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Coherent states and the resonance of a quantum damped oscillator

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A quantum-mechanical model of a damped harmonic oscillator (both with time-independent and time-dependent parameters) is studied in the framework of the linear Schrödinger equation with a Hermitian nonstationary Hamiltonian. Integrals of the motion of this equation and their eigenstates, including coherent states, are constructed. The influence of an external harmonic force to the time evolution of various average values calculated over coherent states is considered, including the resonance case. The specific symmetry of the Hamiltonian leading to the new concept of loss-energy states is discussed.

I. INTRODUCTION

Coherent states, which were used already by Schrödinger in the first papers on quantum mechanics,[1] now serve as a very convenient tool for solving various problems in almost all fields of physics, especially in quantum optics,[2,3] in the theories of superfluidity and superconductivity,[4] in the theory of elementary particles,[5] etc. Therefore, during recent years a great number of papers were devoted to various generalizations of Glauber's[2] coherent states and to the construction of such states for concrete quantum systems; see, e.g., Refs. 6 and 7 and references therein.

A method of constructing coherent states for an arbitrary quantum dynamical system (i.e., a system described by an equation of the type $i\,\partial\psi/\partial t=\hat{H}\psi$, where $\hat{H}$ is a certain operator) based on the employment of quantum integrals of the motion was proposed by Malkin and Man'ko.[8] (For the first time the significance of time-dependent quantum integrals of the motion for solving the Schrödinger equation was emphasized by Lewis and Riesenfeld.[9]) This method was used in Ref. 10 for the detailed study of multidimensional quantum systems with time-dependent Hamiltonians which were the most general quadratic forms of the operators of coordinates and momenta. Besides, in Ref. 10 some special cases of general quadratic systems, such as a charged particle in time-dependent electromagnetic fields, were considered in detail. In Ref. 11 the integrals of the motion method were used to derive new equations for the Green's function and the density matrix of an arbitrary quantum system. These equations were solved in the case of the most general quadratic time-dependent Hamiltonian, and the results were applied, specifically, to the problem of a charged particle moving with damping in crossed magnetic and electric fields. The anharmonic oscillator was studied with the aid of the integrals of the motion method in Ref. 12.

In the present paper we apply the general method of Ref. 10 to construct coherent states and some other systems of solutions in the case of a very interesting example of a quadratic nonstationary quantum system, namely, in the case of the system which can be considered as a quantum model of a damped harmonic oscillator describing this oscillator in terms of pure quantum-mechanical states, i.e., with the aid of the Schrödinger equation. This equation was considered earlier in many papers,[13-21] but the solutions for all values of parameters, as well as coherent states, have not been obtained. To fill this gap is one of the aims of this paper; this is done in Secs. II and III. Moreover, we consider very interesting group-theoretical aspects of the problem, and consider in this connection in Sec. IV the new concept of loss-energy states. Finally, considering the physical significance of the model and its relation to other models describing quantum dissipative systems, we discuss in Secs. V and VI the general problem of the quantization of a given classical system.

II. INTEGRALS OF THE MOTION AND COHERENT STATES OF A DAMPED FORCED OSCILLATOR WITH TIME-DEPENDENT PARAMETERS

Let us consider the Hamiltonian

$$\hat{H}(t)=\tfrac{1}{2}\left[\hat{p}^2e^{-2\Gamma(t)}+\omega_0^2(t)\,e^{2\Gamma(t)}\hat{x}^2\right]-f(t)\,e^{2\Gamma(t)}\hat{x}\,. \quad (1)$$

It leads to the following equations of the motion:

$$\begin{aligned}&\dot{x}=pe^{-2\Gamma(t)},\\&\dot{p}=-\omega_0^2(t)\,e^{2\Gamma(t)}x+f(t)\,e^{2\Gamma(t)},\\&\ddot{x}+2\dot{\Gamma}(t)\,\dot{x}+\omega_0^2(t)\,x=f(t)\,.\end{aligned} \quad (2)$$

Therefore, the quantum system with such a Hamiltonian can be considered as a quantum analog of a classical damped forced harmonic oscillator with time-dependent parameters. For the first time, Hamiltonian (1) (in the case of constant parameters) was suggested independently by Caldirola[13] and Kanai.[14]

Following Ref. 10, we are first of all looking for the integrals of the motion, i.e., the operators $\hat{I}(t)$ satisfying the equation $[i(\partial/\partial t)-\hat{H},\hat{I}]=0$ (we suppose $\hbar=1$). For the systems with quadratic Hamiltonians all integrals of the motion can be constructed from two (we consider a one-dimensional problem) independent linear integrals of the motion of the form $\hat{I}(t)=a(t)\hat{x}+b(t)\hat{p}+\delta(t)$.

Calculating the commutator $[\hat{H},\hat{I}]$, one obtains the following equation for the coefficients a, b, and δ:

$$\dot{a}=\omega_0^2(t)e^{2\Gamma(t)}b\,,\qquad \dot{b}=-e^{-2\Gamma(t)}a\,,\qquad \dot{\delta}=-f(t)e^{2\Gamma(t)}b\,.$$

Therefore, all linear integrals of the motion have the form

$$\hat{I}(t)=b(t)\hat{p}-\dot{b}(t)e^{2\Gamma(t)}\hat{x}-\int f(t)e^{2\Gamma(\tau)}b(\tau)d\tau \tag{3}$$

(the lower limit of the integral in the right-hand part may be arbitrary), $b(t)$ being an arbitrary solution to the equation

$$\ddot{b}+2\dot{\Gamma}\dot{b}+\omega_0^2(t)b=0\,. \tag{4}$$

Evidently, there exist two and only two independent linear integrals of the motion. If one chooses the function $b(t)$ in the form $b(t)=i2^{-1/2}\epsilon(t)$, where $\epsilon(t)$ is a complex function satisfying Eq. (4) and the additional condition

$$e^{2\Gamma(t)}(\dot{\epsilon}\epsilon^*-\dot{\epsilon}^*\epsilon)=2i \tag{5}$$

[the left-hand part of Eq. (5) does not depend on time due to Eq. (4); ϵ^* means the complex conjugate to ϵ], then one obtains two independent mutually Hermitian conjugate linear integrals of the motion satisfying the relation $[\hat{A}(t),\hat{A}^+(t)]=1$. The operator $A(t)$ thus obtained has the form

$$\begin{aligned}\hat{A}(t)&=(i/\sqrt{2})[\epsilon(t)\hat{p}-\dot{\epsilon}(t)e^{2\Gamma(t)}\hat{x}]+\delta(t)/\sqrt{2}\,,\\ \delta(t)&=-i\int\epsilon(\tau)e^{2\Gamma(t)}f(t)d\tau\,.\end{aligned} \tag{6}$$

The eigenfunctions ψ_α of the operator $\hat{A}(t)$ are just the coherent states. To obtain the explicit expressions for these states, one should solve the system of equations

$$\hat{A}(t)\psi_\alpha(x;t)=\alpha\psi_\alpha(x;t)\,,$$

$$\hat{A}^+(t)\psi_\alpha(x;t)=\frac{\partial}{\partial\alpha}[e^{(1/2)|\alpha|^2}\psi_\alpha(x;t)]\cdot e^{-(1/2)|\alpha|^2}.$$

