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Coulomb scattering in a laser field

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The problem of scattering by a local potential in the presence of an intense radiation field is studied for the case where the potential is Coulombic at great distances. The effect of the Coulomb tail on the asymptotic dynamics is accounted for here through a modification of the form of the wave functions which describe the time evolution of the system in initial and final states. This is in analogy with previous treatments of field-free Coulomb scattering. Starting from the time-dependent picture, the author obtains a time-independent formulation of the problem and then applies it to the derivation of a low-frequency approximation. In the simplest version of this approximation the transition amplitude is represented as the product of a known field-dependent factor (a Bessel function) and the physical field-free scattering amplitude, thus generalizing an earlier result of this type derived for the case of a short-range potential.

I. INTRODUCTION

The problem of electron-atom scattering in the presence of an intense radiation field has received a fair amount of attention in recent years.[1] In the case where the target is an ion with a net charge, additional complications arise as a result of the long range of the Coulomb potential. Our purpose here is to reformulate the scattering theory to account for the effects of the Coulomb tail. For simplicity we ignore the internal structure of the target, representing the scatterer by a local potential, and consider an idealized single-mode radiation field. Even in such a model the prescription for calculation is quite complicated, but it simplifies considerably in the dipole approximation and in the low-frequency limit, as discussed below.

A time-independent method for calculating scattering amplitudes in the presence of an intense laser field has been described previously for the case of short-range potentials.[2] As for the field-free scattering problem[3] a time-independent formulation can be derived from a time-dependent description based on the use of wave packets. (A quantum, rather than classical, treatment of the radiation field is somewhat more convenient for the purpose of arriving at a time-independent formulation, although the two are physically equivalent in the intense-field case considered here.) In order to extend this method to include Coulomb scattering it is necessary to properly account for the role of the Coulomb tail on the propagation of the system during the initial and final stages of the scattering process. In particular, the time-dependent asymptotic solutions which go into the construction of the wave packets should be such that the packets follow Coulomb-modified classical trajectories, as would be expected on physical grounds. Modified asymptotic solutions for Coulomb scattering in the absence of an external field were first introduced, and justified mathematically, by Dollard.[4] In the approach adopted here we introduce a Coulomb modification not of the free-particle solution but of the solution appropriate to an electron moving in an intense radiation field. In passing over to a time-independent formulation we are led to a rule (generalizing one obtained earlier[5,6] for field-free Coulomb scattering) for determining the elements of the scattering matrix by extracting the coefficients of certain singularities in the matrix elements of the resolvent operator. These are branch-point singularities, not the simple poles which arise in the case of short-range potentials. One possible practical procedure for extracting these coefficients would involve an extrapolation from complex energies, along the lines discussed, in the field-free case, by McCartor and Nuttall.[5] An attractive feature of this procedure is that it requires the use of the same basis states as in the short-range case; these are plane waves modified by the presence of the radiation field *alone*.

Section II is devoted to an S-matrix formulation of the problem of bremsstrahlung in the absence of an external radiation field. Our purpose here is to introduce, in the context of a relatively simple and well-understood problem, some of the heuristic methods and specific results to be used later on. The *stimulated* bremsstrahlung process is taken up in Sec. III. The choice of asymptotic states is justified here using physical arguments based on a wave-packet picture. Mathematical questions dealing with the existence of the wave operator and the scattering operator lie outside the scope of the present investigation.[7,8] The assumed existence of the scattering operator im-

plies the existence of certain branch-point singularities in the resolvent. In the dipole approximation these singularities can be exhibited; their coefficient then provides an explicit expression for the scattering matrix valid for real values of the energy. The necessity of extrapolation from complex energies is avoided here, though at the expense of introducing, as input to the calculation of the initial—and final—state wave functions, solutions of the Schrödinger equation in the local potential. The form of this result, derived in Sec. III B, suggests an iterative solution, with the frequency of the field playing the role of the small expansion parameter. The first term in this expansion is of a particularly simple form. As shown in Sec. III C, it can be determined from a knowledge of the same free-free matrix element which appears in the treatment of bremsstrahlung in the absence of the field. Further simplification is obtained from an approximate evaluation of this matrix element. The result, Eq. (3.42), expresses the amplitude for scattering in the presence of the field as a product of a field-dependent factor (a Bessel function) and the physical field-free scattering amplitude, thus generalizing an earlier result derived for scattering by a short-range potential.[9-11]

