Angular-spectral distribution and polarization of synchrotron radiation from a "short" magnet

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Power per unit solid angle, spectrum and polarization as a function of angle, and integrated spectrum are calculated for the radiation from a beam of ultrarelativistic ($\gamma \gg 1$) charged particles in a magnet causing a deflection much smaller than $1/\gamma$, with an arbitrary form of the magnetic field $B(z)$. Some examples are given, and the connection with the "edge effect" is shown.

I. INTRODUCTION

When emitting synchrotron radiation, an ultrarelativistic ($\gamma \gg 1$) charged particle is seen by the observer over a piece of its trajectory of length $L_0 = mc/eB$, where m is the mass, e the charge, B the magnetic field, and MKS units are used. If over such a distance the magnetic field B is uniform, the spectrum of the radiation is the well-known synchrotron spectrum. If it is not, the variation of both B and the direction of motion of the particle will influence the spectrum, which, in the general case, will have to be calculated by numerical methods. In the case in which the length of the magnetic is much shorter than L_0 (which is equivalent to saying that the deflection is $\ll 1/\gamma$; we could call this a "short" or "weak" magnet), the calculation is easier [it is, in a certain sense, the case opposite to the "usual" synchrotron radiation, since only B instead of only $f(\theta, \varphi)$ is varying in Eqs. (2) below].

Historically, the first example of what we would call a (periodic) short magnet (or a succession of short magnets) is the "undulator" described by Ginzburg[1] and by Motz[2,3]: in this case $B(z)$ is sinusoidal, and in each period the maximum deflection is $\ll 1/\gamma$, that is, the spectrum is a single narrow band. Recently, a greater interest has developed in undulators for application to production of narrow-band x and vuv radiation and to "free-electron lasers," and some devices have been constructed[4,5,6] or are under construction.

Another example was the remark by Robinson[7] that in a sufficiently short magnet the radiation would be emitted, at a given wavelength, under a larger angle than in usual synchrotron radiation. Recently it has been also remarked[8,9] that if the length of the magnet is $L < L_0$, the cutoff frequency is greater than the usual "critical frequency"; this would be relevant, in practice, for electrons in low magnetic fields (say 100 G) or for protons. The cutoff frequency for radiation emitted by protons could easily be increased by a factor 10 or 50, getting visible light instead of infrared in 300-GeV machines. With certain approximations, the "edge effect" foreseen in Ref. 8 can also be calculated utilizing the results for "short" magnets (see Sec. VII).

An interesting aspect of "short" magnets is that, in principle, it could be designed to give an arbitrary spectrum (or even two different arbitrary spectra for two orthogonal polarizations).

We want to see now how to calculate the complete angular-spectral distribution and polarization of radiation from a "short" magnet in the general case [that is, for any function $B(z)$; most results reported up to now are only for undulators], and to present simple examples.

II. GENERAL PROCEDURE

The starting point is, as usual,[10,11] Liénard's expression for the far fields emitted by a point charge:

$$\vec{E}(t) = \frac{e}{4\pi\epsilon_0 r c^2}\left(1 + \frac{\vec{n}\cdot\vec{v}}{c}\right)^{-3} \times \left\{\vec{n} \times \left[\left(\vec{n} - \frac{\vec{v}}{c}\right) \times \vec{a}\right]\right\}_{t' = t - r(t')/c}, \quad (1)$$
$$\vec{B}(t) = (1/c)\vec{n} \times \vec{E},$$

with the usual meaning of symbols, and $\vec{n} = -\vec{r}/r$, $\vec{v}(t') = -d\vec{r}/dt'$, $\vec{a}(t') = d\vec{v}/dt'$, θ is the angle between $\vec{n}$ and $\vec{v}$, and φ the angle between the plane containing $\vec{v}$ and $\vec{a}$ and the plane containing $\vec{v}$ and $\vec{n}$. We define, as in Ref. 10, an amplitude $\vec{U}(t)$ proportional to $r\vec{E}$ such that its modulus square is the instantaneous power per unit solid angle, and the modulus square of its Fourier transform is the power per unit bandwidth per unit solid angle, and considering that $\vec{v}\cdot\vec{a} = 0$ and $\gamma \gg 1$ (then the radiation is mainly concentrated within an angle of the order of $1/\gamma$), we can make the approximations:

$$v/c = 1 - 1/2\gamma^2, \quad \theta^2 \ll 1, \quad \sin\theta = \theta, \quad (1 - \cos^2\theta)^{1/2} = \theta,$$