These equations determine the function $\psi_\alpha(x;t)$ up to a factor dependent only on time. This factor can be obtained from the Schrödinger equation and the normalization condition. After all calculations, one can obtain the expression

$$\begin{aligned}\psi_\alpha(x;t)=(\pi\epsilon^2)^{-1/4}\exp\Big[&\frac{i\dot{\epsilon}}{2\epsilon}e^{2\Gamma(t)}x^2+\frac{\sqrt{2}}{\epsilon}\alpha x-\frac{\epsilon^*}{2\epsilon}\alpha^2-\tfrac{1}{2}|\alpha|^2-\frac{x\delta}{\epsilon}-\frac{\epsilon^*}{4\epsilon}\delta^2-\tfrac{1}{4}|\delta|^2\\ &+\frac{\alpha}{\sqrt{2}}\left(\delta^*+\frac{\epsilon^*}{\epsilon}\delta\right)-\frac{i}{2}\int\mathrm{Im}(\delta\dot{\delta}^*)d\tau\Big].\end{aligned} \tag{7}$$

These states satisfy the standard relations[2]

$$\begin{aligned}\langle\alpha|\beta\rangle&=\exp\{-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2+\alpha^*\beta\}\,,\\ \frac{1}{\pi}\int d\,\mathrm{Re}\alpha\,d\,\mathrm{Im}\alpha\,|\alpha\rangle\langle\alpha|&=1\,.\end{aligned} \tag{8}$$

The average values of the coordinate and momentum operators in these states are

$$\begin{aligned}\langle\alpha|\hat{x}|\alpha\rangle&=\sqrt{2}\,\mathrm{Re}(\alpha\epsilon^*)-\mathrm{Re}(\epsilon^*\delta)\,,\\ \langle\alpha|\hat{p}|\alpha\rangle&=e^{2\Gamma(t)}\left[\sqrt{2}\,\mathrm{Re}(\alpha\dot{\epsilon}^*)-\frac{d}{dt}\mathrm{Re}(\epsilon^*\delta)\right].\end{aligned} \tag{9}$$

Since the function $\epsilon^*(t)$ satisfies Eq. (4), and the function $-\mathrm{Re}(\epsilon^*\delta)$ satisfies Eq. (2), these average values change in time according to classical mechanics. For the dispersions one can obtain the formulas

$$\begin{aligned}\langle\alpha|\Delta\hat{x}^2|\alpha\rangle&=\tfrac{1}{2}|\epsilon|^2\,,\\ \langle\alpha|\Delta\hat{p}^2|\alpha\rangle&=\tfrac{1}{2}e^{4\Gamma(t)}|\dot{\epsilon}|^2\,.\end{aligned} \tag{10}$$

Due to Eq. (5),

$$\begin{aligned}\langle\Delta\hat{x}^2\rangle\langle\Delta\hat{p}^2\rangle&=\tfrac{1}{4}e^{4\Gamma(t)}|\epsilon\dot{\epsilon}|^2\\ &\geq\tfrac{1}{4}e^{4\Gamma(t)}[\mathrm{Im}(\dot{\epsilon}\epsilon^*)]^2\geq\tfrac{1}{4}\,.\end{aligned} \tag{11}$$

Note that the dispersions do not depend on α and δ. (The most general nonstationary Hamiltonians preserving the minimal value of $\langle\Delta\hat{x}^2\rangle\langle\Delta\hat{p}^2\rangle$ were studied in Ref. 6.)

Expanding $\psi_\alpha(x;t)$ in a power series of α,

$$\psi_\alpha(x;t)=\exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\psi_n(x;t),$$

$$\psi_n(x;t)=(n!)^{-1/2}\left(\frac{\epsilon^*}{2\epsilon}\right)^{n/2}\psi_0(x;t)H_n\left(\frac{x+\mathrm{Re}(\epsilon^*\delta)}{|\epsilon|}\right), \qquad (12)$$

$[H_n(x)$ is the Hermite polynomial$]$ we obtain the eigenstates of the quadratic integral of the motion

$$\hat{K}(t)=\tfrac{1}{2}(\hat{A}^+\hat{A}+\hat{A}\hat{A}^+)$$
$$=\tfrac{1}{2}[|\epsilon|^2\hat{p}^2+|\dot{\epsilon}|^2e^{4\Gamma}\hat{x}^2-e^{2\Gamma}\mathrm{Re}(\dot{\epsilon}\epsilon^*)(\hat{x}\hat{p}+\hat{p}\hat{x})+\mathrm{Im}(\epsilon^*\delta)\hat{p}-e^{2\Gamma}\mathrm{Im}(\dot{\epsilon}^*\delta)\hat{x}+|\delta|^2],\quad \hat{K}\psi_n=(n+\tfrac{1}{2})\psi_n. \qquad (13)$$

Using Eqs. (5), (9), and (10), one can obtain the formulas

$$|\psi_\alpha(x;t)|^2=(2\pi\langle\Delta\hat{x}^2\rangle)^{-1/2}\exp\left(-\frac{(x-\langle\alpha|\hat{x}|\alpha\rangle)^2}{2\langle\Delta\hat{x}^2\rangle}\right), \qquad (7a)$$

$$|\psi_n(x;t)|^2=(2^n n!)^{-1}|\psi_0(x;t)|^2H_n^2\left(\frac{x-\langle 0|\hat{x}|0\rangle}{[2\langle\Delta\hat{x}^2\rangle]^{1/2}}\right). \qquad (12a)$$

Note that the operator $\hat{A}(t)$ [Eq. (6)] and coherent states [Eq. (7)] can be constructed for the quite arbitrary functions $\Gamma(t)$, $\omega_0(t)$, and $f(t)$. For example, in the case of a motion with damping in a uniform field, i.e., when $\omega_0=0$ and $\Gamma(t)=\gamma t$, the function $\epsilon(t)$ can be chosen as follows:

$$\epsilon(t)=\lambda+(i/2\gamma\lambda)(1-e^{-2\gamma t}), \qquad (14)$$

λ being an arbitrary real number. Another example is the case of a centrifugal potential, when $\omega_0^2<0$. If $\omega_0=\mathrm{const}$, and $\Gamma(t)=\gamma t$, then one can choose the following function $\epsilon(t)$:

$$\epsilon(t)=\tilde{\Omega}^{-1/2}e^{-\gamma t}[\cosh(\tilde{\Omega}t)+i\sinh(\tilde{\Omega}t)],$$
$$\dot{\epsilon}(t)=-\gamma\epsilon+i\tilde{\Omega}\epsilon^*,\quad \tilde{\Omega}=(\gamma^2-\omega_0^2)^{1/2}. \qquad (15)$$

However, the choice of the function $\epsilon(t)$ is not unique, because the function $\tilde{\epsilon}=\xi\epsilon+\eta\epsilon^*$ satisfies the same relations as $\epsilon(t)$, provided complex numbers ξ and η satisfy the condition $|\xi|^2-|\eta|^2=1$. Therefore, applying various linear canonical transformations to operator (6) one can construct various operators of the same type (cf. Ref. 7). In a general case, none of these operators are "better" or "worse" than the others, so that all systems of coherent states corresponding to these operators are equivalent. However, in the most important special case when ω_0 and $\dot{\Gamma}$ are constant, some systems of coherent states stand out against the others. Therefore we are proceeding to the detailed study of this case.