II. BREMSSTRAHLUNG IN A COULOMB FIELD

We are concerned in this section with the scattering of an electron in a potential $V(r)$, with $V\sim g/r$ for large r, in the absence of an external field. In the course of the scattering a single photon is produced. The interaction H' between the electron and the radiation field will be treated to first order. The Hamiltonian of the system is

$$H=K+H_F+H'+V, \tag{2.1}$$

where $K=p^2/2\mu$ is the electron kinetic energy operator, H_F is the free-field Hamiltonian, and H' takes the form

$$H'=-(e/\mu c)\vec{\mathfrak{p}}\cdot\vec{A}+(e^2/2\mu c^2)A^2. \tag{2.2}$$

The vector potential for a single-mode field is

$$\vec{A}=(2\pi\hbar c^2/\omega L^3)^{1/2}(a\vec{\lambda}e^{i\vec{k}\cdot\vec{r}}+a^\dagger\vec{\lambda}^*e^{-i\vec{k}\cdot\vec{r}}), \tag{2.3}$$

where $\omega=kc$, L^3 is the quantization volume, and operator a destroys a photon of polarization $\vec{\lambda}$ and wave number $\vec{k}$. The contribution to the field energy H_F is $\hbar\omega a^\dagger a$. In general, of course, we must sum over modes, with $L^{-3}\Sigma_k\to(2\pi)^{-3}\int d^3k$ in the usual way.

The S-matrix element of interest is

$$S_{fi}=\lim_{\substack{t_1\to\infty\\ t_2\to-\infty}}\langle\Phi_f(t_1)|e^{-iH(t_1-t_2)/\hbar}|\Phi_i(t_2)\rangle. \tag{2.4}$$

We turn now to a discussion of the asymptotic states Φ_i and Φ_f. The system is initially in the state $|\vec{\mathfrak{p}}_i\rangle|0\rangle$, corresponding to an electron with momentum $\vec{\mathfrak{p}}_i$ with no photons present. An approximate description of the time evolution of this state, valid for $t_2\sim-\infty$, is given by

$$|\Phi_i(t_2)\rangle=\exp[-iE_it_2/\hbar +i\xi_i\ln(-4E_it_2/\hbar)]|\vec{\mathfrak{p}}_i\rangle|0\rangle. \tag{2.5}$$

Here we define $E_i=p_i^2/2\mu$ and

$$\xi_i=\mu g/\hbar p_i. \tag{2.6}$$

The time dependence of the final state is taken to be

$$|\Phi_f(t_1)\rangle=\exp[-i(E_f+\hbar\omega)t_1/\hbar -i\xi_f\ln(4E_ft_1/\hbar)]|\vec{\mathfrak{p}}_f\rangle|\vec{k},\vec{\lambda}\rangle, \tag{2.7}$$

corresponding to the presence of a single photon, the electron having momentum $\vec{\mathfrak{p}}_f$. Following Dollard,[4] we have included the logarithmic phase factors in Eqs. (2.5) and (2.7) to account for the effect of the Coulomb tail. In fact, each of these states satisfies an equation of the form

$$\left(K+H_F+\frac{g}{r}-i\hbar\frac{d}{dt}\right)|\Phi(t)\rangle=\left(\frac{g}{r}-\frac{\mu g}{p|t|}\right)|\Phi(t)\rangle. \tag{2.8}$$

We should, of course, be working with a wave packet, obtained by superposition of modified plane waves in such a way that the center of the packet moves according to

$$\langle\vec{\mathfrak{r}}\rangle\cong\frac{\langle\vec{\mathfrak{p}}\rangle}{\mu}t-g\mu\left\langle\frac{\vec{\mathfrak{p}}}{p^3}\right\rangle(\mathrm{sign}t)\ln|t|,$$

the averages being taken over the packet. Such a packet will satisfy an equation of the form (2.8), with $\vec{\mathfrak{p}}$ and $\vec{\mathfrak{r}}$ on the right-hand side replaced by their averages. The right-hand side will then vanish faster than $1/r$, indicating that the Coulomb tail has been accounted for in an approximate way. Dollard was able to show that this is sufficient to guarantee the existence of the limit

$$|u_i^{(+)}\rangle=\lim_{t_2\to-\infty}e^{i(K+V)t_2/\hbar}|\Phi_i(t_2)\rangle; \tag{2.9}$$

this wave function satisfies $(K+V-E_i)|u_i^{(+)}\rangle=0$ with outgoing wave boundary conditions. A wave-packet superposition of states is to be understood in the interpretation of Eq. (2.9).

