Exact two-dimensional plasma pair-correlation function in the Singwi-Tosi-Land-Sjolander approximation. I. k-space solutions and thermodynamic properties

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The Singwi-Tosi-Land-Sjolander (STLS) nonlinear integral equation for the plasma pair-correlation function g is adapted to two-dimensional one-component plasma (ocp) systems whose Coulombic interactions are characterized by the logarthmic potential. The two-dimensional STLS integral equation has the remarkable property that it can be exactly solved. The resulting expressions for $g(k)$ are extensively analyzed and are shown to be negative definite for all values of k and the coupling strength. The implication here is that in the STLS approximation, the ocp liquid can never undergo a phase transition. Exact expressions for g enable us to evolve exact formulas for the correlation energy and heat capacity. Finally, we demonstrate that the STLS approximation scheme exactly reproduces the correct equation of state for the two-dimensional ocp liquid.

I. INTRODUCTION

Over the past decade there has been a growing interest in the two-dimensional plasmas whose interactions are characterized by the logarithmic potential. As a result of theoretical studies of the two-dimensional one-component plasma (ocp), its exact equation of state[1] and thermodynamic limits[2] have been formulated, long-wavelength compressibility-sum rules have been established,[3] and progress has been made in formulating the weak-coupling-limit structure of the equilibrium pair-correlation function g.[4] It is more difficult, however, to obtain a theory which can provide information about the system in the domain where the coupling strength, characterized by the two-dimensional plasma parameter $\gamma=\beta e^2$ (β is the inverse temperature), is not small. To be sure, there is no dearth of strongly coupled plasma approximations[5-8] formulated principally for the physical three-dimensional ocp. In that case, they lead to nonlinear integral equations for the pair-correlation function $g(k)$ which are soluble only by numerical interation. However, none of these approximation schemes has been applied to two-dimensional plasmas. In this paper, we shall undertake such an analysis by examining the two-dimensional version of the approximation scheme proposed (for the three-dimensional ocp) by Singwi, Tosi, Land, and Sjolander (STLS).[5] We shall show that the relevant nonlinear integral equation in g, in contrast to the three-dimensional situation, has the remarkable property that it can be solved exactly for arbitrary values of γ. The primary aim of this paper is the analysis of the exact solution—the first of its kind for any strongly coupled plasma model—and of its implications.

In the STLS scheme, one first calculates the polarizability α from the first Born-Bogoliubov-Green-Kirkwood-Yvon (BBGKY) kinetic equation under the assumption that the proper nonequilibrium part of the two-particle distribution function can be ignored; hence the dielectric function $\epsilon(\vec{k}\omega)$ is expressible as $\epsilon=\epsilon(g)$. Self-consistency is then guaranteed by application of the static fluctuation-dissipation theorem linking ϵ and g. The result is the $\vec{k}$-space nonlinear integral equation

$$G(k)=\frac{-\kappa^2[1+u(k)]}{k^2+\kappa^2[1+u(k)]}\,, \tag{1}$$

where

$$u(k)=\frac{1}{N}\sum_{p}\frac{\vec{k}\cdot\vec{p}}{p^2}G(|\vec{k}-\vec{p}|) \tag{2}$$

is the so-called screening function. The system of volume V consists of N particles immersed in a neutralizing inert background where $n=N/V$ is the density, $G(k)=ng(k)$, and κ is the inverse Debye distance; note the relationship $ng(k)=S(k)-1$ between the pair-correlation function $g(k)$ and the static-structure factor $S(k)$. As we have already pointed out, in the three-dimensional case one must employ numerical-iteration procedures to arrive at an overall solution of Eqs. (1) and (2).[7] For the case of the two-dimensional ocp, however, Eqs. (1) and (2) can be solved easily without resorting to numerical iterations. Of all the strongly coupled plasma approximation schemes[5,6,8] which are nonperturbative in γ, only the STLS approach seems to possess this remarkable feature in two dimensions.

The ocp potential energy is given by

$$U=U_{\text{p-p}}+U_{\text{p-b}}+U_{\text{b-b}}$$
$$=\frac{e^2}{2}\sum_{i\neq j}^{N}\ln\left|\frac{r_{ij}}{R}\right|$$
$$-\frac{Ne^2}{2}\sum_{i=1}^{N}\left(\frac{r_i}{R}\right)^2+\left(\frac{3}{2}+\ln\frac{1}{R}\right)\frac{N^2e^2}{4},$$

where R is the radius of the domain, and p-p, p-b, and b-b refer to particle-particle, particle-background, and background-background interactions. We assume that V and N are large such that n remains finite. The calculations presented in the sequel will then correspond only to p-p interactions, since in a translation-invariant system the p-b contribution has no physical meaning from the point of view of correlation functions, and the b-b contribution is a constant.