III. CASE OF CONSTANT PARAMETERS

If the frequency ω_0 and the friction coefficient $\dot{\Gamma}=\gamma$ are constant, then Hamiltonian (1) satisfies (provided $f=\mathrm{const}$) the interesting relation

$$\hat{H}(t+i\pi/\gamma)=\hat{H}(t). \qquad (16)$$

Evidently, one should take into account this symmetry to construct the "best" systems of coherent states or some other functions.

(i) Let us consider first the case of a weak damping: $\omega_0>\gamma$. Then choosing the function $\epsilon(t)$ in the form

$$\epsilon(t)=\Omega^{-1/2}\exp(-\gamma t+i\Omega t),\quad \Omega=(\omega_0^2-\gamma^2)^{1/2}, \qquad (17)$$

we obtain the following operator $\hat{A}(t)$:

$$\hat{A}(t)=(ie^{i\Omega t}/\sqrt{2\Omega})[e^{-\gamma t}\hat{p}+(\gamma-i\Omega)e^{\gamma t}\hat{x}]+\delta(t)/\sqrt{2}, \qquad (18)$$

$$\delta(t)=-\frac{i}{\sqrt{\Omega}}\int e^{(\gamma+i\Omega)\tau}f(\tau)\,d\tau. \qquad (18a)$$

In the case $f=\mathrm{const}$ the function $\delta(t)$ can be chosen in such a way that this operator possesses the following property:

$$\hat{A}(t+i\pi/\gamma)=-\exp(-\pi\Omega/\gamma)\hat{A}(t),$$
$$\hat{A}^+(t+i\pi/\gamma)=-\exp(\pi\Omega/\gamma)\hat{A}^+(t). \qquad (19)$$

This property distinguishes the operator (18) from all other linear integrals of the motion. Therefore in the case ω_0, $\gamma=\mathrm{const}$ the "best" coherent states are the eigenstates of the operator (18):

$$\psi_\alpha(x;t)=(\Omega/\pi)^{1/4}\exp[-\tfrac{1}{2}i(\Omega+i\gamma)t-\tfrac{1}{2}(\Omega+i\gamma)e^{2\gamma t}x^2 +\sqrt{2\Omega}\,\alpha x e^{(\gamma-i\Omega)t}-\tfrac{1}{2}\alpha^2e^{-2i\Omega t}-\tfrac{1}{2}|\alpha|^2-\sqrt{\Omega}\,xe^{(\gamma-i\Omega)t}\delta(t) -\alpha\sqrt{2\Omega}\,e^{(\gamma-i\Omega)t}z(t)-\tfrac{1}{2}\Omega z^2(t)e^{2\gamma t}+i\phi(t)]\,, \tag{20}$$

where the phase factor $\phi(t)$ is equal to

$$\phi(t)=\tfrac{1}{2}\Omega\,\mathrm{Re}\left(\int e^{-2i\Omega\tau}\delta^2(\tau)d\tau\right) \tag{20a}$$

and the function

$$z(t)=-\mathrm{Re}\left(\frac{\delta(t)}{\sqrt{\Omega}}e^{-(\gamma+i\Omega)t}\right) \tag{20b}$$

is the solution to classical equation (2) with the initial conditions $z(0)=\dot z(0)=0$.

The functions (20) satisfy the relation

$$|\tilde\alpha;x;t+i\pi/\gamma\rangle=|-\tilde\alpha e^{\pi\Omega/\gamma};x;t\rangle\exp(\tfrac{1}{2}i\pi+\pi\Omega/2\gamma)\,,$$
$$|\tilde\alpha;x;t\rangle\equiv\psi_\alpha(x;t)\exp(\tfrac{1}{2}|\alpha|^2)\,. \tag{21}$$

All other systems of coherent states do not possess such a property. The average values of the operators $\hat x$ and $\hat p$ move along classical trajectories in the phase space [see Eq. (9)]. For the widths of the wave packets (20) we obtain the relations

$$\langle\Delta\hat x^2\rangle=\frac{e^{-2\gamma t}}{2\Omega},\quad\langle\Delta\hat p^2\rangle=\frac{\omega_0^2}{2\Omega}e^{2\gamma t}\,,$$
$$\langle\Delta\hat x^2\rangle\langle\Delta\hat p^2\rangle=\frac{\omega_0^2}{4\Omega^2}\geqslant\tfrac{1}{4}\,. \tag{22}$$

Note that if we chose another system of coherent states, then the product of uncertainties would not be constant but would vary in time. For example, we could choose the operator $\hat A(t)$ in the form

$$\hat A(t)=(2\omega_0)^{-1/2}(i\hat P_0+\omega_0\hat X_0)\,, \tag{23}$$

$\hat X_0$ and $\hat P_0$ being the operators of the initial coordinates and momenta [for simplicity we suggest $f(t)=0$]:

$$\hat X_0(t)=-e^{-\gamma t}\frac{\sin(\Omega t)}{\Omega}\hat p+e^{\gamma t}\left[\cos(\Omega t)-\frac{\gamma}{\Omega}\sin(\Omega t)\right]\hat x\,,$$
$$\hat P_0(t)=e^{-\gamma t}[\cos(\Omega t)+(\gamma/\Omega)\sin(\Omega t)]\hat p +(\omega_0^2/\Omega)e^{\gamma t}\sin(\Omega t)\hat x\,, \tag{24}$$
$$\hat X_0(0)=\hat x\,,\quad \hat P_0(0)=\hat p\,.$$

Then at the initial moment $t=0$ the corresponding coherent states would coincide with usual Glauber states,[2] but the product of uncertainties in these states would oscillate in time:

$$\langle\Delta\hat x^2\rangle=\frac{e^{-2\gamma t}}{2\omega_0}\left\{1+\frac{2\gamma}{\Omega}\sin(\Omega t)\left[\cos(\Omega t)+\frac{\gamma}{\Omega}\sin(\Omega t)\right]\right\},$$
$$\langle\Delta\hat x^2\rangle\langle\Delta\hat p^2\rangle=\tfrac{1}{4}[1+(4\omega_0^2\gamma^2/\Omega^4)\sin^4(\Omega t)]\,. \tag{25}$$

Therefore, states (20) are distinguished not only from the group-theoretical point of view, but also from the point of view of the uncertainty relation, so that states (20) are indeed the "best" coherent states. If $\gamma=0$, these states coincide with Glauber ones.