A useful time-independent reformulation of Eq. (2.9) follows from the introduction of the identity

$$e^{i(K+V)t_2/\hbar}=\frac{1}{2\pi i}\int_c e^{iEt_2/\hbar}g(E)dE, \tag{2.10}$$

where $g(E)=(E-K-V)^{-1}$ and where the integration contour runs from $+\infty$ to $-\infty$ above the real axis. To carry out the limit in Eq. (2.9) we write $E=E_0+i\epsilon$,

with $\epsilon \to 0+$, and make use of the relation[12]

$$\lim_{\epsilon\to 0+}\lim_{t_2\to-\infty}\frac{e^{i(E_0-E_i)t_2/\hbar}}{(E_0+i\epsilon-E_i)^{1-i\xi_i}}\,e^{i\xi_i\ln(-4E_it_2/\hbar)} = -2\pi i\,\delta(E_0-E_i)\zeta_i, \tag{2.11a}$$

with

$$\zeta_i \equiv (4E_i)^{i\xi_i}e^{-\pi\xi_i/2}/\Gamma(1-i\xi_i). \tag{2.11b}$$

The existence of the limit in Eq. (2.9) implies the alternative representation

$$|u_i^{(+)}\rangle = \lim_{\epsilon\to 0+}(i\epsilon)^{1-i\xi_i}\,\zeta_i g(E_i+i\epsilon)|\vec{p}_i\rangle. \tag{2.12a}$$

A similar analysis of the final-state wave function leads to

$$\langle u_f^{(-)}| = \zeta_f \lim_{\epsilon\to 0+}(i\epsilon)^{1-i\xi_f}\langle\vec{p}_f|\,g(E_f+i\epsilon). \tag{2.12b}$$

These representations can be verified directly within the time-independent formalism.[13]

Returning now to expression (2.4) for the S-matrix element we introduce

$$e^{-iH(t_1-t_2)/\hbar} = \frac{1}{2\pi i}\int_C e^{-iE(t_1-t_2)/\hbar}\,G(E)\,dE, \tag{2.13}$$

with $G(E)=(E-H)^{-1}$. It will be convenient to expand the resolvent as

$$G = G_1 + G_1H'G_1 + G_1H'GH'G_1, \tag{2.14}$$

with $G_1(E)=(E-K-V-H_F)^{-1}$. Only the second term on the right-hand side of Eq. (2.14) contributes to the photon production process in first order. The limits in Eq. (2.4) can now be carried out using the relations

$$G_1(E)|\vec{p}_i\rangle|0\rangle = |0\rangle\,g(E)|\vec{p}_i\rangle, \tag{2.15a}$$

$$\langle\vec{k},\vec{\lambda}|\langle\vec{p}_f|G_1(E) = \langle\vec{p}_f|\,g(E-\hbar\omega)\langle\vec{k},\vec{\lambda}|, \tag{2.15b}$$

along with Eqs. (2.12). This leads to

$$S_{fi} = -2\pi i\,\delta(E_f+\hbar\omega-E_i)M_{fi}, \tag{2.16a}$$

with

$$M_{fi} = \left(\frac{2\pi\hbar c^2}{\omega L^3}\right)^{1/2}\frac{|e|}{\mu c}\,\vec{\lambda}^*\cdot\langle u_f^{(-)}|\,\vec{p}\,e^{-i\vec{k}\cdot\vec{r}}|u_i^{(+)}\rangle. \tag{2.16b}$$

For a potential of short range, the Lippmann-Schwinger integral equation can be used to derive an expansion, in powers of the frequency, of the matrix element in Eq. (2.16b); the first two terms in the expansion can be evaluated from a knowledge of the on-shell field-free scattering amplitude.[14] Now the Coulomb wave function satisfies the homogeneous version of the integral equation[15] (we shall return to this point in the following) so that the standard derivation for short-range potentials is inapplicable. In fact a classical argument leads to an expansion of a different form in the Coulomb case.[16] Presumably an analysis of the quantum mechanical matrix element would lead to a similar modified low-frequency expansion but we shall not enter into such an analysis at the present time. However, if we confine ourselves to the dipole approximation the leading term in the expansion may be derived without difficulty.[17] Thus, we write

$$\langle u_f^{(-)}|\,\vec{p}\,|u_i^{(+)}\rangle = (E_i-E_f)^{-1}\vec{N}_{fi}, \tag{2.17a}$$

with

$$\vec{N}_{fi} = \langle u_f^{(-)}|[\vec{p},(K+V)]|u_i^{(+)}\rangle. \tag{2.17b}$$

Our approximation for $\vec{N}_{fi}$ consists of setting $\omega=0$ in the energy conservation condition $E_f=E_i-\hbar\omega$. Going back to Eqs. (2.12) we then have

$$\vec{N}_{fi} \cong \xi_i^2 \lim_{\epsilon\to 0+}(i\epsilon)^{2(1-i\xi_i)} \times\langle\vec{p}_f|\,g(E_i+i\epsilon)[\vec{p},(K+V-E_i-i\epsilon)] \times g(E_i+i\epsilon)|\vec{p}_i\rangle. \tag{2.18}$$