and write

$$\vec{U}(t) = C\gamma^3 B\vec{f}(\theta, \varphi) \tag{2a}$$

where B (magnetic field) and $\vec{f}$ must be evaluated at the time $t' = t - r(t')/c$ and $C = (e^2/\pi m)(1/\epsilon_0 c)^{1/2} = 1.74\times 10^{-7}\ \mathrm{kg}^{-1/2}\,\mathrm{m\,sec}^{1/2}\,\mathrm{C}$, or

$$C = \frac{2e^2}{m}\left(\frac{1}{\pi c^3}\right)^{1/2} = 5.5\times 10^{-8}\ \mathrm{cm}^{3/2}\,\mathrm{sec}^{-1/2},$$

$$\vec{f}(\theta, \varphi) = f_\parallel \hat{i}_\parallel + f_\perp \hat{i}_\perp \tag{2b}$$

where $\hat{i}_\parallel$ and $\hat{i}_\perp$ are unit vectors parallel and perpendicular to the acceleration, and

$$f_\parallel = (1+\gamma^2\theta^2)^{-3}(1-\gamma^2\theta^2+2\gamma^2\theta^2\sin^2\varphi),$$
$$f_\perp = -2(1+\gamma^2\theta^2)^{-3}\gamma^2\theta^2\sin\varphi\cos\varphi. \tag{2c}$$

The total power (both polarizations) will be proportional to

$$f^2 = f_\parallel^2 + f_\perp^2 = (1+\gamma^2\theta^2)^{-6} \times[(1-\gamma^2\theta^2)^2 + 4\gamma^2\theta^2\sin^2\varphi]. \tag{2d}$$

In some cases it is more useful to decompose $\vec{f}$ in directions parallel and perpendicular to a plane containing $\vec{v}$ and $\vec{n}$, $\hat{i}_\pi$ and $\hat{i}_\sigma$:

$$\vec{f}(\theta, \varphi) = f_\pi \hat{i}_\pi + f_\sigma \hat{i}_\sigma,$$
$$f_\pi = (1+\gamma^2\theta^2)^{-3}(1-\gamma^2\theta^2)\cos\varphi, \tag{2e}$$
$$f_\sigma = -(1+\gamma^2\theta^2)^{-3}(1+\gamma^2\theta^2)\sin\varphi.$$

These expressions are always valid, but the relation between t' and t is, in general, not simple; it is always true that

$$\frac{dt'}{dt} = \frac{2\gamma^2}{1+\gamma^2\theta^2}, \tag{3}$$

but, in general, $\theta(t')$ is not constant. In our case (short magnet) we suppose $\alpha\gamma \ll 1$, or

$$\frac{e}{mc}\int B(z)\,dz \ll 1, \tag{4}$$

and then we consider θ as constant and write (with a suitable choice of $t = 0$)

$$t'/t = 2\gamma^2/(1+\gamma^2\theta^2), \tag{5}$$

then the time scale of $B(t') = B(z/c)$ has to be compressed by a factor $2\gamma^2/(1+\gamma^2\theta^2)$. The relation between the power seen by the observer (a function of t) and power emitted by the particle (a function of t'), i.e., the power seen by the observer, is $2\gamma^2/(1+\gamma^2\theta^2)$ times bigger and its duration is $2\gamma^2/(1+\gamma^2\theta^2)$ times smaller.

III. ANGULAR DISTRIBUTION AND POLARIZATION

The energy, per unit solid angle, emitted by an electron in the whole "short" magnet is obtained by integrating $U^2(t)$:

$$\frac{dW}{d\Omega} = C^2\gamma^6 f^2(\theta, \varphi)\int B^2(t)\,dt = \frac{C^2}{2c}\gamma^4 f^2(\theta, \varphi)(1+\gamma^2\theta^2)\int B(z)\,dz. \tag{6}$$

This result is valid not only for the energy emitted in both polarizations, but also for each polarization separately if instead of f^2 we write $f_\parallel^2$ or $f_\perp^2$.

If the beam is not modulated (then amplitudes add incoherently), the power emitted by the whole beam (per unit solid angle) is $dP/d\Omega = n\,dW/d\Omega$, where $n = I/e$ is the number of electrons per second. The same applies for the other quantities defined below.