(ii) In the case $\omega_0^2<\gamma^2$ the operators $\hat A(t)$ and $\hat A^+(t)$ turn into the operators [we drop the factor $i(2\Omega)^{-1/2}$]

$$\hat I_\pm(t)=e^{\pm\tilde\Omega t}[e^{-\gamma t}\hat p+(\gamma\mp\tilde\Omega)e^{\gamma t}\hat x]+\tilde\delta_\pm(t)\,,$$
$$\tilde\Omega=(\gamma^2-\omega_0^2)^{1/2}\,, \tag{26}$$

$$\tilde\delta_\pm(t)=-\int e^{(\gamma\pm\tilde\Omega)\tau}f(\tau)\,d\tau\,, \tag{26a}$$

satisfying (if $f=$ const) the relations

$$\hat I_\pm(t+i\pi/\gamma)=-\exp(\pm i\pi\tilde\Omega/\gamma)\hat I_\pm(t)\,. \tag{27}$$

Therefore, in this case there exist besides the systems of coherent states [for example, with the function $\epsilon(t)$ given by Eq. (15)] two other interesting systems of functions, namely, the eigenstates of the integrals of the motion $\hat I_+$ and $\hat I_-$ satisfying the Schrödinger equation (let us consider for simplicity the case $f=$ const):

$$\varphi_K^{(\pm)}(x;t)\equiv|K;\pm;t\rangle=(2\pi)^{-1/2}\exp\left[\frac{i}{2}(\pm\tilde\Omega-\gamma)e^{2\gamma t}x^2+ixe^{\gamma t}\left(Ke^{\mp\tilde\Omega t}+\frac{fe^{\gamma t}}{\gamma\pm\tilde\Omega}\right)\pm\frac{iK^2}{4\tilde\Omega}e^{\mp2\tilde\Omega t} -\frac{iKf}{\omega_0^2}e^{(\gamma\mp\tilde\Omega)t}-\frac{if^2e^{2\gamma t}}{4\gamma(\gamma\pm\tilde\Omega)^2}+(\gamma\mp\tilde\Omega)\frac{t}{2}\right]. \tag{28}$$

These states form two complete sets

$$\langle K'; +; t|K; -; t\rangle = (4\pi i\tilde{\Omega})^{-1/2}\exp(-iKK'/2\tilde{\Omega}), \tag{29}$$

$$\langle K'; \pm; t|K; \pm; t\rangle = \delta(K-K'),$$

and satisfy the relations

$$\left|K; \pm; t+\frac{i\pi}{\gamma}\right\rangle = |-Ke^{\mp i\pi\tilde{\Omega}/\gamma}; \pm; t\rangle \exp\left(\frac{i\pi}{2\gamma}(\gamma\mp\tilde{\Omega})\right),$$
$$\hat{I}_{\pm}\varphi_K^{(\pm)} = K\varphi_K^{(\pm)}. \tag{30}$$

(iii) In the case $\omega_0^2=\gamma^2$ the operators $\hat{I}_+$ and $\hat{I}_-$, and consequently the states $\varphi_K^{(+)}$ and $\varphi_K^{(-)}$, coincide:

$$\hat{I}_+ = \hat{I}_- = \hat{I}_0(t) = e^{-\gamma t}\hat{p} + \gamma e^{\gamma t}\hat{x} - (f/\gamma)e^{\gamma t} \quad (f=\text{const}), \tag{31}$$
$$\hat{I}_0(t+i\pi/\gamma) = -\hat{I}_0(t).$$

Therefore, one can construct the eigenstates of the operator $\hat{I}_0$:

$$|K; t\rangle = (2\pi)^{-1/2}\exp(-\tfrac{1}{2}i\gamma e^{2\gamma t}x^2 + iKxe^{\gamma t} + (if/\gamma)xe^{2\gamma t} - (iKf/\omega_0^2)e^{\gamma t} - (if^2/4\gamma^2)e^{2\gamma t} - \tfrac{1}{2}iK^2t + \tfrac{1}{2}\gamma t), \tag{32}$$

satisfying the relations

$$\hat{I}_0|K; t\rangle = K|K; t\rangle,$$
$$|K; t+i\pi/\gamma\rangle = |-K; t\rangle\exp(\tfrac{1}{2}i\pi + \pi K^2/2\gamma), \tag{33}$$
$$\langle K; t|K'; t\rangle = \delta(K-K').$$

Another integral of the motion in this case is (we suppose again for simplicity $f=$ const)

$$\hat{I}_1(t) = te^{-\gamma t}\hat{p} - (1-\gamma t)e^{\gamma t}\hat{x} + (fe^{\gamma t}/\gamma^2)(1-\gamma t). \tag{34}$$

It is not transformed into itself by the translation $t\to t+i\pi/\gamma$. States (33) are the limit cases of the states

$$|\tilde{K}; \pm; t\rangle \equiv |K; \pm; t\rangle\exp(\mp iK^2/4\tilde{\Omega}),$$
$$\langle\tilde{K}'; +; t|\tilde{K}; -; t\rangle = (4\pi i\tilde{\Omega})^{-1/2}\exp[i(K-K')^2/4\tilde{\Omega}] \tag{28a}$$

when $\tilde{\Omega}\to 0$.

The linear integrals of the motion and Gaussian wave packets in the case $\omega_0>\gamma$, $f=0$ were constructed in Ref. 20 (see also Refs. 13 and 21), but the states satisfying relations of the type (21), (30), and (32), as well as the solutions in the case $\omega_0<\gamma$, were not considered earlier.

(iv) The last special case is $\omega_0=0$. Gaussian wave packets in this case were constructed in Ref. 21 in which the external uniform field was described by the scalar potential Ex, and in Ref. 22 in which the uniform field was described by the vector potential $A(t)=-\int E(\tau)\,d\tau$. This problem was also considered in Ref. 23. Coherent states in the case under study are given by formula (7) with the function $\epsilon(t)$ given by Eq. (14). In the case $f=$ const we can also construct the states analogous to states (28). The integrals of the motion $\hat{I}_+$ and $\hat{I}_-$ in this case turn into

$$\hat{I}_+(t) = \hat{p} - (f/2\gamma)e^{2\gamma t}, \quad \hat{I}_+(t+i\pi/\gamma) = \hat{I}_+(t),$$
$$\hat{I}_-(t) = \hat{p}e^{-2\gamma t} + 2\gamma\hat{x} - ft. \tag{35}$$

Therefore, if $f\neq 0$ only the eigenstates of the operator $\hat{I}_+$ are transformed into themselves when t is replaced by $t+i\pi/\gamma$:

$$\psi_p(x; t) = (2\pi)^{-1/2}\exp[ipx + (ifx/2\gamma)e^{2\gamma t} + (ip^2/4\gamma)e^{-2\gamma t} - ipft/2\gamma - (if^2/16\gamma^3)e^{2\gamma t}], \quad \hat{I}_+\psi_p = p\psi_p,$$
$$\psi_p(x; t+i\pi/\gamma) = \psi_p(x; t)\exp(\pi pf/2\gamma^2). \tag{36}$$

IV. LOSS-ENERGY STATES

The invariance of the Hamiltonian with respect to the translation $t\to t+i\pi/\gamma$ in the case of constant parameters ω_0 and γ, f leads to the existence of the solutions of the Schrödinger equation satisfying the relation

$$\psi(x; t+i\pi/\gamma) = \lambda\psi(x; t).$$

Such states are evident generalizations of the Bloch states in the case of spatially periodic Hamiltonians: $H(\vec{x}+\vec{a}; t) = \hat{H}(\vec{x}; t)$, and the quasienergy states[24] in the case of time-periodic Hamiltonians: $\hat{H}(t+T)=\hat{H}(t)$, $\mathrm{Im}T=0$. We call the new type of states *loss-energy states* because they arise in studying quantum analogs of dissipative systems. Let us write $\lambda=\exp[(\pi/\gamma)E]$.