This reduces to

$$\vec{N}_{fi} \cong (\vec{p}_f-\vec{p}_i)f(\vec{p}_f,\vec{p}_i), \tag{2.19}$$

where, for $p_f^2/2\mu = p_i^2/2\mu = E_i$ and $\vec{p}_f \neq \vec{p}_i$,

$$f(\vec{p}_f,\vec{p}_i) = \xi_i^2 \lim_{\epsilon\to 0+}(i\epsilon)^{2(1-i\xi_i)} \times\langle\vec{p}_f|\,g(E_i+i\epsilon)|\,\vec{p}_i\rangle \tag{2.20}$$

represents the physical scattering amplitude.[6] The approximate bremsstrahlung amplitude then takes the form

$$M_{fi} \cong \left(\frac{2\pi\hbar c^2}{\omega L^3}\right)^{1/2}\frac{|e|}{\mu c}\,\vec{\lambda}^*\cdot(\vec{p}_f-\vec{p}_i) \times(E_i-E_f)^{-1}f(\vec{p}_f,\vec{p}_i). \tag{2.21}$$

We conclude this section with a remark on the integral equation for the Coulomb wave function which will be relevant to our later discussion. Suppose we introduce the form

$$g(E) = g_0(E)+g_0(E)Vg(E), \tag{2.22}$$

where $g_0(E)=(E-K)^{-1}$, into Eq. (2.10); this may be combined with Eq. (2.9) for the wave function. The contribution arising from the first term on the right-hand side of Eq. (2.22) is

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}dE_0\lim_{t_2\to-\infty}\frac{e^{i(E_0-E_i)t_2/\hbar}}{E_0+i\epsilon-E_i} \times e^{i\xi_i\ln(-4E_it_2/\hbar)}|\vec{p}_i\rangle.$$

To interpret this expression we replace the plane-wave state $|\vec{p}_i\rangle$ by a wave-packet state. The logarithmic phase factor then introduces a rapid oscillation, which leads to vanishing contribution

in the limit $t_2 \to -\infty$; the singularity $(E_0+i\epsilon-E_i)^{-1}$ is too weak to prevent this.[6] Taking into account the second term in Eq. (2.22) we obtain the homogeneous integral equation

$$|u_i^{(+)}\rangle = \lim_{\epsilon \to 0+} g_0(E_i+i\epsilon)V|u_i^{(+)}\rangle . \tag{2.23}$$

This result may be verified directly by showing that the right-hand side satisfies the Schrödinger equation and has the correct asymptotic form.[15,18]

III. STIMULATED BREMSSTRAHLUNG

A. General formulation

We begin with a description of the asymptotic states appropriate to the case where the electron interacts with an intense laser field as it scatters from the potential $V(r)$. The field might be represented as a pulse of radiation, moving along the z axis, say, with vanishing intensity for large values of $|z-ct|$. A wave-packet solution[19] may be constructed from a superposition of states, each corresponding to a definite field frequency and asymptotic electron momentum. In analogy with Eqs. (2.5) and (2.7) we represent these elementary states as

$$|\Phi_i(t_2)\rangle = \exp[-iE_{n_i\vec{p}_i}t_2/\hbar + i\xi_i \ln(-4E_it_2/\hbar)]|\psi_{n_i\vec{p}_i}\rangle, \tag{3.1a}$$

$$|\Phi_f(t_1)\rangle = \exp[-iE_{n_f\vec{p}_f}t_1/\hbar - i\xi_f \ln(4E_ft_1/\hbar)]|\psi_{n_f\vec{p}_f}\rangle. \tag{3.1b}$$

The electron-field states $|\psi_{n\vec{p}}\rangle$, introduced previously,[2,11] may be thought of as evolving from the unperturbed state $|n;\vec{p}\rangle$ as the interaction is switched on adiabatically. Here $|n;\vec{p}\rangle$ corresponds to an electron with momentum $\vec{p}$, the field consisting of n photons each with polarization $\vec{\lambda}$ and wave number $\vec{k}$. The interacting states satisfy the time-independent Schrödinger equation

$$H_0|\psi_{n\vec{p}}\rangle = E_{n\vec{p}}|\psi_{n\vec{p}}\rangle, \tag{3.2}$$

where

$$H_0 = K+H_F+H', \tag{3.3}$$

$$E_{n\vec{p}} = p^2/2\mu + n\hbar\omega + \Delta, \tag{3.4a}$$

and

$$\Delta = (e^2/2\mu c^2)(4\pi n\hbar c^2/\omega L^3). \tag{3.4b}$$

(In the intense field limit considered here photon depletion effects are negligible. The level shift Δ is then state independent and plays no essential role.[20]) The asymptotic states defined in Eqs. (3.1) then satisfy equations of the form

$$\left(H_0+\frac{g}{r}-i\hbar\frac{d}{dt}\right)|\Phi(t)\rangle = \left(\frac{g}{r}-\frac{\mu g}{p|t|}\right)|\Phi(t)\rangle. \tag{3.5}$$

We see, as discussed earlier in connection with Eq. (2.8), that wave packets built up from the functions $\Phi(t)$ will represent asymptotic solutions, the effect of the Coulomb tail being accounted for by inclusion of the logarithmic phase.