This paper is divided into two main parts. The first part (Sec. II) deals with the pair-correlation function: the development of an exact algebraic equation for it (Sec. II A), its behavior at long and short wavelengths (Sec. II B), the analysis of the general-solution curves $G(k;\gamma)$ for arbitrary values of k and γ (Sec. II C), explicit formulas for G when $\gamma=0$, $\gamma=2$, and $\gamma\gg 1$ (Sec. II D), and numerical results (Sec. II E). The second part (Sec. III) deals with thermodynamic properties. Closed algebraic expressions for the correlation energy and heat capacity for arbitrary coupling strength are derived in Sec. III A, and the equation of state[1] is developed in Sec. III B. We shall see that the STLS approximation scheme exactly reproduces the correct equation of state[1] for the two-dimensional ocp. A following paper[9] discusses the configuration-space results of this study.

II. PAIR-CORRELATION FUNCTION

A. Exact k-space solution

We introduce the dimensionless wave vectors $\vec{x}=\vec{k}/\kappa$ and $\vec{y}=\vec{p}/\kappa$ ($\kappa=\sqrt{2\pi n\gamma}$ is the two-dimensional inverse Debye length) and make the change of variables $\vec{z}=\vec{x}-\vec{y}$. In two dimensions, Eqs. (1) and (2) then become

$$G(x)=\frac{1+u(x)}{x^2+1+u(x)}\,, \tag{3}$$

$$u(x)=\gamma\int_0^\infty dz\, zG(z)\frac{1}{2\pi}\int_0^{2\pi}d\phi\,\frac{x^2-xz\cos\phi}{x^2+z^2-2xz\cos\phi}\,, \tag{4}$$

where $\cos\phi=\vec{x}\cdot\vec{z}/xz$. Noting that

$$\frac{1}{2\pi}\int_0^{2\pi}d\phi\,\frac{x^2-xz\cos\phi}{x^2+z^2-2xz\cos\phi}=\frac{1}{2}\left(1+\frac{x^2-z^2}{|x^2-z^2|}\right)$$
$$=\theta(x-z)\,, \tag{5}$$

where θ is the unit step function, Eq. (4) simplifies to

$$u(x)=\gamma\int_0^x dz\,zG(z)\,, \tag{6}$$

whence

$$u'(x)=\gamma xG(x)\,. \tag{7}$$

We note that for $G(x)$ to be physically acceptable, it must tend to zero faster than $1/x^2$ as $x\to\infty$. This implies from Eq. (3) that $u(x\to\infty)=-1$.

The reduction of Eq. (2) to the simple indefinite integral is a salient feature of the STLS approximation scheme for the two-dimensional ocp. It is precisely this feature which renders Eqs. (3) and (4) exactly solvable. To proceed, we combine Eqs. (3) and (7) and obtain the first-order differential equation

$$\frac{G'}{G(1+G)[2+\gamma(1+G)]}=-\frac{1}{x}\,, \tag{8}$$

with the solution

$$\frac{G}{1+G}\left(\frac{\gamma}{2}+\frac{1}{1+G}\right)^{\gamma/2}=\frac{A}{x^{\gamma+2}}\,. \tag{9}$$

The integration constant A is readily determined by observing from Eqs. (3) and (6) that $u(x\to 0)\sim-\frac{1}{2}\gamma x^2$ and $G(x\to 0)\sim -1+x^2$. Taking the latter into account in analyzing Eq. (9) in the $x\to 0$ limit, compatibility requires that $A=-1$, whence

$$\frac{G}{1+G}\left(\frac{\gamma}{2}+\frac{1}{1+G}\right)^{\gamma/2}=\frac{-1}{x^{\gamma+2}}\,. \tag{10}$$

