PHYSICAL REVIEW A VOLUME 20, NUMBER 1 JULY 1979

Generalized low-frequency approximation for scattering in a laser field. II

Leonard Rosenberg*
Joint Institute for Laboratory Astrophysics, University of Colorado, and National Bureau of Standards, Boulder, Colorado 80309
(Received 12 February 1979)

A time-independent formulation of the problem of scattering in an intense radiation field is used as the basis for a derivation of a low-frequency approximation in which field-free scattering amplitudes enter into the determination of transition amplitudes in the presence of the field. A single-mode field of arbitrary polarization is assumed and the scatterer is represented by a local, short-range potential. In the present derivation the dipole approximation for the field is not assumed. As a result, recoil corrections to earlier versions of the low-frequency approximation are obtained in explicit form. The low-frequency approximation for bremsstrahlung in the absence of a background field, derived some time ago by Low and others, appears as a limiting case of the result obtained here.

I. INTRODUCTION

In a previous paper[1] (referred to in the following as I) a time-independent formulation of the problem of scattering in the presence of an intense radiation field was applied to the derivation of a low-frequency approximation. The result obtained in I provides a generalization to arbitrary polarization of the low-frequency approximation for a linearly polarized field derived (using time-dependent methods) by Kroll and Watson.[2] These earlier results were obtained within the dipole approximation. This amounts to ignoring recoil corrections, of the order v/c, which are expected to be small in many of the situations of interest.[3] Nevertheless, it seems desirable to determine these recoil corrections explicitly, not only for the purpose of obtaining accurate numerical estimates of these effects but also as a way of sharpening our calculational tools. Here, by following the approach taken in I but without imposing the dipole approximation, we obtain a generalized version of the low-frequency approximation which reduces to the form obtained earlier in the static limit. We retain the assumptions of a single-mode field and a local, short-range potential.

It is helpful to keep in mind the close analogy which exists between these intense-field results and the original low-frequency theorems for bremsstrahlung in the absence of an external field. The first result of this latter type was derived by Low using the Lippmann-Schwinger equation.[4] Subsequently, Low's nonrelativistic result was rederived with recoil corrections included.[5,6] As will be seen explicitly, these earlier versions are recovered in the weak-coupling limit of the low-frequency approximation derived below. This is to be expected, since the particle-field interaction is treated here nonperturbatively; by expanding the transition amplitude in powers of the electronic charge we recover, as the leading term, the result of first-order perturbation theory.

Some properties of the asymptotic states of the particle-field system and of the field-free scattering amplitude which will be required are collected in Sec. II. With these properties in hand the actual derivation, given in Sec. III, is quite straightforward. I conclude by pointing out how the limiting cases mentioned above can be recovered from the generalized version derived here.

II. SOME USEFUL IDENTITIES

A. Particle-field solutions

The asymptotic states appropriate to scattering in a laser field, which play a role analogous to the plane waves of field-free scattering, satisfy the Schrödinger equation

$$\left(\frac{(\vec{p}-e\vec{A}/c)^2}{2\mu}+\hbar\omega a^\dagger a\right)|\psi_{n\vec{p}}\rangle = E_{n\vec{p}}|\psi_{n\vec{p}}\rangle . \tag{2.1}$$

The vector potential for the single-mode field considered here is

$$\vec{A}=(2\pi\hbar c^2/\omega L^3)^{1/2}(a\vec{\lambda}e^{i\vec{k}\cdot\vec{r}}+a^\dagger\vec{\lambda}^*e^{-i\vec{k}\cdot\vec{r}}) . \tag{2.2}$$