We remark, parenthetically, that the asymptotic wave functions defined in Eqs. (3.1) can be expressed in terms of certain asymptotic time-evolution operators, in analogy with earlier work.[4,21] Thus, in the absence of the radiation field the wave function of Eq. (2.5) can be expressed as $|\Phi_i(t)\rangle = U_{as}(t)|\vec{p}_i\rangle|0\rangle$, where $U_{as}(t)$ is Dollard's asymptotic evolution operator

$$U_{as}(t) = U_0(t)\exp[i(\mu g/\hbar p)\ln(-4Kt/\hbar)], \tag{3.6}$$

with $U_0(t) = \exp(-iKt/\hbar)$. As a natural generalization appropriate to the case where the field is present we take $|\Phi_i(t)\rangle = U_{as}(t)|\vec{p}_i\rangle|n_i\rangle$; $U_{as}(t)$ is of the form shown in Eq. (3.6), where now $U_0(t)$ is the evolution operator for the "dressed" electron. The equivalence with the original form (3.1a) may be established through a constructive procedure[21] in which U_0 is represented as

$$U_0(t) = \exp[-i(K+H_F)t/\hbar]Z(t).$$

Then $Z(t)$ satisfies

$$i\hbar\frac{d}{dt}Z(t) = H'(t)Z(t),$$

where $H'(t)$ is the electron-field interaction in the interaction representation, its time dependence being generated by the free Hamiltonian $K+H_F$. The differential equation for $Z(t)$ is easily solved by exponentiation. The desired initial condition is imposed by assuming that the interaction $H'(t)$ is switched off adiabatically for $t \to -\infty$. The result (omitting details here) is that Eq. (3.1a) is recovered along with a rule for constructing the states $|\psi_{n\vec{p}}\rangle$ (as an expansion in unperturbed states). This rule agrees with that obtained in Ref. 11 through a solution of Eq. (3.2).

We assume that the asymptotic solutions discussed above are sufficiently accurate to ensure the existence of the limits in the definition (2.4) of the S matrix. To pass over to a time-independent form we make use of the identity (2.13). Then, with the asymptotic states given by Eqs. (3.1) the S matrix for scattering from an initial electron-field state $|n_i;\vec{p}_i\rangle$ to a final state $|n_f;\vec{p}_f\rangle$ becomes

$$S_{fi}=\lim_{\substack{t_2\to-\infty\\ t_1\to\infty}}\langle\psi_{n_f\vec{p}_f}|\frac{1}{2\pi i}\int_c dE\,e^{i\xi_f\ln(4E_f t_1/\hbar)}$$

$$\times e^{-i(E-E_{n_f\vec{p}_f})t_1/\hbar}$$

$$\times G(E)\,e^{i(E-E_{n_i\vec{p}_i})t_2/\hbar}$$

$$\times e^{i\xi_i\ln(-4E_i t_2/\hbar)}|\psi_{n_i\vec{p}}\rangle. \tag{3.7}$$

The resolvent can be expressed as

$$G=G_0+G_0TG_0, \tag{3.8}$$

where $G_0(E)=(E-H_0)^{-1}$ and T satisfies

$$T(E)=V+VG_0(E)T(E). \tag{3.9}$$

The contribution to S_{fi} coming from the first term G_0 in Eq. (3.8) can be evaluated by making use of the eigenfunction expansion

$$G_0(E)=\sum_n\int d^3p\,\frac{|\psi_{n\vec{p}}\rangle\langle\psi_{n\vec{p}}|}{E-E_{n\vec{p}}}, \tag{3.10}$$

and the orthonormality relation

$$\langle\psi_{n'\vec{p}'}|\psi_{n\vec{p}}\rangle=\delta_{n'n}\delta(\vec{p}'-\vec{p}). \tag{3.11}$$

We then see that the contribution from G_0 vanishes due to rapid oscillations introduced by the logarithmic phases. The absence of the "no scattering" contribution to the S matrix has its counterpart in the case of field-free Coulomb scattering.[6] [The effect is similar to that responsible for the appearance of the *homogeneous* integral equation for the continuum Coulomb wave function, as indicated by the argument leading to Eq. (2.23).] We then obtain, with the aid of Eq. (2.11) to evaluate the limits, the result

$$S_{fi}=-2\pi i\delta(E_{n_f\vec{p}_f}-E_{n_i\vec{p}_i})M_{fi}, \tag{3.12}$$

where

$$M_{fi}=\lim_{\epsilon\to 0+}\zeta_f\zeta_i(i\epsilon)^{2-i(\xi_i+\xi_f)}$$

$$\times\langle\psi_{n_f\vec{p}_f}|G(E_{n_i\vec{p}_i}+i\epsilon)|\psi_{n_i\vec{p}_i}\rangle. \tag{3.13a}$$