B. Long- and short-wavelength behavior

Explicit formulas for G, valid at small and large values of x, can now be derived. From Eqs. (3) and (6) one obtains the long-wavelength expression

$$G(x\to 0)=-1+x^2-(1-\tfrac{1}{2}\gamma)x^4$$
$$+(1-\tfrac{5}{4}\gamma+\tfrac{1}{4}\gamma^2)x^6-\cdots\,, \tag{11}$$

while at short wavelengths we have from Eq. (10)

$$G(x\to\infty) = -\left(\frac{2}{\gamma+2}\right)^{\gamma/2}\frac{1}{x^{\gamma+2}} + \cdots . \qquad (12)$$

Equation (12) reveals that for $\gamma > 0$, $G(x)$ is indeed normalizable. The $\gamma = 0$ case, i.e., the Debye limit, is known to be invalid for $x\to\infty$ (or $r\to 0$) as a result of the nonuniformity of the γ expansion. As to its small-x behavior, we note from Eq. (11) that $G(x\to 0)|_{\rm STLS}$ does not entirely satisfy the compressibility-sum-rule requirement[3]

$$G(x\to 0) = -1 + x^2 - (1 - \tfrac{1}{4}\gamma)x^4 + \cdots .$$

This defect of the STLS scheme is a problem in three dimensions as well, and therefore should not be a cause for great surprise. We recall that the situation was, however, remedied by Vashishta and Singwi[5] in a later work by introducing into the screening function an *ad hoc* density-derivative correction. This kind of correction might be entirely feasible also in the two-dimensional case—perhaps without seriously impairing the mathematical tractability of the original two-dimensional screening function (6).

An important corollary of Eq. (11) is a statement concerning the behavior of the "screened" effective potential $\Phi(k)$ surrounding a test particle in the system. The static dielectric function $\epsilon(k,0)=\epsilon(x)$ is expressible in terms of $G(x)$ with the aid of the static fluctuation-dissipation theorem. Using Eq. (3), we obtain

$$\epsilon(x) = \frac{1}{1-[1+G(x)]/x^2} = \frac{1+x^2+u(x)}{x^2+u(x)}, \qquad (13)$$

and therefore the effective potential

$$\Phi(k) = \frac{2\pi e^2}{k^2}\,\frac{1}{\epsilon(k0)} \qquad (14)$$

becomes

$$\beta n\Phi(x) = \frac{1}{x^2}\left(1 - \frac{1+G(x)}{x^2}\right) = \frac{1}{x^2}\,\frac{x^2+u(x)}{1+x^2+u(x)} . \qquad (15)$$

Thus at $x=0$, Eq. (11) provides

$$\beta n\Phi(0) = 1 - \tfrac{1}{2}\gamma . \qquad (16)$$

At $\gamma=0$, one deals with the screened Debye potential, and $\beta n\Phi(x=0)=1$ indicates that $\int d\vec{r}\,\Phi(r)$ is finite (while $\int d\vec{r}\ln r$ for the bare potential is infinite) and positive, i.e., has the same sign as the bare potential. This is of course the manifestation of the screened character of the effective potential. We see, however, that at $\gamma=2$ the situation changes. For $\gamma>2$, $\int d\vec{r}\,\Phi(r)<0$, while evidently $\Phi(r=0)>0$. This is possible only if $\Phi(r)$, rather than being screened, assumes an oscillatory character and develops negative domains. This behavior will be discussed in greater detail in Paper II. Returning to Eq. (13), we also note that, in virtue of what is said in Sec. II C about the behavior of $G(x)$, $1/\epsilon(x)$ does not develop a pole for any value of γ, even though the compressibility is negative for $\gamma>2$. This is in contrast to conjectured[10] behavior, where the two phenomena have been proposed to be linked. The error obviously lies in predicting a pole for a finite value of x from the $x\to 0$ behavior.