The total energy emitted by one particle is obtained by integrating $f^2(\theta, \varphi)(1+\gamma^2\theta^2)$ over the whole solid angle (the variable θ is changed to $y = 1+\gamma^2\theta^2$, then $dy = 2\gamma^2\theta\,d\theta$, with $\int dy$ extending from 1 to ∞):

$$W = (\pi C^2/6c)\gamma^2\int B^2(z)\,dz. \tag{7}$$

The result is the same as the energy emitted in any magnet (the difference is that in a "long" magnet this energy is swept over different directions as the particle is deflected).

Let us consider now the polarization characteristics: Eqs. (2a)–(2c) show the instantaneous distribution, which also represents the polarization distribution of the energy emitted in a plane trajectory.

We remark that $f_\perp = 0$ when $\sin\varphi\cos\varphi = 0$, or on the axes $\varphi = 0$ and $\varphi = \pi/2$, while $f_\parallel = 0$ when $\varphi = \arcsin(1/\sqrt{2})(1 - 1/\gamma^2\theta^2)^{1/2}$. At the intersection of these two curves, at $\theta = 1/\gamma$, $\varphi = 0$, and $\varphi = \pi$, the intensity is (obviously) zero. In all directions the radiation is linearly polarized. If we integrate $f_\parallel^2$ and $f_\perp^2$ over the whole solid angle (as for Eq. 7) we find that, if we call $W_\parallel$ and $W_\perp$ the energies emitted in the two polarizations (and $W = W_\parallel + W_\perp$),

$$W_\parallel/W = \tfrac{7}{8}$$

and

$$W_\perp/W = \tfrac{1}{8}. \tag{8}$$

This result [valid for any form of the field $B(z)$] is the same as for the usual synchrotron radiation. The difference is in the angular distribution of the perpendicular polarization, which has in this case a fourfold symmetry, while in the usual (uniform field) case the symmetry is twofold.

The radiation is concentrated mainly within $\theta < 1/\gamma$ [see Eqs. (2c), (2d), and (6)] and the perpendicular polarization is distributed mainly at (relatively) large angles: only $\frac{7}{32}$ of the total intensity is out of the cone $\theta = 1/\gamma$; $\frac{17}{112}$ for the parallel polarization and $\frac{11}{16}$ for the perpendicular one (then out of $\theta = 1/\gamma$ the ratio of perpendicular to parallel intensities is $\frac{11}{17}$).

Looking at Eq. (2e) we can see, for example, that if B is helical, the polarization of the radiation will be circular at $\theta = 0$, linear on the circle $\theta = 1/\gamma$, and, in general, elliptical, with ratio of amplitudes

$$f_\pi/f_\sigma = (1-\gamma^2\theta^2)/(1+\gamma^2\theta^2) \qquad (9)$$

(which generalizes the result of Ref. 12). For any "elliptical" field B (superposition of helical and plane sinusoidal) we have linear polarization on the circle $\theta = 1/\gamma$ and circular in two symmetrical points inside it.

IV. SPECTRUM

The spectral distribution $dW/d\Omega\, d\nu$ of the energy (per unit solid angle $d\Omega$) of the radiation seen by the observer (which, multiplied by n, gives the power per unit bandwidth and $d\Omega$ for the whole beam) is obtained by Fourier-transforming $U(t)$:

$$\frac{dW}{d\Omega d\nu} = C^2\gamma^6 f^2 \mathfrak{F}^2\{B(t)\}$$
$$= \tfrac{1}{4}C^2\gamma^2 f^2(1+\gamma^2\theta^2)B\left(\frac{1+\gamma^2\theta^2}{2\gamma^2}\nu\right), \qquad (10)$$

where by $\mathfrak{F}^2\{B(t)\}$ we mean the Fourier transform (modulus square) of $B(t)$, and by

$$B\left(\frac{1+\gamma^2\theta^2}{2\gamma^2}\nu\right)$$

we mean a function obtained by the following procedure: take $B(t') = B(z/c)$, make its Fourier transform (FT) and substitute the frequency variable ν' with $\nu(1+\gamma^2\theta^2)/2\gamma^2$ (in the case we have the advantage, useful for numerical calculations, that we make the FT with respect to a variable t' which is independent of θ).