This generalizes Eq. (2.20) for the Coulomb scattering amplitude in the absence of the field, reducing to it for $H'\to 0$. An alternative version of Eq. (3.13a) is

$$M_{fi}=\lim_{\epsilon\to 0+}\zeta_f\zeta_i(i\epsilon)^{-i(\xi_i+\xi_f)}\langle\psi_{n_f\vec{p}_f}|T(E_{n_i\vec{p}_i}+i\epsilon)|\psi_{n_i\vec{p}_i}\rangle. \tag{3.13b}$$

As a calculational prescription this form requires the evaluation of the T-matrix element using, e.g., the Lippmann-Schwinger equation (3.9), with $\epsilon>0$. The physical amplitude is determined by analytic continuation, with the removal of the singular phase factors. The possibility of numerical implementation of this idea, for the case of field-free scattering, was discussed by McCartor and Nuttall.[5]

B. Dipole approximation

If we ignore the effect of electron recoil in the emission and absorption process the general expression given previously[11] for the state $\psi_{n\vec{p}}$ as an expansion in unperturbed states simplifies to

$$|\psi_{n\vec{p}}\rangle=|\vec{p}\rangle\sum_m\gamma_m|n+m\rangle. \tag{3.14}$$

The expansion coefficients are given by

$$\gamma_m(\vec{p}\cdot\vec{\lambda})=e^{-im\theta}J_{-m}(\rho), \tag{3.15}$$

where J_{-m} is the Bessel function of the first kind, and ρ and θ are real parameters defined by[20]

$$(\tfrac{1}{2}\hbar\omega)\rho e^{i\theta}=(2\pi\hbar e^2 n/\mu^2\omega L^3)^{1/2}(\vec{p}\cdot\vec{\lambda}). \tag{3.16}$$

With the neglect of recoil effects the general expressions (3.13) for the transition matrix can be put in more explicit form, leading to particularly simple approximations in the low-frequency domain, as we now show.

We begin by introducing a low-frequency approximation for the free Green's function $G_0(E_{n_i\vec{p}_i}+i\epsilon)$, obtained from the eigenfunction expansion (3.10) by setting $\omega=0$ in the energy denominator. Thus, we define

$$G_0^i=\sum_n\int d^3p\,\frac{|\psi_{n\vec{p}}\rangle\langle\psi_{n\vec{p}}|}{E_i+i\epsilon-p^2/2\mu}. \tag{3.17}$$

The error in this approximation is

$$G_0-G_0^i=G_0W_iG_0^i, \tag{3.18}$$

with

$$W_i=\sum_n\int d^3p\,|\psi_{n\vec{p}}\rangle(n-n_i)\hbar\omega\langle\psi_{n\vec{p}}|. \tag{3.19}$$

Using the Graf addition formula for the Bessel function the sum over photon states can be performed,[11] leading to

$$G_0^i=\int d^3p\,\frac{|\vec{p}\rangle\langle\vec{p}|}{E_i+i\epsilon-p^2/2\mu}\equiv g_0(E_i+i\epsilon); \tag{3.20}$$

this is to be understood as a unit operator in the space of photon states. To include the effect of the potential we introduce

$$G^i=G_0^i+G_0^iVG^i, \tag{3.21}$$

which, according to Eqs. (2.22) and (3.20) is just $g(E_i+i\epsilon)$. Comparison of Eq. (3.21) with the corresponding equation

$$G=G_0+GVG_0 \tag{3.22}$$

for the exact resolvent leads to the identity

$$G=G^i+(1+GV)(G_0-G_0^i)(1+VG^i). \tag{3.23}$$

This reduces, with the aid of Eqs. (3.18), (3.21), and (3.22) to

$$G=G^i+G\,W_iG^i. \tag{3.24a}$$

A similar analysis leads to

$$G=G^f+G^fW_fG, \tag{3.24b}$$

which can be inserted into Eq. (3.24a) to give

$$G=G^i+G^fW_iG^i+G^fW_fGW_iG^i. \tag{3.25a}$$

The equivalent form

$$G=G^f+G^fW_fG^i+G^fW_fGW_iG^i \tag{3.25b}$$

is obtained by inserting the expression (3.24a) into Eq. (3.24b).