By analogy with quasistationary states satisfying the relation $\psi(t+T)=\psi(t)\exp(-iET)$, we call the value E in the relation

$$\psi(x;\,t+i\pi/\gamma)=\psi(x;\,t)\exp[(\pi/\gamma)E] \tag{37}$$

the loss energy. Note that Hamiltonians with imaginary periods are well known in physics. For example, the one-dimensional Schrödinger equation can be solved exactly in the case of Hulthen's potential

$$V(r)=-V_0\exp(-r/a)/[1-\exp(-r/a)],$$

Morse's potential

$$V(x)=V_0(e^{-2ax}-2e^{-ax}),$$

and some others of a similar kind (see, e.g., Ref. 25, problems 68 and 70). The solutions in all known cases satisfy the relation

$$\psi_n(x+2\pi i/a)=\lambda_n\psi_n(x)\ ,$$

i.e., they can be considered as the Bloch states with an imaginary period.

Therefore, we can suppose that for any potential $V(\vec{r})$ with an imaginary period the solutions of the Schrödinger equation can be chosen in the form $\psi(\vec{r})=\exp(\vec{\alpha}\,\vec{r})g(\vec{r})$, where $g(\vec{r})$ is a periodic function with an imaginary period. [In the known cases[25] $g(\vec{r})$ is a hypergeometric function of a periodical variable.] However, this supposition requires a more detailed study. Indeed, the existence of the usual Bloch states is the consequence of the fact that all irreducible unitary representations of the Abelian translational group are one-dimensional, while in the case of the Hamiltonians with imaginary periods we have to use nonunitary representations of the translational group, among which there are not only one-dimensional but also multidimensional representations. As far as we know the symmetry concerned was not discussed earlier anywhere. (In the case of the Hulthen potential another group of symmetry, namely, SO(2, 1) group, was discussed in Ref. 26.)

Although some properties of the loss-energy states are similar to those of the quasienergy states, there are also certain essential differences. These differences arise owing to the fact that many properties of the solutions to the Schrödinger equation have been proved using the reality of the time variable, while the concept of loss-energy states is based on the translations of this variable to an imaginary value. The detailed study of loss-energy states of quadratic systems will be given in another paper, so that here we consider only two examples.

First of all let us note that the integral of the motion $\hat{K}(t)$ (13) in the case of the operator $\hat{A}(t)$ given by Eq. (18) satisfies the same relation (16) as the Hamiltonian itself:

$$\hat{K}(t)=\Omega^{-1}\left[\hat{H}(t)+\tfrac{1}{2}\gamma(\hat{x}\hat{p}+\hat{p}\hat{x})-\frac{\gamma f}{2\omega_0^2}\hat{p}+\frac{f^2}{2\omega_0^2}e^{2\gamma t}\right],$$
$$\hat{K}(t+i\pi/\gamma)=\hat{K}(t)\,. \tag{38}$$

Therefore the eigenstates of the operator $\hat{K}(t)$—the functions (12)—are the loss-energy states:

$$\psi_n(x;\,t)=\frac{\exp(-i\Omega nt)}{(2^n n!)^{1/2}}\psi_0(x;\,t)H_n(\sqrt{\Omega}\,e^{\gamma t}[x-f/\omega_0^2])\,, \tag{39}$$

$$\psi_n(x;\,t+i\pi/\gamma)=\psi_n(x;\,t)\exp[(n+\tfrac{1}{2})(\Omega+i\gamma)\pi/\gamma]\,. \tag{40}$$

The function $\psi_0(x;\,t)$ is given by Eq. (20) with $\alpha=0$. The loss-energy spectrum is discrete:

$$E_n=(n+\tfrac{1}{2})(\Omega+i\gamma)\,,\quad n=0,1,2,\ldots\,. \tag{41}$$

It is essential that the loss energies are complex, the imaginary part being related to the damping coefficient γ. The integral of the motion (38) was obtained in the case $f=0$, also in Refs. 16, 17, and 20. States (39) in the case $f=0$ were constructed in Refs. 18, 20, and 21, and in the case of an arbitrary function $f(t)$—in Ref. 17. These states were called "pseudostationary"[14,21] or "quasistationary."[17] However, the symmetry of the Hamiltonian and of the operator $\hat{K}(t)$, as well as relation (40), were not discussed.

Formula (39) is valid only in the case $\omega_0>\gamma$. The case $\omega_0\leq\gamma$ is even more interesting from the viewpoint of loss-energy states, and this case was not studied earlier in any paper. If $\omega_0=\gamma$, the loss-energy states can be easily constructed from states (32):

$$U_K^{(\pm)}(x;\,t)=|K;\,t\rangle\pm|-K;\,t\rangle=(2\pi)^{-1/2}\exp[\tfrac{1}{2}\gamma t-iK^2t-\tfrac{1}{2}i\gamma e^{2\gamma t}x^2+(if/\gamma)xe^{2\gamma t}-(if^2/4\gamma^2)e^{2\gamma t}]$$
$$\times\{\exp[iKe^{\gamma t}(x-f/\omega_0^2)]\pm\exp[-iKe^{\gamma t}(x-f/\omega_0^2)]\}\,. \tag{42}$$

The loss-energy spectrum in this case consists of two continuous branches:

$$E_K^{(\pm)}=\tfrac{1}{2}(K^2\pm i\gamma),\quad 0\leqslant K<\infty\,. \tag{43}$$

The functions $U_K^{(\pm)}$ are the eigenstates of the operator

$$\hat{K}'=\Omega\hat{K}\,,\quad \hat{K}'U_K^{(\pm)}=\tfrac{1}{2}K^2U_K^{(\pm)}\,.$$

In the case $0<\omega_0<\gamma$ the loss-energy states are again the eigenstates of the operator $\hat{K}'(t)$; they can be expressed in terms of the parabolic cylinder functions. The explicit form of these states is given in Ref. 27; here we give only the formula for the loss-energy spectrum:

$$E_V^{(\pm)}=V\tilde{\Omega}\pm\tfrac{1}{2}i\gamma\,,\quad -\infty<V<\infty\,. \tag{44}$$

The most interesting and complicated case is, strange though it may seem, the case $\omega_0=0$. In this case there exist many different complete sets of loss-energy states corresponding to different branches of loss-energy spectrum. The details are given in Ref. 27.

