \sigma_{12}) \left(\left\langle \vec{r} \middle| \begin{matrix} J & M \\ 0 & 0 \end{matrix} \right\rangle\right)\right\}, \tag{3.20}$$

or in shorthand notations,

$$\left\langle \begin{matrix} J & M \\ 0 & 0 \end{matrix} \middle| H_B \middle| \begin{matrix} J & M \\ 0 & 0 \end{matrix} \right\rangle = \int d^3r_1 d^3r_2 \frac{\alpha}{2r_{12}} \times |F(r_1)F(r_2)|^2 \mathrm{Tr}\begin{Bmatrix} J & M \\ 0 & 0 \end{Bmatrix}, \tag{3.21}$$

where ( ) and Tr{ } denote a matrix and the trace of a matrix, respectively. Similarly we can obtain for the triplet

$$\left\langle \begin{matrix} J & M \\ 1 & 1 \end{matrix} \middle| H_B \middle| \begin{matrix} J & M \\ 1 & 1 \end{matrix} \right\rangle = \int d^3r_1 d^3r_2 \frac{\alpha}{2r_{12}} |F(r_1)F(r_2)|^2 \mathrm{Tr}\begin{Bmatrix} J & M \\ 1 & 1 \end{Bmatrix}. \tag{3.22}$$

To compute the relativistic hyperfine splitting

(3.16), we first calculate the difference between the traces in (3.21) and (3.22) and then perform the integration. The difference between the traces can be obtained more easily by making use of the fact that the trace of two matrices is independent of the order of multiplication.

Hence the hyperfine splitting (3.16) of the ground state of muonic helium can be written as

$$\Delta\nu = \tfrac{128}{3}\alpha\pi C_1 C_2^* \int_0^\infty dr_1 \int_0^\infty dr_2 \, |F(r_1)F(r_2)|^2 \times r_1^2 r_2^2 r_</r_>^2 , \tag{3.23}$$

which is expressible in terms of hypergeometric function ${}_2F_1$. The result is

$$\Delta\nu = \tfrac{1}{3}(\Delta\nu)_F \frac{(1-\gamma_1)(1-\gamma_2)}{(Z_1\alpha)^2(Z_2\alpha)^2} \frac{\Gamma(2\gamma_1+2\gamma_2+1)}{\Gamma(2\gamma_1+1)\Gamma(2\gamma_2+1)} \frac{a^{2\gamma_1-1}b^{2\gamma_2-1}}{(a+b)^{2\gamma_1+2\gamma_2-2}} \times [(\gamma_1+1)\,{}_2F_1(1, 2\gamma_1+2\gamma_2+1; 2\gamma_2+3; b/(a+b)) + (\gamma_2+1)\,{}_2F_1(1, 2\gamma_1+2\gamma_2+1; 2\gamma_1+3; a/(a+b))] \tag{3.24}$$

where $a = 2M_1(Z_1\alpha)$, $b = 2M_2(Z_2\alpha)$, and γ_i were defined previously in (3.6).

To calculate the relativistic correction to the desired order, we expand the expression (3.24) in terms of $(Z_1\alpha)$, $(Z_2\alpha)$, and $(1/M_2)$. After a little algebra we obtain

$$\Delta\nu = (\Delta\nu)_F\{1 + \tfrac{3}{2}(Z_1\alpha)^2 - \tfrac{1}{3}(Z_2\alpha)^2 + O((Z_2\alpha)^2/M_2)\} , \tag{3.25}$$

where we can identify the first correction term $\frac{3}{2}(Z_1\alpha)^2$ with the Breit term.[15] The second correction term $-\frac{1}{3}(Z_2\alpha)^2$ gives the contribution due to the muon.

IV. RADIATIVE CORRECTIONS

We consider now the processes involving the creation of virtual electron-positron pairs and the emission and reabsorption of virtual photons by the electron and the muon themselves.