The spectral distribution of the energy collected over the whole solid angle (in practice a solid angle $\gg \pi/\gamma^2$ around $\theta = 0$) can be calculated by integrating Eq. (10) with respect to $d\Omega = \theta\, d\theta d\varphi$:

$$\frac{dW}{d\nu} = \frac{1}{4}C^2\gamma^2\int_0^{2\pi} d\varphi \int_0^\infty \theta\, d\theta f^2(1+\gamma^2\theta^2)^2 \times B^2\left(\frac{1+\gamma^2\theta^2}{2\gamma^2}\nu\right)$$
$$= \frac{1}{4}\pi C^2\int_1^\infty y^2\langle f^2\rangle(y)B^2\left(\frac{\nu}{2\gamma^2}y\right)dy, \qquad (11)$$

where $\langle f^2\rangle$ is the average of f^2 over φ, and $y = 1 + \gamma^2\theta^2$.

Also, Eqs. (10) and (11) can be referred either to both polarizations, or to each polarization separately, using $f_\parallel^2$ and $f_\perp^2$ instead of f^2. We have:

$$\langle f^2\rangle = y^{-6}(y^2 - 2y + 2),$$
$$\langle f_\parallel^2\rangle = y^{-6}(\tfrac{1}{2}y^2 - y + \tfrac{3}{2}), \qquad (12)$$
$$\langle f_\perp^2\rangle = y^{-6}(\tfrac{1}{2}y^2 - y + \tfrac{1}{2}).$$

The polarization ratio $W_\parallel/W_\perp$ [see Eq. (8)] will be different from 7 if only a part of the spectrum is taken (>7 at higher frequencies, <7 in the lower part of the spectrum).

Even in a plane field $B(z)$, the spectra of the two orthogonal linearly polarized components are not the same because of the different angular distributions of the two polarizations (and the dependence of the time scale t on θ).

An alternative approach to get $dW/d\nu$ would be to consider the Fourier components of $B(z)$; each of them is an (infinitesimal) undulator, for which $dW/d\nu$ can be calculated, and then the result integrated over all components.

V. UNDULATOR

To illustrate the preceding statement and to make a first example of a short magnet let us take an undulator, with period λ_0 and length $\Lambda \gg \lambda_0$:

$$B(t') = B_0 \cos 2\pi\nu_0 t' \operatorname{rect}(\lambda_0/\Lambda)\nu_0 t' \qquad (13)$$

where $\nu_0 = c/\lambda_0 = \nu_m/2\gamma^2$, and rect $x = 1$ for $-\frac{1}{2} < x < \frac{1}{2}$ and $= 0$ elsewhere. Its FT is

$$B(\nu') = \frac{\Lambda}{\lambda_0\nu_0}\ \operatorname{sinc}\ \frac{\Lambda}{\lambda_0}\left(\frac{\nu'}{\nu_0} - 1\right), \qquad (14)$$

where sinc $x = (\sin\pi x)/\pi x$. Squaring and approximating for $\Lambda/\lambda_0 \to \infty$ ($\operatorname{sinc}^2(\Lambda/\lambda_0)x \to (\lambda_0/\Lambda)\delta(x)$) we obtain

$$B^2(\nu') = B_0^2\frac{\Lambda}{\lambda_0}\ \delta\left(\frac{\nu'}{\nu_0} - 1\right). \qquad (15)$$

Substituting into Eq. (10) and integrating over angles, we obtain

$$\frac{dW_{\text{und}}}{d\nu} = \frac{\pi C^2}{2c}\gamma^2 B_0^2\Lambda\frac{\nu}{\nu_m^2}\left(1 - 2\frac{\nu}{\nu_m} + 2\frac{\nu^2}{\nu_m^2}\right),\ \left(\frac{\nu}{\nu_m} < 1\right), \qquad (16)$$

where $\nu_m = 2\gamma^2\nu_0$ is the maximum frequency contained in the spectrum (the one corresponding to $\theta = 0$).

If we consider again a general short magnet $B(z)$, and write $B(t')$ as a sum of contributions of undulators of square amplitude $B^2(\nu_0)(c/\Lambda)\,d\nu_0$, then the total energy per unit $d\nu$ is

$$\frac{dW}{d\nu} = \tfrac{1}{4}C^2\int_1^\infty \frac{\nu^2}{\nu_m^2}\left(1 - 2\frac{\nu}{\nu_m} + 2\frac{\nu^2}{\nu_m^2}\right)B^2\left(\frac{\nu}{2\gamma^2}\frac{\nu_m}{\nu}\right)d\frac{\nu_m}{\nu}, \qquad (17)$$

which is identical with Eq. (11) except for the physical meaning of the integration variable.