In conclusion of this section, we would like to emphasize that loss-energy states can be constructed not only in the case of dissipative systems with Hamiltonians of the type (1), but also in the case of quite usual Schrödinger, Dirac, or Klein-Gordon equations describing the motion of charged particles in uniform but time-dependent electric and magnetic fields of the type $E(t)=E_0\tanh(\alpha t)$. Although the solutions in these cases are known (see, e.g., Ref. 28), their invariance in respect to imaginary translations in time was not noted earlier. It is possible that just the existence of such an additional symmetry enables to obtain exact solutions in the case discussed.

V. RESONANCE IN COHERENT STATES

In this section we obtain the formulas for the energy of a damped oscillator being swung by the harmonic force

$$f(t)=F_0\sin(\omega t+\varphi)\,,\quad F_0,\,\omega,\,\varphi=\text{const}\,.$$

We define the energy W as $W=\frac{1}{2}(\dot{x}^2+\omega_0^2x^2)$. Then the quantum-mechanical average of this value in an arbitrary coherent state (20) changes in time as follows:

$$\langle\alpha|\hat{W}|\alpha\rangle=W_{cl}(t)+(\omega_0^2/2\Omega)e^{-2\gamma t}\,. \tag{45}$$

$W_{cl}(t)$ is the energy of the classical damped oscillator moving along the trajectory determined by Eq. (9). Therefore, the steady regimes (when $t\gg\gamma^{-1}$) of the classical oscillator and of its quantum-mechanical model in a coherent state coincide. However, for small values of time, $t\lesssim\gamma^{-1}$, the behaviors of quantum and classical oscillators may be different. To show this let us consider the case of the strict resonance: $\omega=\Omega$. Usually we are interested in the values averaged over the period $2\pi/\omega$. We shall designate such average values by the symbol $\langle\!\langle\ \rangle\!\rangle$. Let us suppose that $\gamma\ll\omega$. Choosing the function $\delta(t)$ in such a manner that $\delta(0)=0$ and neglecting the terms of the order γ/ω, one can obtain the following expression for $\langle\!\langle W(t)\rangle\!\rangle$ in the coherent state given by Eq. (20):

$$\langle\!\langle W(t)\rangle\!\rangle=\omega_0e^{-2\gamma t}(|\alpha|^2+\tfrac{1}{2})+(1-e^{-\gamma t})^2F_0^2/8\gamma^2 -(\tfrac{1}{2}\omega_0)^{1/2}\,\mathrm{Re}(\alpha e^{i\varphi})(e^{-\gamma t}-e^{-2\gamma t})F_0/\gamma\,,\quad \gamma\ll\omega\,. \tag{46}$$

If we chose the coherent states corresponding to the operator $\hat{A}(t)$ (23), then to the same approximation we would obtain again formula (46).

Let us consider the case $\alpha=0$, $F_0^2/8\gamma^2<\frac{1}{2}\omega_0$, i.e., the case of a very weak external force, when the energy of the steady forced oscillations is less than the initial energy of the quantum oscillator.

Then the energy of the classical oscillator

$$\langle\!\langle W^{(cl)}(t)\rangle\!\rangle=(1-e^{-\gamma t})^2F_0^2/8\gamma^2$$

monotonically increases, while the energy of the quantum oscillator

$$\langle\!\langle W^{(q)}(t)\rangle\!\rangle=(1-e^{-\gamma t})^2F_0^2/8\gamma^2+\tfrac{1}{2}\omega_0e^{-2\gamma t}$$

first decreases, and only when $\gamma t\gtrsim 1$ it becomes to increase. These differences of the behaviors of the functions $\langle\!\langle W^{(q)}(t)\rangle\!\rangle$ and $\langle\!\langle W^{(cl)}(t)\rangle\!\rangle$ are shown in the case $F_0^2/8\gamma^2=\frac{1}{4}\omega_0$ in Fig. 1.

Using Eqs. (7a) and (22) one can easily verify that for $t\to\infty$

$$|\psi_\alpha(x;t)|^2\to\delta(x-\langle\alpha|\hat{x}|\alpha\rangle)\,. \tag{47}$$

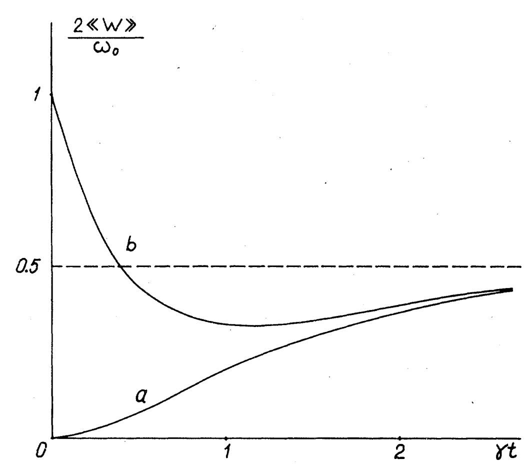

FIG. 1. Time dependences of the averaged over the period energies $\langle\langle W(t)\rangle\rangle$ of the classical (a) and quantum (b) oscillators in the case $\alpha=0$; $F_0^2/(8\gamma^2)=\frac{1}{4}\omega_0$.

This means that for large $t \gg \gamma^{-1}$ the wave function of the damped oscillator is "spread" nearly uniformly over all energy eigenstates $|n_0\rangle$ of the usual (undamped) harmonic oscillator [the explicit form of the states $|n_0\rangle$ is given by Eq. (39) with $\gamma = f = 0$]. For example, in the simplest case $\alpha = 0$ and $f(t) = 0$ one has

$$\langle 2n_0+1|0\rangle = 0\,,$$
$$\langle 2n_0|0\rangle = (\omega_0\Omega)^{1/4}\left(\frac{2(2n_0)!}{\omega_0+\mu(t)}\right)^{1/2}(n_0!)^{-1} \times\left(\frac{\omega_0-\mu(t)}{\omega_0+\mu(t)}\right)^{n_0} 2^{-n_0}\exp(\tfrac{1}{2}\gamma t + 2i\omega_0 n_0 t)\,,$$
$$\mu(t) = (\Omega + i\gamma)e^{2\gamma t}\,. \tag{48}$$

Therefore, for $\gamma t \gg 1$ (we suppose $\gamma \ll \omega_0$)

$$|\langle 2n_0|0\rangle|^2 \approx 2e^{-\gamma t}|e^{-2\gamma t} - \tfrac{1}{2}|^{2n_0}(2n_0)!/(n_0!)^2\,. \tag{49}$$

For $n_0 \gg 1$ one obtains, using Stirling's formula,

$$|\langle 2n_0|0\rangle|^2 \approx \left(\frac{4}{\pi n_0}\right)^{1/2} e^{-\gamma t}|1-2e^{-2\gamma t}|^{2n_0}\,. \tag{50}$$

Consequently, although the average value of the operator $\hat{W}$ tends to zero when $t\to\infty$ [if $f(t)=0$], the probabilities $|\langle 2n_0|0\rangle|^2$ are not equal to zero even for $n_0 \gg 1$; moreover, they have the same order of magnitude as the value $|\langle 0_0|0\rangle|^2$.