A. External potentials for the muon and the electron

In order to find the radiative correction to the hyperfine splitting it is necessary to specify the forms of the external potentials observed by the electron and the muon. We consider first the four-component external potential of the electron

$$eA(\vec{r}_1) = \left\{ -\frac{Z\alpha}{r_1} + \frac{\alpha}{r_{12}} , e\mu_2\vec{\sigma}_2 \times \vec{\nabla}_1\left(\frac{1}{r_{12}}\right) \right\} , \tag{4.1}$$

where μ_2 is the muon magneton, and $\vec{\sigma}_2$ the muon Pauli spin operator. The first component of (4.1) is the Coulomb potential produced by the nuclear charge and the muon charge while the rest are the magnetic dipole potential produced by the muon magnetic moment. On averaging over the muon coordinates, we obtain

$$eA^{(E)}(\vec{r}_1) = -Z\alpha/r_1 + (\alpha/r_1)[1 - e^{-br_1}(1 + \tfrac{1}{2}br_1)] ,$$
$$e\vec{A}^{(H)}(\vec{r}_1) = e\mu_2\vec{\sigma}_2 \times \vec{\nabla}\{1/r_1)[1 - e^{-br_1}(1 + \tfrac{1}{2}br_1)]\} , \tag{4.2}$$

where $b = 2M_2Z_2\alpha$, and we have used the Pauli-Schrödinger wave function instead of Dirac wave-function. In calculations of other corrections to the hyperfine structure, relativistic modifications to the wave functions may be ignored[16,17] with an error of order $\alpha(Z\alpha)^3(\Delta\nu)_F$.

The four-component external potential of the muon is given by similar expressions as (4.2).

B. Vacuum polarization correction

In this subsection we consider the corrections due to processes involving the creation of virtual electron-positron pairs.

The polarization potential can be regarded as consisting of a Coulomb part, which is independent of the magnetic moment, and a magnetic part, which involves the magnetic moment linearly. The magnetic part is simply a modification of the magnetic field and can accordingly be expected to produce a modification in the hyperfine structure. There is, however, also an effect from the Coulomb part which arises from the fact that the lower-energy hyperfine structure state are the more tightly bound and, therefore, spend more time in the region where the vacuum polarization potential is large. These two effects are, in fact, exactly equal[16] up to the order of $\alpha(Z\alpha)$ and α^2.

The energy shift due to vacuum polarization is given by

$$\Delta E_{\mathrm{vp}} = \langle 0|\gamma \cdot eA^{\mathrm{vp}}|0\rangle , \tag{4.3}$$

where the four-component potential A^{vp} is[18]

$$A_\mu^{\mathrm{vp}}(\vec{q}) = \frac{\alpha}{2\pi}CA_\mu(\vec{q}) + \frac{\alpha}{2\pi}\vec{q}^2\int_0^1 dv \frac{2v^2(1-\frac{1}{3}v^2)}{4+\vec{q}^2(1-v^2)}A_\mu(\vec{q}) + \delta A_\mu^{\mathrm{vp}}(\vec{q}) . \tag{4.4}$$

Here the first term, which contains the infinite constant factor C, is the well-known charge renormalization term and is henceforth ignored. The last term $\delta A_\mu^{\mathrm{vp}}$ is finite and entirely free of effects to be attributed to renormalization. It is, however, at least a factor $\alpha(Z\alpha)$ smaller than the leading term of $A_\mu^{\text{vp}}(\vec{q})$. So the first and the third terms of (4.4) will be ignored.

Now we calculate the vacuum-polarization contribution of the electron in the field of the α particle and the muon. We can show that in position space the Coulomb part of (4.4) has the form

$$eA_0^{\text{vp}}(\vec{r}_1) = -\frac{\alpha}{2\pi}\int_0^1 dv\,\frac{2v^2(1-\frac{1}{3}v^2)}{(1-v^2)}\left[Z\alpha\frac{e^{-sr_1}}{r_1} - b^4\alpha\left(\frac{e^{-br_1}}{2b(s^2-b^2)} + \frac{1}{(s^2-b^2)^2}\,\frac{e^{-sr_1}-e^{-br_1}}{r_1}\right)\right], \tag{4.5}$$

where $s = 2(1-v^2)^{-1/2}$. There exist spurious poles in the above integral. We can see, however, the integrand has finite limits at the spurious poles. The divergent integrals due to this origin will cancel each other when taken together. Hence in the subsequent calculation, we will retain only finite expressions. The total vacuum-polarization energy shift of the electron can now be written as

$$\Delta E_{\text{vp}}^{(1)} = 2\int d^3r_1\bar{\psi}_1\gamma_0 eA_0^{\text{vp}}\psi_1\,, \tag{4.6}$$

where $\bar{\psi}$ is the adjoint (not Hermitian) conjugate of ψ. As stated before, in the calculation of corrections to the hyperfine structure, relativistic modifications to the wave functions may be ignored with error of order $\alpha(Z\alpha)^3(\Delta\nu)_F$. We will thus use the Pauli-Schrödinger wave functions throughout this subsection. These wave functions are given in the Appendix.