C. Analysis of $g(k)$ solution curves

As to the nature of the solution for arbitrary values of x and γ, one should bear in mind that, while Eq. (10) possesses more than one root, a root can be a solution only if it satisfies Eq. (3) with u given by Eq. (6); thus not all of the roots of Eq. (10) can survive. Now the behavior of the solution $G(x)$ can be further assessed, first by observing that $G(x)$ must be a continuous function of x. To prove this, assume that $G(x)$ is discontinuous at some point $x=x_0$. This implies from Eq. (3) that $u(x_0)$ must be discontinuous, requiring, in turn, according to Eq. (6), that G exhibit δ-function behavior. However, the structure of Eq. (3) does not provide for such singular behavior (compare the explicit solution (10)]. Having ruled out any discontinuities in G, we next analyze Eq. (8) to determine if a continuous-solution curve $G(x)$ can ever cross the x axis from the $G<0$ region into the $G>0$ region. Supposing that such an excursion can take place, the point x_1 satisfying the equation $G(x_1)=0$ must, of course, lie on the trajectory of $G(x)$. Inspection of Eq. (8) and the higher-order differential equations generated from it reveals that if $G(x_1)=0$, then $G'(x_1)=0$, $G''(x_1)=0$, etc. Hence a continuous-solution curve can never cross the x axis from the $G<0$ region into the $G>0$ region. Then, since all solutions to Eq. (10) are continuous, it follows that they must be negative definite with $G(0)=-1$. This is important from the point of view of possible phase-transition phenomena associated with the occurrence of poles in $G(x)$. In view of the fact that $-1\le G(x)\le 0$ always, it can never develop a pole in the STLS approximation. Evidently, in this approximation the *two dimensional* ocp *remains a fluid for all values of* γ.

D. Solutions for specific γ values

Explicit solutions of Eq. (10) for the algebraically tractable cases $\gamma=0$, $\gamma=2$, and $\gamma\gg 1$ are expected to represent reasonably well the behavior of G over the entire range of γ values. One readily obtains

$$G(x;\gamma=0)=-1/(1+x^2)\,, \tag{17}$$

$$G(x;\gamma=2)=-1+x^2/(1+x^4)^{1/2} \tag{18}$$

We note that

$$\epsilon(x;\gamma=2)=\frac{1}{1-1/(1+x^4)^{1/2}}\,,$$
$$\beta n\Phi(x;\gamma=2)=\frac{1}{x^2}\left(1-\frac{1}{(1+x^4)^{1/2}}\right), \tag{19}$$

and

$$\epsilon(x\to 0;\ \gamma=2)=2/x^4\,,$$

and thus ϵ does not have the usual $1/x^2$ perfect screening structure. This is in accordance with what has been stated above about the change of behavior at $\gamma=2$.

Exact explicit expressions can even be obtained from Eq. (11) for $\gamma=1$, 4, and 6, albeit one encounters cubic or quartic equations for these latter γ values. In the strong-coupling limit ($\gamma\to\infty$), analysis of Eq. (10) gives

$$\frac{G^2}{(1+G)^2}\exp\left(\frac{2}{1+G}\right)=\begin{cases}0, & q=k/\sqrt{\pi n}=\sqrt{2\gamma}\,x>2\\ \infty, & q<2\end{cases} \tag{20}$$

which in turn implies the strong-coupling-approximation formula

$$G(q;\gamma\to\infty)=-\theta(2-q)\,. \tag{21}$$

Note the change of scale in k as one goes from small to large γ values. For γ small, the scale is governed by κ; at large γ, the scale is governed by $(\pi n)^{1/2}$. For this reason, the dimensionless variable $q=k/\sqrt{\pi n}=\sqrt{2\gamma}\,x$ has been introduced in Eq. (20) in favor of x.

E. Numerical results

The algebraic equation (10) has been solved numerically and the solutions are displayed in Figs. 1 and 2. Our numerical results have been obtained by the iteration scheme

$$C=\frac{(2\gamma/q^2)^{1+\gamma/2}}{[\frac{1}{2}\gamma+1/(1+G)]^{\gamma/2}}\,,\quad G=\frac{C}{1-C} \tag{22}$$

for $0<\gamma\leqslant 6.0$ and by a systematic use of the function

$$f=\ln G-2\ln(1+G)+\gamma\left(\frac{2}{1+G}+\gamma\right)+\ln\left[\frac{\gamma^2}{q^4}\left(\frac{2\gamma}{q^2}\right)^{\gamma}\right]=0$$