Let us calculate also the transition amplitude $\langle\beta|\alpha\rangle$ between the coherent state $|\alpha\rangle$ [Eq. (20)] of the damped oscillator and the coherent state $|\beta\rangle$ of the usual undamped oscillator with the mass m_0 and the frequency ω_0. The state $|\beta\rangle$ has the following form[2,3]:

$$\langle x|\beta\rangle = (m_0\omega_0/\pi)^{1/4}\exp[-\tfrac{1}{2}m_0\omega_0 x^2 + (2m_0\omega_0)^{1/2}\beta x - \tfrac{1}{2}\beta^2 - \tfrac{1}{2}|\beta|^2]\,. \tag{51}$$

A simple calculation yields the following expression for $\langle\beta|\alpha\rangle$:

$$\langle\beta|\alpha\rangle = \sqrt{2}\,[(m_0\omega_0\tilde{m}\Omega)^{1/4}/(\mu(t)+m_0\omega_0)^{1/2}] \times\exp\{-\tfrac{1}{2}(|\alpha|^2+|\beta|^2) - \tfrac{1}{2}(\beta^{*2}+\tilde{\alpha}^2) - \tfrac{1}{2}\Omega z^2(t)e^{2\gamma t} - (2\tilde{m}\Omega)^{1/2}\tilde{\alpha}z(t) + i\phi(t) + \tfrac{1}{2}[\mu(t)+m_0\omega_0]^{-1}[(2\tilde{m}\Omega)^{1/2}\tilde{\alpha} - (\tilde{m}\Omega)^{1/2}\tilde{\delta} + (2m\omega_0)^{1/2}\beta^*]^2\}\,;$$
$$\tilde{m} = e^{2\gamma t}\,,\quad \tilde{\alpha} = \alpha e^{-i\Omega t}\,,\quad \tilde{\delta} = \delta e^{-i\Omega t}\,. \tag{52}$$

If one expands this formula in a power series of β^*,[2,3] then the following expression for the values $\langle n_0|\alpha\rangle$ ($|n_0\rangle$ means the nth eigenstate of the undamped oscillator) can be written

$$\langle n_0|\alpha\rangle = \sqrt{2}\,\frac{(m_0\omega_0\tilde{m}\Omega)^{1/4}}{\{n_0!\,[\mu(t)+m_0\omega_0]\}^{1/2}} \times\exp\left(-\tfrac{1}{2}[|\alpha|^2+\tilde{\alpha}^2+\tilde{m}\Omega z^2(t)] - (2\tilde{m}\Omega)^{1/2}\tilde{\alpha}z(t) + i\phi(t) + \frac{\tilde{m}\Omega(\sqrt{2}\,\tilde{\alpha}-\tilde{\delta})^2}{2[\mu(t)+m_0\omega_0]}\right)\left(\frac{\mu(t)-m_0\omega_0}{\mu(t)+m_0\omega_0}\right)^{n_0/2} 2^{-n_0/2} \times H_{n_0}\left(\frac{2(m_0\omega_0\tilde{m}\Omega)^{1/2}(\sqrt{2}\,\tilde{\alpha}-\tilde{\delta})}{[\mu^2(t)-m_0^2\omega_0^2]^{1/2}}\right)\,. \tag{53}$$

Let us note that for large $t\to\infty$ the function $\delta(t)$ behaves as $\exp(\gamma t)$ [see Eq. (18a)]. Therefore, if $m_0 = 1$, as in the previous case, then the argument of the Hermite polynomial has the asymptotic form $-2\tilde{\delta}e^{-\gamma t}$, and formula (53) can be simplified (we neglect the terms of the order γ/ω_0):

$$\langle n_0|\alpha\rangle = \left(2^{1-n_0}\frac{e^{-\gamma t}}{n_0!}\right)^{1/2} \times\exp\{-\tfrac{1}{2}[\mathrm{Im}(\sqrt{2}\,\tilde{\alpha}-\tilde{\delta})]^2 + i\,\mathrm{Re}\,\tilde{\alpha}\,\mathrm{Im}(\tilde{\alpha}-\sqrt{2}\,\tilde{\delta}) + i\,\mathrm{Im}\tilde{\delta}\,\mathrm{Re}\tilde{\delta} + i\phi(t)\}H_{n_0}(-2\tilde{\delta}e^{-\gamma t})\,. \tag{54}$$

Suppose now that at the moment $t_0 \gg \gamma^{-1}$ the external force is switched off together with the friction force. Then one may think that the real mass of the oscillator is $m_0 = \exp(2\gamma t_0)$ [using the terminology of the quantum field theory, one would say that $m = 1$ is the mass of an "undressed" oscillator, and $m_0 = \exp(2\gamma t)$ is the mass of a "dressed" oscillator arising due to the interaction with the external world; the interaction is described phenomenologically by the friction coefficient γ]. In this case one must calculate not the values $\langle n_0|\alpha\rangle$ [Eq. (54)], but the values $\langle\tilde{n}_0|\alpha\rangle$, where $|\tilde{n}_0\rangle$ is the eigenstate of the undamped oscillator with the mass $\exp(2\gamma t_0)$. The formula for the values $\langle\tilde{n}_0|\alpha\rangle$ is very different from Eq. (54) (evidently, one should calculate this transition amplitude at the moment

$t=t_0$, since for $t>t_0$ the probabilities would not change in time):

$$\langle \tilde{n}_0 | \alpha \rangle = (2^{n_0} n_0!)^{-1/2} \exp\left[-\tfrac{1}{4} |\sqrt{2}\tilde{\alpha} - \tilde{\delta}|^2 + \frac{i}{\sqrt{2}} \operatorname{Im}(\tilde{\alpha}\tilde{\delta}^*) + \tfrac{1}{2} i \operatorname{Re}\tilde{\delta} \operatorname{Im}\tilde{\delta} + i\phi(t) \right] H_{n_0}\left[-\tilde{\delta} \left(\frac{2\omega_0}{i\gamma} \right)^{1/2} \right], \quad (55)$$

$t=t_0 \gg \gamma^{-1}$.

These examples show that the physical interpretation of the results obtained may be different depending on the choice of the states which are assumed to be "physical" ones. Therefore, we have to discuss the problem of the physical significance of the model considered above.

VI. DISCUSSION

In the previous sections we have obtained for the first time several new complete systems of solutions of the Schrödinger equation with Hamiltonian (1) for all possible relations between the parameters γ and ω_0, and we think the results obtained are quite interesting themselves. Now we want to discuss the physical content of the model considered above, especially its relation to the real quantum damped oscillator.