Since the hyperfine dependence is taken from the linear magnetic corrections to either wave functions, we have

$$\delta_1(\Delta\nu)_F^{\text{vp}} = 4\int d^3r_1\bar{\psi}_1\gamma_0 eA_0^{\text{vp}}\delta_M\psi_1\,, \tag{4.7}$$

where $\delta_M\psi_1$ is the S-wave part of the first-order magnetic dipole perturbation to the wave function ψ_1. The reason for taking only the S-wave part is the following. Since we seek only terms linear in the magnetic moment and the vacuum-polarization potential $A_0^{\text{vp}}(\vec{r})$ is spherically symmetric, only the S-wave part of the magnetic wave functions can contribute. Consequently, in all expressions we will retain only the S-part of the magnetic wave function. The magnetic correction $\delta_M\psi_1$ to the wave function ψ_1 is calculated in the Appendix.

Substituting (4.5) for A_0^{vp} in (4.7) we obtain

$$\begin{aligned}\delta_1(\Delta\nu)_F^{\text{vp}} = \int_0^1 dv\,\frac{v^2(3-v^2)}{(1-v^2)}\Bigg\{&\alpha(Z\alpha)\left(-\frac{4}{3\pi}\right)\int\psi_1\frac{e^{-sr_1}}{r_1}\delta_M\psi_1 d^3r_1\\ &+\alpha^2\Bigg[\left(\frac{2}{3\pi}\right)\frac{b^3}{(s^2-b^2)}\int\psi_1 e^{-br_1}\delta_M\psi_1 d^3r_1 + \left(\frac{4}{3\pi}\right)\frac{b^4}{(s^2-b^2)^2}\int\psi_1\frac{e^{-sr_1}}{r_1}\delta_M\psi_1 d^3r_1\\ &-\left(\frac{4}{3\pi}\right)\frac{b^4}{(s^2-b^2)^2}\int\psi_1\frac{e^{-br_1}}{r_1}\delta_M\psi_1 d^3r_1\Bigg]\Bigg\}.\end{aligned} \tag{4.8}$$

There are four integrals inside the curly brackets of (4.8). The first integral is evaluated with the result

$$\begin{aligned}I_1' &\equiv -\left(\frac{4}{3\pi}\right)\int\psi_1\frac{e^{-sr_1}}{r_1}\delta_M\psi_1 d^3r_1\\ &= (\Delta\nu)_F\left(\frac{8M_1}{3\pi}\right)\left[\frac{(a+b)(b-a)s+(2b-a)(a+b)^2}{2b(s+a)(s+a+b)^2} + \left(\frac{a(a^2+ab+3b^2)}{2b(a+b)} + a\ln\frac{(a+b)(s+a)}{a(s+a+b)}\right)\frac{1}{(s+a)^2} - \frac{a^2}{(s+a)^3}\right]\\ &= (\Delta\nu)_F\left[\left(\frac{2}{3\pi}\right)\frac{(1-v^2)[B+2(1-v^2)^{1/2}]}{[B+(1-v^2)^{1/2}]^2} + O(Z_1\alpha\ln(Z_1\alpha))\right],\end{aligned} \tag{4.9}$$

where $B = (1/2\alpha)(M_1/M_2)$ is a dimensionless constant. The second integral is

$$\begin{aligned}I_2' &\equiv \left(\frac{2}{3\pi}\right)b\int\psi_1 e^{-br_1}\delta_M\psi_1 d^3r_1\\ &= (\Delta\nu)_F\left(\frac{4M_1}{3\pi}\right)\left(-\frac{b^3(10a^3+38a^2b+39ab^2+5b^3)}{(a+b)^4(a+2b)^3} - \frac{2ab}{(a+b)^3}\ln\frac{(a+b)^2}{a(a+2b)}\right)\\ &= (\Delta\nu)_F[-(5/12\pi)B + O(Z_1\alpha\ln(Z_1\alpha))].\end{aligned} \tag{4.10}$$