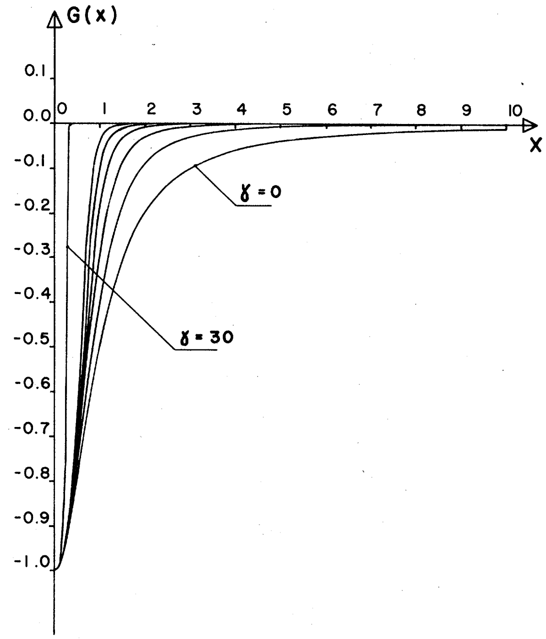

FIG. 1. Fourier-transformed pair-correlation function $G(x)=ng(k/\kappa)$ for $\gamma=0$, 1, 2, 3, 4, 5, and 30. Note that the scale is set by the Debye wave number κ.

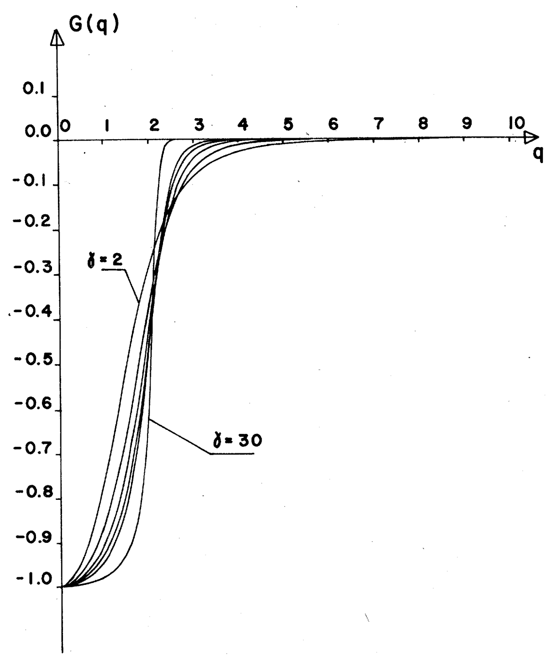

FIG. 2. Fourier-transformed pair-correlation function $G(q)=n\,g(k/\sqrt{\pi n})$ for $\gamma=2$, 4, 6, 8, 10, and 30. Note that the scale is set by the inverse ion-circle radius $\sqrt{\pi n}$.

for $\gamma > 6.0$. The precision obtained is estimated to 10^{-10}. We observe that as γ increases, the function $G(q;\gamma)$ more closely approaches the q axis over most of the region. As soon as $\gamma = 6.0$, the function may be well approximated by the step function $-\theta(2-q)$ consistent with the asymptotic result (21).

III. THERMODYNAMIC PROPERTIES

A. Correlation energy

In this section we first derive a closed-form expression for the correlation energy E_c (per particle) for the two-dimensional ocp in the STLS approximation. Starting with[11]

$$\beta E_c = -\tfrac{1}{2}\gamma n \int d\vec{r}\, g(r) \ln r \equiv -\gamma E(\gamma), \tag{23}$$

and introducing the function

$$H(r) = \int_0^r dr'\, r' g(r')$$

$$= \gamma \int_0^r dr'\, r' \int_0^\infty dx\, x G(x) J_0(x\kappa r')$$

$$= r\left(\frac{\gamma}{2\pi n}\right)^{1/2} \int_0^\infty dx\, G(x) J_1(x\kappa r), \tag{24}$$

$$H(0) = 0, \quad H(\infty) = -1/2\pi n, \tag{25}$$

our task amounts to calculating

$$E(\gamma) = \pi n \int_0^\infty dr\, H'(r) \ln r$$

$$= -\tfrac{1}{2} \lim_{z\to\infty} \left(\ln z + 2\pi n \int_0^z dr\, \frac{H(r)}{r}\right). \tag{26}$$

Upon combining Eqs. (24) and (25), we obtain

$$E(\gamma) = -\tfrac{1}{2} \lim_{z\to\infty} \left(\ln z + \int_0^\infty dx\, \frac{G(x)}{x}\, [1 - J_0(x\kappa z)]\right). \tag{27}$$