Here $\omega=kc$ and L^3 is the quantization volume. The polarization vector is represented as

$$\vec{\lambda}=\hat{e}_1\cos\chi+i\hat{e}_2\sin\chi ; \tag{2.3}$$

$\hat{e}_1$ and $\hat{e}_2$ are orthonormal and orthogonal to $\vec{k}$. It follows that $\vec{k}\cdot\vec{\lambda}=0$, $\vec{\lambda}\cdot\vec{\lambda}^*=1$, and $\vec{\lambda}\cdot\vec{\lambda}=\cos2\chi$. The state $|\psi_{n\vec{p}}\rangle$ is that which evolves from the unperturbed state $|n;\vec{p}\rangle$ (corresponding to the field containing n photons, the electron having momentum $\vec{p}$) as the field is switched on adiabatically. The solution of Eq. (2.1) was discussed in I within the approximations that photon depletion effects may be ignored and corrections of order $\hbar\omega/\mu c^2$

are negligible. Here I record those properties of the solution required in the following.

The energy eigenvalue is

$$E_{n\vec{p}} = p^2/2\mu + n\hbar\omega + \Delta \,, \tag{2.4}$$

with the continuum level shift given by

$$\Delta = (e^2/2\mu c^2)(4\pi n\hbar c^2/\omega L^3) \,. \tag{2.5}$$

The eigenstates may be expanded in terms of the unperturbed states as

$$|\psi_{n\vec{p}}\rangle = \sum_{m=-\infty}^{\infty} \gamma_m |n+m; \vec{p} - m\hbar\vec{k}\rangle \,; \tag{2.6}$$

the expansion coefficients γ_m are defined by the integral representation

$$\gamma_m(\hat{\rho}, \alpha) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \exp[im\phi + i\rho\sin(\theta+\phi) + i\alpha\sin 2\phi] \,. \tag{2.7}$$

Here I make use of the following notation: I introduce

$$\eta = \hbar\omega - \vec{p}\cdot\hbar\vec{k}/\mu \,, \tag{2.8}$$

in terms of which the complex number $\hat{\rho}$ is defined as

$$\tfrac{1}{2}\eta\hat{\rho} = (2\pi\hbar e^2 n/\mu^2\omega L^3)^{1/2}(\vec{p}\cdot\vec{\lambda}) \,; \tag{2.9}$$

ρ and θ represent the magnitude and phase, respectively, of $\hat{\rho}$. We also have

$$\alpha = \Delta\cos 2\chi/2\eta \,. \tag{2.10}$$

The essential properties of the expansion coefficients γ_m are given as follows.

(i) Recursion relation: According to Eq. (2.7) we may write

$$2m\gamma_m = (i\pi)^{-1}\int_{-\pi}^{\pi} d(e^{im\phi})\exp[i\rho\sin(\theta+\phi) + i\alpha\sin 2\phi] \,. \tag{2.11}$$

Integrating by parts, we immediately obtain

$$2m\gamma_m + \rho[e^{i\theta}\gamma_{m+1} + e^{-i\theta}\gamma_{m-1}] + 2\alpha[\gamma_{m+2} + \gamma_{m-2}] = 0 \,. \tag{2.12}$$

(ii) Addition formula:

$$\sum_{l=-\infty}^{\infty} \gamma^*_{l-n'}(\hat{\rho}', \alpha')\gamma_{l-n}(\hat{\rho}, \alpha) = \gamma_{n'-n}(\hat{\rho} - \hat{\rho}', \alpha - \alpha') \,. \tag{2.13}$$

Here the primes on $\hat{\rho}$ and α indicate the appearance of the momentum $\vec{p}'$ rather than $\vec{p}$ in the definitions (2.9) and (2.10). Equation (2.13) is readily verified by using the integral representation to evaluate the left-hand side. The sum over l can then be performed and the complex numbers $\hat{\rho}$ and $\hat{\rho}'$ combined by vector addition. The formula (2.13) represents a generalization of the well-known Graf addition formula for Bessel functions,[7] to which it reduces when α and α' are set equal to zero. We have, in fact,

$$\gamma_m(\hat{\rho}, 0) = e^{-im\theta}J_{-m}(\rho) \,. \tag{2.14}$$

This limiting case is appropriate for a circularly polarized field [$\chi = \frac{1}{4}\pi$ in Eq. (2.10)] or when the dipole approximation is used. In either case the A^2 term in the particle-field interaction plays no significant role.