The third integral is the same as the first. The fourth integral is

$$I_4' \equiv -\left(\frac{4}{3\pi}\right)\int \psi_1 \frac{e^{-br_1}}{r_1}\delta_M\psi_1 d^3r_1$$
$$=(\Delta\nu)_F\left(\frac{8M_1}{3\pi}\right)\left(\frac{b(3a^3+16a^2b+21ab^2+3b^3)}{2(a+2b)^2(a+b)^3}+\frac{a}{(a+b)^2}\ln\frac{(a+b)^2}{a(a+2b)}\right)$$
$$=(\Delta\nu)_F\left[\left(\frac{1}{2\pi}\right)B+O(Z_1\alpha\ln(Z_1\alpha))\right]. \tag{4.11}$$

Define $I_1'=(\Delta\nu)_F I_1$, $I_2'=(\Delta\nu)_F I_2$, and $I_4'=(\Delta\nu)_F I_4$. We can then rewrite (4.8) in terms of I_1, I_2, and I_4 as

$$\delta_1(\Delta\nu)_F^{\mathrm{vp}}=(\Delta\nu)_F\left[\alpha(Z\alpha)\int_0^1 dv\frac{v^2(3-v^2)}{1-v^2}I_1+\alpha^2\left(I_2\int_0^1 dv\frac{v^2(3-v^2)}{1-v^2}\frac{b^2}{s^2-b^2}-\int_0^1 dv\frac{v^2(3-v^2)}{1-v^2}\frac{b^4}{(s^2-b^2)^2}I_1\right.\right.$$
$$\left.\left.+I_4\int_0^1 dv\frac{v^2(3-v^2)}{1-v^2}\frac{b^4}{(s^2-b^2)^2}\right)\right]. \tag{4.12}$$

Since we want to calculate the correction up to the order of $\alpha(Z\alpha)(\Delta\nu)_F$ or $\alpha^2(\Delta\nu)_F$, we retain only leading terms in the above integrals. The final result obtained is

$$\delta_1(\Delta\nu)_F^{\mathrm{vp}}=(\Delta\nu)_F[\epsilon_e+O(\alpha^2)(Z_1\alpha)], \tag{4.13}$$

where

$$\begin{aligned}\epsilon_e=&\ \alpha(Z\alpha)[\tfrac{3}{4}+(14/3\pi)B-2B^2+(4/\pi)B^3-2B^4+(2/\pi)B(2-B^2-2B^4)L_B]\\&+\alpha^2(-\tfrac{3}{4}+4B^2+\tfrac{20}{3}B^4-(1/12\pi)B\{193/3+108B^2+4(16-15B^2-24B^4)[1/(1-B^2)]\\&-(68-220B^2+29B^4+132B^6)[1/3(1-B^2)^2]+(70-99B^2-228B^4)L_B\\&+4B^2(16-15B^2-24B^4)[L_B/(1-B^2)]+B^4(42-17B^2-28B^2)[L_B/(1-B^2)^2]\}),\end{aligned} \tag{4.14}$$

where L_B is defined as

$$L_B=\frac{1}{(1-B^2)^{1/2}}\ln\left(\frac{1-(1-B^2)^{1/2}}{B}\right). \tag{4.15}$$

We consider now the vacuum-polarization contribution from the muon. By a similar procedure as in the electron case, we have

$$\begin{aligned}\delta_2(\Delta\nu)_F^{\mathrm{vp}}=&\int_0^1 dv\frac{v^2(3-v^2)}{(1-v^2)}\left\{\alpha(Z\alpha)\left(-\frac{4}{3\pi}\right)\int\psi_2\frac{e^{-sr_2}}{r_2}\delta_M\psi_2 d^3r_2\right.\\&+\alpha^2\left[\left(\frac{2}{3\pi}\right)\frac{a^2}{s^2-a^2}\int\psi_2 e^{-ar_2}\delta_M\psi_2 d^3r_2+\left(\frac{4}{3\pi}\right)\frac{a^4}{(s^2-a^2)^2}\int\psi_2\frac{e^{-sr_2}}{r_2}\delta_M\psi_2 d^3r_2\right.\\&\left.\left.-\left(\frac{4}{3\pi}\right)\frac{a^4}{(s^2-a^2)^2}\int\psi_2\frac{e^{-ar_2}}{r^2}\delta_M\psi_2 d^3r_2\right]\right\}\\=&(\Delta\nu)_F[\epsilon_\mu+O(\alpha^2)(Z_1\alpha)],\end{aligned} \tag{4.16}$$

where the subscript 2 denotes the muon. The magnetic correction $\delta_M\psi_2$ are defined in the Appendix. Integrals in (4.16) can be evaluated similarly as in the case of (4.8) by retaining only the leading terms.