We proceed by introducing the representation (3.25a) into the expression (3.13a) for the transition matrix element. [The following analysis can be repeated, starting instead with Eq. (3.25b); the same final result is obtained in either case.] We write

$$M_{fi}=M_{fi}^{(1)}+M_{fi}^{(2)}, \tag{3.26}$$

where $M_{fi}^{(1)}$ is the contribution arising from the first term G^i in Eq. (3.25a). It can be evaluated, with the aid of the addition formula for the Bessel functions, as

$$M_{fi}^{(1)}=\gamma_{n_f-n_i}([\vec{\mathfrak{p}}_i-\vec{\mathfrak{p}}_f]\cdot\vec{\lambda})\lim_{\epsilon\to 0+}\zeta_f\zeta_i \times(i\epsilon)^{2-i(\xi_f+\xi_i)}\langle\vec{\mathfrak{p}}_f|g(E_i+i\epsilon)|\vec{\mathfrak{p}}_i\rangle. \tag{3.27}$$

In the case where the electron gains or loses a net amount of energy to the field, i.e., $n_f\neq n_i$, $M_{fi}^{(1)}$ vanishes since, for $E_f\neq E_i$, the matrix element $\langle\vec{\mathfrak{p}}_f|g(E_i+i\epsilon)|\vec{\mathfrak{p}}_i\rangle$ is not sufficiently singular to compensate for the vanishing factors of ϵ in Eq. (3.27). More explicitly, we can use Eq. (2.12a) to write

$$M_{fi}^{(1)}=\gamma_{n_f-n_i}\lim_{\epsilon\to 0+}\zeta_f(i\epsilon)^{1-i\xi_f}\langle\vec{\mathfrak{p}}_f|u_i^{(+)}\rangle. \tag{3.28}$$

From the integral equation (2.23) we have

$$\langle\vec{\mathfrak{p}}_f|u_i^{(+)}\rangle=\frac{1}{E_i-E_f+i\epsilon}\langle\vec{\mathfrak{p}}_f|V|u_i^{(+)}\rangle. \tag{3.29}$$

This expression is bounded for $E_f\neq E_i$ so that the limit in Eq. (3.28) is zero. For the case $n_f=n_i$ we may use Eq. (2.20) for the physical Coulomb scattering amplitude; Eq. (3.27) then becomes

$$M_{fi}^{(1)}=\gamma_0([\vec{\mathfrak{p}}_i-\vec{\mathfrak{p}}_f]\cdot\vec{\lambda})f(\vec{\mathfrak{p}}_f,\vec{\mathfrak{p}}_i)\delta_{n_fn_i}. \tag{3.30}$$

[The same result for $n_f=n_i$ is obtained by making use, in Eq. (3.29), of Altick's evaluation[22] of the matrix element $\langle\vec{\mathfrak{p}}_f|V|u_i^{(+)}\rangle$ in the limit $p_f\to p_i$. For a potential of short range this matrix element represents the physical scattering amplitude, while in the Coulomb case a singular phase factor appears and is cancelled in Eq. (3.28).]

The remaining contribution $M_{fi}^{(2)}$ to the transition matrix element arises from the last two terms in Eq. (3.25a). When these are inserted in Eq. (3.13a) we obtain a result of the form

$$M_{fi}^{(2)}=\langle\psi_{n_f\vec{\mathfrak{p}}_f}^{(-)}|\,W_i+W_fG(E_{n_i\vec{\mathfrak{p}}_i})W_i|\psi_{n_i\vec{\mathfrak{p}}_i}^{(+)}\rangle. \tag{3.31}$$

Here we have defined

$$|\psi_{n_i\vec{\mathfrak{p}}_i}^{(+)}\rangle=\lim_{\epsilon\to 0+}\zeta_i(i\epsilon)^{1-i\xi_i}G^i|\psi_{n_i\vec{\mathfrak{p}}_i}\rangle. \tag{3.32}$$

Using the expansion (3.14) and the identification of G^i with $g(E_i+i\epsilon)$ the limit can be evaluated in terms of the Coulomb wave function represented in Eq. (2.12a), with the result

$$|\psi_{n_i\vec{\mathfrak{p}}_i}^{(+)}\rangle=|u_i^{(+)}\rangle\sum_m\gamma_m(\vec{\mathfrak{p}}_i\cdot\vec{\lambda})|n_i+m\rangle. \tag{3.33}$$

In a similar way we find

$$|\psi_{n_f\vec{\mathfrak{p}}_f}^{(-)}\rangle=|u_f^{(-)}\rangle\sum_m\gamma_m(\vec{\mathfrak{p}}_f\cdot\vec{\lambda})|n_f+m\rangle. \tag{3.34}$$

We note that (within the dipole approximation) a result of the same form as that shown in Eqs. (3.30) and (3.31) can be derived from the scattering formalism set up previously for the case of short-range potentials.[2] Thus all of the specifically Coulombic effects are contained in the scattering amplitude and wave functions which appear in these expressions.

One can think of the operators W_i and W_f as measuring an average fluctuation energy of the field in interaction with the electron. Treating these operators as small perturbations in the low-frequency domain we can set up a procedure for determining the resolvent in Eq. (3.31) based on iteration of Eqs. (3.24). In Sec. III C we analyze the "Born" approximation associated with this iterative procedure.