VI. EXAMPLES

As an example, let us take a simple analytical form of $B(z)$, i.e., a Lorentzian field of length $2L$ [full width at half maximum (FWHM)]:

$$B(z)=B_0/(1+z^2/L^2)\,; \tag{18}$$

the FT of $B(t')$ is $(\nu'>0)$

$$B(\nu')=(\pi/c)B_0 L e^{-2\pi(L/c)\nu'}\,. \tag{19}$$

Then, from Eqs. 6, 7, 10, and 11, we have

$$\frac{dW}{d\Omega}=\frac{\pi C^2}{4c}\,\gamma^4 B_0^2 L(1+\gamma^2\theta^2)f^2\,, \tag{20}$$

$$W=\frac{\pi C^2}{12c}\gamma^2 B_0^2 L\,, \tag{21}$$

$$\frac{dW}{d\Omega d\nu}=\frac{\pi^2C^2}{4c^2}\gamma^2B_0^2L^2(1+\gamma^2\theta^2)^2f^2 \times\exp\left(-4\pi\frac{L(1+\gamma^2\theta^2)}{2\gamma^2c}\nu\right)\,, \tag{22}$$

$$\begin{aligned}\frac{dW}{d\nu}&=\frac{\pi^3C^2}{4c^2}B_0^2L^2\int_1^\infty(y^{-2}-2y^{-3}+2y^{-4})e^{-xy}\,dy\\&=\frac{\pi^3C^2}{4c^2}B_0^2L^2[\tfrac{2}{3}e^{-x}(1+x+\tfrac{1}{2}x^2)+x(1+x-\tfrac{1}{3}x^2)\\&\quad\times\mathrm{Ei}(-x)]\,,\end{aligned} \tag{23}$$

where $x=4\nu/\nu_1$, $\nu_1=2\gamma^2c/\pi L$, and

$$\mathrm{Ei}(x)=\int_{-\infty}^{x}\frac{e^x}{x}\,dx$$

(exponential integral).

As another example to compare (and to show that qualitatively the results are the same), let us take a Gaussian field, also of length ~$2L$ ($1.66L$ FWHM):

$$B(z)=B_0e^{-z^2/L^2}\,. \tag{24}$$

In this case the FT of $B(t')$ is

$$B(\nu')=\sqrt{\pi}\,B_0\exp\{-[\pi(L/c)\nu']^2\}\,, \tag{25}$$

and then,

$$\frac{dW}{d\Omega}=\frac{\sqrt{\pi}C^2}{2\sqrt{2}\,c}\,\gamma^4B_0^2L(1+\gamma^2\theta^2)f^2\,, \tag{26}$$

$$W=\frac{\pi^{3/2}C^2}{6\sqrt{2}\,c}\,\gamma^2B_0^2L\,, \tag{27}$$

$$\frac{dW}{d\Omega d\nu}=\frac{\pi C^2}{4c^2}\gamma^2B_0^2L^2(1+\gamma^2\theta^2)f^2 \times\exp\left\{-2\left[\frac{\pi L(1+\gamma^2\theta^2)}{2\gamma^2c}\nu\right]^2\right\}, \tag{28}$$

$$\begin{aligned}\frac{dW}{d\nu}&=\frac{\pi^2C^2}{4c^2}B_0^2L^2\int_1^\infty(y^{-2}-2y^{-3}+2y^{-4})e^{-x^2y^2}\,dy\\&=\frac{\pi^2C^2}{4c^2}B_0^2L^2[\tfrac{1}{3}e^{-x^2}(1+4x^2)+x\sqrt{\pi}(1+\tfrac{4}{3}x^2)\,\mathrm{erfc}x\\&\quad-x^2\mathrm{Ei}(-x^2)]\,,\end{aligned} \tag{29}$$

where $x=\sqrt{2}\,\nu/\nu_1$, $\nu_1=2\gamma^2c/\pi L$, and

$$\mathrm{erfc}x=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-x^2}\,dx=1-\mathrm{erf}x$$

(complementary error function).