Therefore we can write down the total vacuum-polarization correction to the hyperfine splitting of the ground state of muonic helium as

$$\delta(\Delta\nu)_F^{\mathrm{vp}}\equiv\delta_1(\Delta\nu)_F^{\mathrm{vp}}+\delta_2(\Delta\nu)_F^{\mathrm{vp}}$$
$$=(\Delta\nu)_F[\epsilon_e+\epsilon_\mu+O(\alpha^2)(Z_1\alpha)], \tag{4.17}$$

where ϵ_e and ϵ_μ are given in (4.14) and (4.16), respectively.

C. Self-energy correction

In this subsection we treat processes involving the emission and reabsorption of virtual photons by the electron or the muon themselves. These processes will alter the dynamic responses of the electron and the muon to each other's magnetic field and the results are the same as if they had "anomalous" magnetic moments. For an external field which is not magnetic but electrostatic, these processes again change the responses from that predicted by the single-particle Dirac theory, and the principal effect is the so-called "Lamb shift."[19]

By using the effective charge Z_1 for the electron and simply assuming

$$eA(\vec{r}_1) = [-Z_1\alpha/r_1, e\mu_2\vec{\sigma}_2\times\vec{\nabla}_1(1/r_1)], \qquad (4.18)$$

instead of the more detailed potential (4.1), the self-energy contribution has the same form as that of muonium,[9] i.e.,

$$\delta_1(\Delta\nu)_F^{se} = (\Delta\nu)_F[(\alpha/2\pi) - 0.328\,478(\alpha/\pi)^2 + 1.188(\alpha/\pi)^3 + \alpha(Z_1\alpha)(\ln 2 - \tfrac{13}{4})]. \qquad (4.19)$$

where the last term in the square brackets has been referred to as "binding correction."

We now consider the self-energy contribution from the muon. Because the muon is about 200 times heavier than the electron, the dependence of the muon wave function on the hyperfine state is expected to be about 200 times smaller than that of the electron. Aside from the contribution by the explicit appearance of a magnetic potential, the rest part of the self-energy contribution would be about a factor of $1/m_\mu$ smaller than that of the electron. Therefore the muonic contribution to the self-energy contribution reads simply[9,20]

$$\delta_2(\Delta\nu)_F^{se} = (\Delta\nu)_F[\alpha/2\pi + 0.765\,782(\alpha/\pi)^2 + 24.448(\alpha/\pi)^3 + 66\times10^{-9}], \qquad (4.20)$$

where the last term is the hadronic contribution.

We summarize the results of this subsection by writing down the self-energy contribution to the hyperfine splitting of the ground state of muonic helium as

$$\begin{aligned}\delta(\Delta\nu)_F^{se} &= \delta_1(\Delta\nu)_F^{se} + \delta_2(\Delta\nu)_F^{se}\\ &= (\Delta\nu)_F[\alpha/\pi + 0.437\,304(\alpha/\pi)^2 + 25.636(\alpha/\pi)^3 + 66\times10^{-9} + \alpha(Z_1\alpha)(\ln 2 - \tfrac{13}{4})]. \end{aligned}\qquad (4.21)$$

V. RECOIL AND OTHER CORRECTIONS

Since we have correlated wave functions and have included exact nonrelativistic reduced mass correction in the lowest-order calculation, all the nonrelativistic effects of recoil are absorbed in $\langle\delta^3(r_{12})\rangle$. By including quantum electrodynamics we will, however, expect further recoil corrections. These corrections are all produced by processes in which the electron and the muon interact twice, either through the exchange of two transverse photons or through one transverse photon and one instantaneous Coulomb interaction. By treating the core $\alpha^{++}\mu^-$ as a pseudonucleus, we may consider the muonic helium as a muonium isotope. Therefore, the recoil correction is similar to that of muonium[9,21] and is estimated to be

$$\delta(\Delta\nu)_F^{rec} = -(\Delta\nu)_F(3\alpha/\pi)[m_e/(m_\mu + m_\alpha)] \times \ln[(m_\mu + m_\alpha)/m_e]. \qquad (5.1)$$