C. Low-frequency approximation

The specification of the order of the terms in arriving at a low-frequency approximation requires some care since the limit $\omega\to 0$ would be a singular one for the expansion coefficients γ_m. The singularity appears in the definition (3.16) for ρ, with γ_m a function of ρ as shown in Eq. (3.15). The infrared difficulties encountered in perturbation theory are avoided by treating the ρ dependence (and hence the ω dependence) of the coefficients γ_m exactly. If multiphoton effects are to be significant in practice the parameter ρ must be of order unity or greater, and in fact this range

is experimentally accessible.[1] For definiteness in the following we take ρ and the expansion coefficients, which are smooth functions of ρ, to be of order unity. Then W_i and W_f in Eq. (3.31) are of first order. The "Born" approximation is $M_{fi}^{(2)} \cong B_{fi}$, with

$$B_{fi} = \langle \psi^{(-)}_{n_f \vec{p}_f} | W_i | \psi^{(+)}_{n_i \vec{p}_i} \rangle . \tag{3.35}$$

This term is not necessarily of first order since the matrix element can be singular [as may be anticipated from the appearance of the ω^{-1} singularity in the bremsstrahlung amplitude shown in Eq. (2.21)].

The Born term may be expressed in the more convenient form

$$B_{fi} = \vec{X} \cdot \langle u_f^{(-)} | (\vec{p}_i - \vec{p}) | u_i^{(+)} \rangle , \tag{3.36}$$

where

$$\vec{X} = -\left(\frac{2\pi\hbar e^2 n}{\mu^2 \omega L^3}\right)^{1/2} \times \{\vec{\lambda} \gamma_{n_f - n_i + 1}([\vec{p}_i - \vec{p}_f] \cdot \vec{\lambda}) + \vec{\lambda}^* \gamma_{n_f - n_i - 1}([\vec{p}_i - \vec{p}_f] \cdot \vec{\lambda})\} . \tag{3.37}$$

This result is easily verified using the addition formula and recursion relation satisfied by the Bessel functions [as shown in more detail in the passage from Eq. (3.26) to Eq. (3.29) of Ref. 11]. With the inclusion of the term shown in Eq. (3.30) the low-frequency approximation becomes

$$M'_{fi} \cong \gamma_0([\vec{p}_i - \vec{p}_f] \cdot \vec{\lambda}) f(\vec{p}_f, \vec{p}_i) \delta_{n_f n_i} + \vec{X} \cdot \langle u_f^{(-)} | (\vec{p}_i - \vec{p}) | u_i^{(+)} \rangle . \tag{3.38}$$

The simplifying feature of the above result lies in the decoupling of the dynamics of the electron interaction with the field and with the center of force. Further simplification is possible, along the lines discussed in connection with the corresponding weak-field bremsstrahlung amplitude shown in Eq. (2.16b). Thus, for $n_f \neq n_i$ the orthogonality of the wave functions allows us to write

$$M_{fi} \cong -\vec{X} \cdot \langle u_f^{(-)} | \vec{p} | u_i^{(+)} \rangle . \tag{3.39}$$

This becomes, with the additional approximation shown in Eqs. (2.17a) and (2.19),

$$M_{fi} \cong -\vec{X} \cdot (\vec{p}_f - \vec{p}_i)(E_i - E_f)^{-1} f(\vec{p}_f, \vec{p}_i) . \tag{3.40}$$

From the definition (3.37) and the Bessel-function recursion relation we have

$$\vec{X} \cdot (\vec{p}_i - \vec{p}_f) = \gamma_{n_f - n_i}([\vec{p}_i - \vec{p}_f] \cdot \vec{\lambda})(n_f - n_i)\hbar\omega . \tag{3.41}$$

This, along with the energy-conservation condition $E_i - E_f = (n_f - n_i)\hbar\omega$ leads to the approximation

$$M_{fi} \cong \gamma_{n_f - n_i}([\vec{p}_i - \vec{p}_f] \cdot \vec{\lambda}) f(\vec{p}_f, \vec{p}_i) . \tag{3.42}$$

This provides a generalization of Eq. (2.21), reducing to it in the appropriate weak-coupling limit.

For the case $n_f = n_i$ we keep only the first term in Eq. (3.38); this term is of order unity while the second term is of first order in the frequency.[23] Thus Eq. (3.42) holds for this case as well.

For a potential of *short* range the free-free matrix element appearing in Eq. (3.38) can be expanded in powers of the frequency and the coefficients of the first two terms can be expressed in terms of the on-shell scattering amplitude. This provides an approximation for M_{fi}, an improvement over Eq. (3.42), which can be shown to be equivalent to that obtained previously.[11] As indicated earlier in the discussion following Eq. (2.16b) such a procedure must be modified in the Coulomb case. We hope to return at a future time to further consideration of this point.

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