VII. CONNECTION WITH THE "EDGE EFFECT"

In the case[8] of a charged particle entering (or leaving a "long" uniform magnet, when the length L of the edge (fall-off distance of the magnetic field) is $\ll L_0$, the field $U(t)$ seen by the observer is composed of two parts: a sharp rise, with a rise time ~$L/2\gamma^2c$, followed by a fall, with fall time ~$1/\nu_c$ where ν_c is the usual critical frequency (or the reverse sequence in the case of the particle leaving the magnet). In the case that one is interested only in the higher part of the spectrum, with frequencies $\gg\nu_c$, one can make the approximation that, after the rapid rise, the amplitude remains constant (neglecting the contribution of the slower fall). In this case the calculation procedure can be the same as with a short magnet, except that in this case the calculated lower part of the spectrum, up to frequencies above ν_c, is meaningless. In order to calculate the spectrum, it can be useful to express the field $B(z)$ as the integral of its derivative with respect to z,

$$B(z)=\int_{-\infty}^{z}B'(z)\,dz=B'(z)*h(z)\,,$$

where $*$ represents convolution and h is Heaviside's step function, and use the convolution theorem in the Fourier transformation (the step function contributes a $1/\nu^2$ factor in the power spectrum).

As particular examples, let us take an arctangent and an error function.

First example: arctangent function.

$$B(z)=B_0\left(\frac{1}{2}+\frac{1}{\pi}\arctan\frac{z}{L}\right)=\frac{B_0}{\pi L}\,\frac{1}{1+z^2/L^2}*h(z)\,,$$

$$B\left(\frac{y\nu}{2\gamma^2}\right)=\frac{B_0}{\pi}\,\frac{\gamma^2}{\nu y}\,\exp\left(-2\pi\frac{L}{c}\,\frac{y\nu}{2\gamma^2}\right),$$

$$\frac{dW}{d\Omega d\nu}=\frac{C^2}{4\pi^2}\,\frac{\gamma^6}{\nu^2}f^2B_0^2\exp\left(-4\pi\frac{L}{c}\,\frac{y\nu}{2\gamma^2}\right),$$

$$\frac{dW}{d\nu}=\frac{C^2}{4\pi}\,\frac{\gamma^4}{\nu^2}\,B_0^2S(x)=\frac{\pi C^2}{c^2}B_0^2L^2\frac{1}{x^2}S(x)\,,$$

$$S(x) = \int_1^\infty \langle f^2\rangle e^{-xy}\,dy$$

$$= \tfrac{1}{10} e^{-x}(\tfrac{7}{3} - \tfrac{13}{3}x + \tfrac{17}{2}x^2 - 3x^3 + \tfrac{1}{2}x^4)$$

$$+ \tfrac{1}{6}x^3(1 - \tfrac{1}{2}x + \tfrac{1}{10}x^2)\,\mathrm{Ei}(-x)\,,$$

where $x = 4\nu/\nu_1$, and $\nu_1 = 2\gamma^2 c/\pi L$.

Second example: error function.

$$B(z) = \tfrac{1}{2}B_0\left(1 + \mathrm{erf}\frac{z}{L}\right) = \frac{B_0}{L\sqrt{\pi}}\, e^{-z^2/L^2} * h(z)\,,$$

$$B\left(\frac{y}{2\gamma^2}\nu\right) = \frac{B_0}{\pi^2}\frac{\gamma^4}{\nu^2 y^2}\exp\left[-\pi^2\left(\frac{Ly\nu}{2\gamma^2 c}\right)^2\right],$$

$$\frac{dW}{d\Omega d\nu} = \frac{C^2}{4\pi^2}\frac{\gamma^6}{\nu^2} f^2 B_0^2 \exp\left[-\left(\pi\frac{Ly\nu}{2\gamma^2 c}\right)^2\right],$$

$$\frac{dW}{d\nu} = \frac{C^2}{4\pi}\frac{\gamma^4}{\nu^2} B_0^2 S(x) = \frac{\pi C^2}{8c^2} B_0^2 L^2 \frac{1}{x^2} S(x)\,,$$

$$S(x) = \int_1^\infty \langle f^2\rangle e^{-x^2 y^2}\,dy$$

$$= \tfrac{1}{15} e^{-x^2}(\tfrac{7}{2} - \tfrac{13}{2}x^2 + 8x^4) + \tfrac{2}{3}\sqrt{\pi}\,x^3(1 + \tfrac{4}{3}x^2)\,\mathrm{erfc}\,x$$

$$+ x^4\,\mathrm{Ei}(-x^2)\,,$$

where $x = \sqrt{2}\,\nu/\nu_1$ and $\nu_1 = 2\gamma^2 c/\pi L$.

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Note added in proof. The first observation of synchrotron radiation from protons and of the "edge effect" has been made at CERN recently, and is described in a paper by Bossart *et al.*[13]

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