We discuss now some of the effects which may also affect the hyperfine splitting of muonic helium. We first consider the effect of the finite size of the α particle. We have treated the α particle as a point nucleus in all our calculations. With a nucleus of finite size, however, the lower-energy hyperfine state (which is the more tightly bound) spends more time inside the nucleus and therefore sees a smaller effective nuclear charge. This is expected to raise the lower-energy level and can accordingly reduce the hyperfine splitting. Also with a nucleus of finite size, the spatial integral $\langle\delta^3(r_{12})\rangle$ will be altered because the finite-size effect modifies the wave functions. We will show that these two effects do not contribute to the order of interest.

Consider the α particle as a sphere of radius R, with the charge uniformly spreading over its volume. This gives rise to the potential

$$V(r) = \begin{cases} -(Z\alpha/R)(\tfrac{3}{2} - \tfrac{1}{2}r^2/R^2) & \text{for } r \le R\\ -Z\alpha/r & \text{for } r > R\end{cases}, \qquad (5.2)$$

which deviates from a simple Coulomb potential by

$$\delta H_c = \begin{cases} Z\alpha/r - (Z\alpha/R)(\tfrac{3}{2} - \tfrac{1}{2}r^2/R^2) & \text{for } r \le R\\ 0 & \text{for } r > R\end{cases}. \qquad (5.3)$$

This small perturbation shifts the hyperfine states by different amounts, depending on $\langle\psi_1|\delta H_c|\delta_M\psi_1\rangle$, where ψ_1 and $\delta_M\psi_1$ are the wave function and its magnetic correction defined in the Appendix. Since $aR \sim 10^{-4}$ and $bR \sim 10^{-2}$, the wave functions inside the nucleus assume the simple forms

$$\psi_1 \sim (a^3/8\pi)^{1/2}[1 + O(aR)], \qquad (5.4)$$

and

$$\delta_M\psi_1 \sim -M_1(2\pi a^3)^{-1/2} b E^M [1 + O(1/M_2)\ln(Z_1\alpha)]. \qquad (5.5)$$

By using (5.4) and (5.5), we can show that the hyperfine splitting will be reduced by an amount $(\Delta\nu)_F O(aR)(bR)$, which does not contribute to the order of interest.

We consider now the correction due to the change in the spatial integral $\langle\delta^3(r_{12})\rangle$. We can show that the ground-state solution of the Schrödinger equation for the potential (5.2) has the form

$$\psi_f(\vec{r}) = \begin{cases} (A^3/\pi)^{1/2}[1 + O(AR)^{7/2}]\exp[\tfrac{1}{2}(AR)^{1/2} - AR - \tfrac{1}{2}(AR)^{1/2}(r/R)^2] & r \le R\\ (A^3/\pi)^{1/2}[1 + O(AR)^{7/2}]\exp(-Ar) & r > R\end{cases} \qquad (5.6)$$

while for a point Coulomb potential it reads

$$\psi_p(\vec{r}) = (A^3/\pi)^{1/2} e^{-Ar}, \qquad (5.7)$$

where A denotes $\frac{1}{2}a$ or $\frac{1}{2}b$ for the electron or the muon, respectively. By assuming simple-product wave functions for the muonic helium atom, we can show that

$$\langle \delta^3(\vec{r}_{12}) \rangle_f = \langle \delta^3(\vec{r}_{12}) \rangle_p [1 + O((aR + bR)^{7/2})]. \qquad (5.8)$$

Hence the finite-size effect of the α particle does not contribute to the hyperfine splitting to the order of interest.

By a similar argument, we can show that a modification of the wave functions due to nuclear polarization does not contribute to the order of interest either.