Let γ_0 designate $\gamma_0 \ll 1$, $\kappa_0^2 = 2\pi n \gamma_0$. Then by writing Eq. (27) in the convenient form

$$E(\gamma) - E(\gamma_0) = -\tfrac{1}{2} \lim_{z\to\infty} \left[\int_0^\infty dx\, \frac{1}{x}\left(G(x) + \frac{1}{1+x^2}\right) - \int_0^\infty dx\, \frac{J_0(x\kappa z)}{x}\,[G(x)+1] - \int_0^\infty dx\, \frac{J_0(x\kappa_0 z)}{x}\left(\frac{1}{1+x^2} - 1\right) + \int_0^\infty dx\, \frac{1}{x}\,[J_0(x\kappa z) - J_0(x\kappa_0 z)] \right], \tag{28}$$

one can readily see that of the four right-hand-side integrals, only the first and the fourth contribute. We evaluate the first integral by observing from the differential equation (8) that

$$\int_0^\infty dx\, \frac{1}{x}\left(G(x) + \frac{1}{1+x^2}\right) = \frac{1}{2}\left(\int_{-1}^0 \frac{dG}{(1+2/\gamma)+G} - \int_{-1}^0 \frac{dG}{1+G} + \int_{-1}^0 \frac{dG_0}{1+G_0}\right) = \tfrac{1}{2}\ln(1+\tfrac{1}{2}\gamma). \tag{29}$$

Thus,

$$E(\gamma) - E(\gamma_0) = -\tfrac{1}{4}[\ln(1+\tfrac{1}{2}\gamma) + \ln(\gamma_0/\gamma)]. \tag{30}$$

To calculate E_0, substitute into Eq. (23) the two-dimensional Debye-Hückel pair correlation $g_0(r) = -\gamma_0 K_0(\kappa_0 r)$. This gives[4]

$$E(\gamma_0) = -\pi n \gamma_0 \int_0^\infty dr\, r K_0(\kappa_0 r) \ln r$$

$$= \tfrac{1}{4}(2C + \ln \pi n + \ln \tfrac{1}{2}\gamma_0), \tag{31}$$

where $C = 0.57\cdots$ is the Euler constant. We note that in previous work by Deutsch and Lavaud[4] the authors have incorrectly computed the right-hand-side integral of Eq. (31), so that their expression for $E(\gamma_0)$ is incorrect. Upon combining Eqs. (30) and (31), one obtains

$$E(\gamma) = \frac{1}{4}\left(2C + \ln \pi n + \ln \frac{\gamma}{2+\gamma}\right), \tag{32}$$

whence

$$\beta E_c = -\frac{\gamma}{4}\left(2C + \ln \pi n + \ln \frac{\gamma}{\gamma+2}\right). \tag{33}$$

The total heat capacity per particle (at constant volume) can now be easily calculated:

$$c_v = [1 + \gamma/2(\gamma+2)]\, k_B. \tag{34}$$

Equations (33) and (34) are the desired result and are valid for arbitrary values of γ. Equation (34) reveals that c_v/k_B increases monotonically with γ starting from its perfect-gas value of 1 and reaching a maximum value of $\frac{3}{2}$ in the strong-coupling limit. This is to be expected since the $\frac{3}{2}$ value for the heat capacity lies between the perfect-gas heat capacity, 1, and that of a two-dimensional harmonic crystal, 2. Our energy expression (33) is compatible with the recently established lower-bound condition[2]

$$\beta E_c \geq -B = -\tfrac{1}{4}\gamma(\tfrac{3}{2}+\ln\pi n)\,. \tag{35}$$

Finally, note that in the strong-coupling limit, Eqs. (33) and (34) become

$$\beta E_c = -\tfrac{1}{4}\gamma(2C+\ln\pi n - 2/\gamma)\,, \tag{36}$$

$$c_v = \frac{3}{2} - 1/\gamma \quad (\gamma \to \infty)\,. \tag{37}$$

One can recover the very same $O(\gamma)$ terms in Eq. (36) by substituting the strong-coupling step-function approximation (21) for $G(x)$ into Eq. (27).