Before discussing this problem let us note that there exists the real physical problem which can be described exactly by means of the Schrödinger equation with Hamiltonian (1). This is the problem of the motion of a particle with the time-dependent mass $m(t)=m_0 \exp[2\Gamma(t)]$ in a uniform nonstationary gravitational field. Such a problem can arise, for example, in the gravitational theory and in studying the early stages of the evolution of the universe; see, e.g., Ref. 29. Besides, unstable systems can be also described with the aid of the concept of the time-dependent mass.[30]

As to the real dissipative systems, they are described usually in terms of the density-matrix formalism, which explicitly takes into account the physical nature of the dissipation—the interaction of the system under study with a large heat reservoir [see papers[31-36] and references therein]. Besides this approach there exist also several other models describing quantum dissipative systems. One of them is the model considered above; it was studied also in Refs. 13–23, 37, and 38. In Refs. 39 and 21 the friction was introduced into quantum mechanics with the aid of nonlinear generalizations of the Schrödinger equation. An approach based on exploiting non-Hermitian Hamiltonians was studied in Ref. 40.

We would like to emphasize that all these models by no means contradict the postulates of quantum mechanics (although in some papers the opposite claim can be found). For example, Hamiltonian (1) is Hermitian, all solutions of the Schrödinger equation with this Hamiltonian are normalized, and the uncertainty relations for the operators of the coordinate and the generalized momentum are fulfilled [see Eq. (11)]. Of course, some properties of the model of a quantum damped oscillator considered above seem rather strange and unexpected. For example, we see that for $t \gg \gamma^{-1}$ the fluctuations of the coordinate disappear, and high-energy levels (in the representation of the quantum numbers of the undamped oscillator) are excited. But there is no reason to say that these results are wrong. To give the correct conclusion one should analyze in detail the problem of measurements of observables in the case of dissipative systems and the problem of the correlation between the observables and quantum-mechanical operators in this case. For example, what operator corresponds to the energy of the damped oscillator—the operator $\hat{W}$ given in Sec. V, and how can this energy be measured? Or another example: We see that $\langle \Delta \hat{x}^2 \rangle \langle \Delta \hat{\dot{x}}^2 \rangle \to 0$ when $t \to \infty$. To conclude whether this relation is incompatible with quantum mechanics or not, one should understand what do we measure in experiments—the velocity or the generalized momentum? If we measure the generalized momentum, then there is no contradiction with quantum mechanics. Hamiltonian (1) is incorrect only if we can measure the velocity and if the relation $\langle \Delta \hat{x}^2 \rangle \langle \Delta \hat{\dot{x}}^2 \rangle \geq \hbar^2/4m^2$ must hold ($m=$const). But in such a case, why can we not measure simultaneously all three components of the velocity in the presence of a magnetic field? Although these problems were discussed slightly in some papers,[16,37,38,41] their complete solutions have not been obtained.

If all the models mentioned above are correct (in the sense that they do not contradict the principles of quantum mechanics) and nonetheless lead to different physical results, although all of them were constructed to describe the same system (a quantum damped oscillator), one can ask what is the reason for such a strange situation. We believe that this situation is by no means strange but, on the contrary, quite natural, because it is the consequence of the general fact of the nonuniqueness of the quantization of the given classical system.

Indeed, what do we mean when we say that we quantize the given classical system described by an equation of the motion of the type $\ddot{x}=F(x, \dot{x}, t)$? One of the possible definitions based on Feynman's approach to quantum mechanics[42] may be the following. To quantize the classical system means to replace the classical trajectory $x(t)$ by the transition amplitude (the Green's function) $G(x, t; x_1, t_1)$. But to calculate Feynman's integral to determine the function G one should know the classical Lagrangian (or Hamiltonian) leading to the given classical equation of the motion. At the same time it is well known that the same classical equation of the motion can be obtained from many quite different Lagrangians.[41,43,44] It can be shown that different Lagrangians lead to substantially different Green's functions; see Ref. 45, in which the problem of the nonuniqueness of the quantization of the given classical system was considered in detail. Therefore, to know only the classical equations of the motion is not sufficient to quantize the classical system uniquely, but some additional information is needed. In the case of the usual quantum systems (without dissipation) this additional information consists of the implicitly imposed requirement that the Hamiltonian must coincide with the energy of the system: $\hat{H}=\hat{p}^2/2m+V(x)$. When this requirement cannot be fulfilled, as in the case of dissipative systems, for example, then we have no generally accepted rule of quantization, and different methods of quantization lead to various different quantum models of the same classical system.

For example, if one requires that the equation of the motion $\ddot{x}+2\gamma\dot{x}+\omega_0^2x=0$ would be the consequence of a certain system of equations describing the interaction of the usual (undamped) oscillator with some reservoir consisting of a large number of particles in the equilibrium states, then the quantization leads to the density-matrix description of the quantum damped oscillator given in Refs. 31–36. Making other additional conjectures on the physical sense of this equation, one can obtain other quantum models corresponding to the same classical equation—in particular, the models considered in Refs. 13–23, 37, and 38 and in the present paper or the models given in Refs. 39 or 40. All of them are correct, but they describe different physical systems. In particular, the model based on Hamiltonian (1), perhaps describes the motion of the particle with a time-variable mass rather than the system in Refs. 31–36. In any case all the models mentioned above deserve studying because for each of them there undoubtedly exists a real physical problem to which this model can be applied.

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APPENDIX

Let us consider some curious consequences of the formulas of Sec. II in the case of the free motion with damping, i.e., when $\omega_0=f(t)=0$. According to Eqs. (10) and (14), in the limit $t\to\infty$ the widths of the Gaussian wave packets (coherent states) tend to the constant value $\sigma^2=\frac{1}{2}[\lambda^2+(2\gamma\lambda)^{-2}]$. Therefore, they cannot be less than $\sigma^2_{\rm min}=(2\gamma)^{-1}$. In the dimensional units one obtains $\langle\Delta\hat{x}^2\rangle_{\rm min}=\hbar/2m\gamma$, where m is the mass of the particle. If one supposes that the friction arises due to radiation losses, then $\gamma=2e^2\omega_0^2/3mc^3$, where ω_0 is a certain "proper frequency" of the particle. It is natural to assume $\omega_0=mc^2/\hbar$. Then one obtains the result

$$(\langle\Delta\hat{x}^2\rangle_{\rm min})^{1/2}=(\hbar/mc)(\tfrac{3}{4}\hbar c/e^2)^{1/2}\approx 10\hbar/mc\,. \quad (A1)$$

It is well known that particles cannot be localized in the regions with the dimensions less than Compton's length because of the effect of pair creation. Formula (A1) shows that the radiation damping increases the lower limit of the region of the possible localization of the particle to a whole order. Of course, these reasonings are very speculative, but perhaps they contain a grain of truth.

Moreover, it is well known that in the case of the free motion without friction any wave packet diffuses because of the uncertainty relations. Perhaps the interaction of the particle with the vacuum fluctuations of the electromagnetic field (the radiation friction and the damping coefficient γ can be considered as the phenomenological description of this interaction) prevents such a diffusion.

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