VI. RESULTS

For the ground state of the muonic atom, the lowest-order hyperfine splitting $(\Delta\nu)_F$ has been given in (2.3) by using correlated wave functions which explicitly contain interparticle coordinates. The corrections to the lowest-order hyperfine splitting arising from the relativistic and radiative effects have been calculated to order α^2, and an estimate of the recoil effect has been given. The theoretical expression for the hyperfine splitting can thus be summarized as

$$\Delta\nu = (\Delta\nu)_F \{1 + \delta^{\text{rel}} + \delta^{\text{rad}} + \delta^{\text{rec}}\}, \qquad (6.1)$$

where $(\Delta\nu)_F$ has been given in (2.3), and the corrections are

$$\delta^{\text{rel}} = \tfrac{3}{2}(Z_e\alpha)^2 - \tfrac{1}{3}(Z_\mu\alpha)^2,$$

$$\delta^{\text{rad}} = \delta^{\text{vp}} + \delta^{\text{se}},$$

$$\delta^{\text{rec}} = -(3\alpha/\pi)[m_e/(m_\mu + m_\alpha)] \ln[(m_\mu + m_\alpha)/m_e]. \qquad (6.2)$$

Here the vacuum polarization contribution is

$$\delta^{\text{vp}} = \epsilon_e + \epsilon_\mu \qquad (6.3)$$

with ϵ_e and ϵ_μ given in (4.14) and (4.16), and the self-energy contribution is

$$\delta^{\text{se}} = \alpha/\pi + 0.437\,304(\alpha/\pi)^2 + 25.636(\alpha/\pi)^3 + 66\times10^{-9} + \alpha(Z_e\alpha)(\ln 2 - \tfrac{13}{4}), \qquad (6.4)$$

where the first four terms are due to the anomalous magnetic moments and the last term due to the electrostatic binding effect.

High-order corrections in (6.1) are calculated with effective charges $Z_\mu = 2$ and $Z_e = 1$, and the results are summarized in Table III with the total theoretical value given as $\Delta\nu = 4465.1 \pm 1.0$ MHz. We see that even with variational wave functions of up to 496 terms the extrapolated $(\Delta\nu)_F$ still has an uncertainty of about 1 MHz, which dominates the accuracy in the calculated hyperfine structure $\Delta\nu$.

TABLE III. Hyperfine splitting of the ground state of the muonic helium atom. [a]

	Contributions (MHz)	(ppm)
Lowest-order HFS: $(\Delta\nu)_F$	4455.2 ± 1.0	
Relativistic correction	0.040	8.9
Radiative correction		
Anomalous magnetic moment:		
Electron	5.166	1159.6
Muon	5.195	1165.9
Vacuum polarization:		
Electron	0.088	19.7
Muon	0.096	21.6
Binding	−0.606	−136.0
Recoil correction	−0.037	−8.3
Total calculation HFS: $\Delta\nu$	4465.1 ± 1.0	

[a] The fundamental physical constants used here are the electron mass $m_e = 0.511\,0034$ MeV, the muon mass $m_\mu = 105.659\,48$ MeV, the mass of the α particle $m_\alpha = 4.002\,603$ amu; 1 amu $= 931.501\,6$ MeV, 1 a.u. $= 27.211\,608$ eV $= 6.579\,684\,13\times10^9$ MHz, and the fine structure constant $\alpha = 1/137.035\,982$.

As an improvement of the present calculation, one may use variational wave functions of still higher number of terms. Or correlated wave functions of different types may be tried. However, the size of the computer and loss of significant figures in the minimization procedure limit the improvement that can be achieved this way. Alternatively, the standard perturbative method may be used. Or one may try to optimize the variational wave function with respect to the hyperfine interaction Hamiltonian itself. These other approaches are being probed by the present authors.

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APPENDIX: WAVE FUNCTIONS AND THEIR MAGNETIC CORRECTIONS

For the ground state of the muonic helium atom, the uncorrelated wave functions of the electron and the muon are

$$\psi_1 = (a^3/8\pi)^{1/2} e^{-ar_1/2},$$

$$\psi_2 = (b^3/8\pi)^{1/2} e^{-br_2/2}, \qquad (\text{A1})$$

multiplied by appropriate spin wave functions. Here $a = 2M_1(Z_1\alpha)$, $b = 2M_2(Z_2\alpha)$, and the subscripts

1 and 2 refer to the electron and the muon, respectively. The wave functions (A1) are eigenfunctions of the nonrelativistic Hamiltonians

$$H_i^c = P_i^2/2M_i - Z_i\alpha/r_i\,, \tag{A2}$$

with eigenvalues

$$E_i^c = -\tfrac{1}{2}M_i(Z_i\alpha)^2, \quad i=1,2\,, \tag{A3}$$

where M_i are the reduced masses with respect to the α particle, and Z_i are the effective charges.