B. Equation of state

We close this section by demonstrating that the STLS approximation scheme reproduces the correct two-dimensional ocp equation of state.[1] Its derivation from Eq. (29) is straightforward. First calculate the free energy $F(\gamma)$:

$$\begin{aligned} F(\gamma) &= F_0 + \frac{1}{\gamma}\int_0^\gamma d\gamma' N E_c(\gamma') \\ &= F_0 - \frac{Ne^2}{4}\bigg\{1 + 2C + \ln\pi N - \ln V \\ &\qquad + \ln\bigg[\frac{\gamma}{\gamma+2}\left(\frac{2}{\gamma+2}\right)^{2/\gamma}\bigg]\bigg\}\,. \end{aligned} \tag{38}$$

The calculation of the pressure $P = -[\partial F(\gamma)/\partial V]_\beta$ immediately results in the equation of state

$$\beta P/n = 1 - \tfrac{1}{4}\gamma\,. \tag{39}$$

Equation (39) valid for arbitrary values of γ, is exact.[1] Its linear structure is a salient feature of two-dimensional plasmas.

IV. CONCLUSIONS

In this paper we have solved exactly the STLS nonlinear integral equation for the Fourier transform of the two-dimensional ocp equilibrium pair-correlation function $G(k)$. Our analysis of the integral equation has revealed that $-1 \leq G(k;\gamma) \leq 0$ always; the absence of singularities implies that the two-dimensional ocp remains a fluid for all values of γ in the STLS approximation. Explicit expressions for $G(k)$ in Eqs. (17), (18), and (21) were obtained from Eq. (10) for $\gamma = 0$, $\gamma = 2$, and $\gamma \gg 1$. These, together with supplementary numerical solutions (Figs. 1 and 2), provide information about $G(k;\gamma)$ over a wide range of k and γ values. At long wavelengths, $G_{\rm STLS}\,(k\to 0;\gamma)$ fails to satisfy the compressibility sum rule, a defect inherent in the origin STLS approach in three dimensions as well.

Our exact results for the pair-correlation function enabled us to arrive successfully at closed formulas for the correlation energy and specific heat at constant volume, Eqs. (33) and (34). This latter equation reveals that c_v/k_B increases monotonically with γ starting from its perfect-gas value of 1 and reaching a maximum value of $\frac{3}{2}$ in the strong-coupling limit. Finally, we have demonstrated that the STLS approach exactly reproduces the equation of state for the two-dimensional ocp.

This is the first time that an exact solution for a reasonable approximation scheme has been obtained for arbitrary values of γ. The results derived in this paper, together with those pertaining to the configuration-space representation in the subsequent paper, constitute a valuable testing ground for many strongly coupled two-dimensional ocp conjunctures and approximations. It is especially interesting to see how the smooth analytic behavior of all the thermodynamic quantities as functions of γ, and also the lack of any abrupt change of behavior for the pair-correlation function at any particular value of γ, are compatible with a change of some important characteristics of the system at $\gamma = 2$.

Finally, one can speculate whether the lack of a phase transition, i.e., the lack of real poles in $1/\epsilon$, is a real feature of the two-dimensional ocp or merely an aberration of the two-dimensional STLS model. Although the question of the possibility of a phase transition in a two-dimensional Coulomb system is still, to a great extent, unresolved,[2] we know from our preliminary investigation of the two-dimensional version of the more sophisticated Totsuji-Ichimaru[6] strongly coupled plasma scheme that in this latter case the pair-correlation function $G(k)$ does not share the negative-definite property with its STLS counterpart, and indeed seems to allow for a phase transition for some critical value of γ ($\gg 4$). Thus it appears to be a fair inference to attribute the lack of a phase transition to the model rather than to the physical system. Nevertheless, it should be kept in mind that the inverse compressibility in the model studied becomes 0 at $\gamma = 2$ and negative for $\gamma > 2$. The corresponding change in the physical nature of the system, even though it is not normally classified as a phase transition, is quite dramatic; it should also be noted that a similar change of behavior has been observed in three-dimensional physical systems[12] as the ocp compressibility becomes negative prior to liquid-solid phase transition. Thus, in this respect, the STLS model of the two-dimensional ocp appears to faithfully represent actual physical systems; so does, presumably, the STLS model in three dimensions. Whether, on the other hand, the STLS scheme in three dimensions can lead to an actual phase transition (corresponding to real poles in $1/\epsilon$) cannot be decided on the basis of the scant numerical data available.[7]

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[11]The potential $-e^2\ln r$ is obviously indeterminate to an arbitrary constant, say $e^2\ln R$, which also fixes the scale of r. As long as R is independent of both the volume and the temperature of the system, a convention we adopt here, its choice has no bearing on the calculation of thermodynamic quantities, and it will not be explicitly displayed in this section. For some further comments on this question, the reader might consult Ref. 11 of Paper II.

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