The magnetic corrections $\delta_M\psi_i$ are the first-order magnetic dipole perturbation (linear in the magnetic interaction) to the ψ_i given in (A1), due to the perturbing Hamiltonian

$$H^M = \frac{\alpha^2}{4}\frac{m_1}{m_2}\left[-\tfrac{8}{3}\pi\delta^3(\vec{r}_{12})(\vec{\sigma}_1\cdot\vec{\sigma}_2) + \frac{1}{r_{12}^3}\left((\vec{\sigma}_1\cdot\vec{\sigma}) - \frac{3(\vec{\sigma}_1\cdot\vec{r}_{12})(\vec{\sigma}_2\cdot\vec{r}_{12})}{r_{12}^2}\right)'\right]. \tag{A4}$$

Here m_i are the rest masses, and $\vec{\sigma}_i$ the Pauli spin operators.

Writing $H_i = H_i^c + H_i^M$ and $E_i = E_i^c + E_i^M$, then $\delta_M\psi_i$ satisfies

$$(H_i^c - E_i^c)\delta_M\psi_i = (E_i^M - H_i^M)\psi_i \tag{A5}$$

and

$$\int \psi_i\delta_M\psi_i d^3r_i = 0\,. \tag{A6}$$

The action of the magnetic potential on the Coulomb S state will introduce some D state which cannot contribute to the hyperfine splitting. Therefore only the S state part of $\delta_M\psi$ will be dealt with. Furthermore, for spherically symmetric functions only the first term of (A4) contributes. Hence for the electron we have the following wave equation to be solved:

$$[-(1/2M_1)\vec{\nabla}_1^2 - Z_1\alpha/r_1 + \tfrac{1}{2}M_1(Z_1\alpha)^2]\delta_M\psi_1 = (E_1^M - H_1^M)\psi_1\,, \tag{A7}$$

where

$$H_1^M = -\tfrac{2}{3}\pi\alpha^2\left(\frac{m_1}{m_2}\right)\int \psi_2^\dagger(\vec{\sigma}_1\cdot\vec{\sigma}_2)\delta^3(\vec{r}_{12})\psi_2 d\tau_2\,, \tag{A8}$$

$$E_1^M = -\tfrac{2}{3}\pi\alpha^2\left(\frac{m_1}{m_2}\right)\int \psi_1^\dagger\psi_2^\dagger(\vec{\sigma}_1\cdot\vec{\sigma}_2)\delta^3(\vec{r}_{12})\psi_1\psi_2 d\tau_1 d\tau_2\,. \tag{A9}$$

Here the integrations in (A8) and (A9) include the spin coordinates implicity.

Using the wave functions defined in (A1), we find the solution of the inhomogeneous differential equation (A7) to be

$$\delta_M\psi_1 = -M_1\left(\frac{2}{\pi a^3}\right)^{1/2} E_1^M e^{-ar_1/2} \times\left\{\frac{1}{r_1} - a\left[\gamma + \ln\left(\frac{ab}{a+b}r_1\right) - Ei(-br_1)\right] + \frac{a(a^2+3ab+5b^2)}{2b(a+b)} - \frac{a^2}{2}r_1 - e^{-br_1}\left(\frac{1}{r_1} + \frac{(a+b)^2}{2b}\right)\right\}, \tag{A10}$$

where γ is the Euler's constant, and $Ei(x)$ the exponential-integral function.

Similarly we find the magnetic correction $\delta_M\psi_2$ to the wave function ψ_2 of the muon as

$$\delta_M\psi_2 = -M_2\left(\frac{2}{\pi b^3}\right)^{1/2} E_2^M e^{-br_2/2} \times\left\{\frac{1}{r_2} - b\left[\gamma + \ln\left(\frac{ab}{a+b}r_2\right) - Ei(-ar_2)\right] + \frac{b(b^2+3ba+5a^2)}{2a(a+b)} - \frac{b^2}{2}r_2 - e^{-ar_2}\left[\frac{1}{r_2} + \frac{(a+b)^2}{2a}\right]\right\}. \tag{A11}$$

Note that $E_1^M = E_2^M = E^M$ and

$$(E^M)_{\text{singlet}} - (E^M)_{\text{triplet}} = (\Delta\nu)_F\,. \tag{A12}$$

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