B. Field-free scattering

The amplitude for scattering by a local, short-range potential V in the absence of an external field can be obtained by solving the Lippmann-Schwinger integral equation

$$t(e; \vec{p}', \vec{p}) = V(\vec{p}' - \vec{p}) + \int d^3p'' V(\vec{p}' - \vec{p}'') \frac{1}{e - p''^2/2\mu} t(e; \vec{p}'', \vec{p}) \,. \tag{2.15}$$

Here $V(\vec{p}' - \vec{p}) \equiv \langle \vec{p}' | V | \vec{p}\rangle$ and the usual limiting procedure in which the energy e is given a small positive imaginary part is to be understood. The physical "on-shell" amplitude is obtained by setting $p'^2/2\mu = p^2/2\mu = e$. When the scattering takes place within a larger system one generally requires a knowledge of the off-shell amplitude. The striking feature of the low-frequency approximations derived previously, as well as that obtained below, is that the amplitude need only be known on the energy shell. In the course of our derivation it will be necessary to evaluate a quantity of the form

$$\int d^3p'' t(p^2/2\mu; \vec{p}', \vec{p}'') \times \frac{\mathcal{E} + \vec{\mathfrak{v}}\cdot\vec{p}''}{(p^2/2\mu - p''^2/2\mu)^2} t(p^2/2\mu; \vec{p}'', \vec{p}) \,,$$

where p' and p differ by a first-order quantity, in terms of the amplitude t and certain derivatives of that amplitude, correct to first order in the small quantities $\mathcal{E}$ and $\vec{\mathfrak{v}}$. The following identities are useful in this connection.

(iii) Energy increment: It follows directly from the integral equation (2.15) that, to first order in $\mathcal{E}$,

$$t(p^2/2\mu + \mathcal{E}; \vec{p}', \vec{p}) - t(p^2/2\mu; \vec{p}', \vec{p}) = -\int d^3p'' t(p^2/2\mu; \vec{p}', \vec{p}'') \times \frac{\mathcal{E}}{(p^2/2\mu - p''^2/2\mu)^2} t(p^2/2\mu; \vec{p}'', \vec{p}) \,. \tag{2.16}$$

It will be convenient to express the left-hand side of Eq. (2.16) in terms of certain scalar variables employed by Heller.[6] Thus, suppose we express the amplitude $t(e;\vec{q}',\vec{q})$, for arbitrary values of the energy and momentum variables, as $t(\nu,\tau,\xi,\xi')$, where

$$\nu=\tfrac{1}{2}(q^2/2\mu+q'^2/2\mu),\quad \tau=(\vec{q}'-\vec{q})^2,$$
$$\xi=e-q^2/2\mu,\quad \xi'=e-q'^2/2\mu. \tag{2.17}$$

We expand the amplitudes on the left-hand side of Eq. (2.16) about their values for $\nu=\bar{\nu}\equiv\frac{1}{2}(p^2/2\mu+p'^2/2\mu)$, $\tau=\bar{\tau}\equiv(\vec{p}'-\vec{p})^2$, $\xi=0$, $\xi'=0$. This leads to an alternative version of Eq. (2.16), somewhat more convenient for our later purposes, of the form

$$\int d^3p''\,t(p^2/2\mu;\vec{p}',\vec{p}'')\frac{\mathcal{E}}{(p^2/2\mu-p''^2/2\mu)^2}\times t(p^2/2\mu;\vec{p}'',\vec{p})=-\mathcal{E}\left(\frac{\partial t}{\partial\xi}+\frac{\partial t}{\partial\xi'}\right); \tag{2.18}$$

the derivatives are evaluated at $\{\bar{\nu},\bar{\tau},0,0\}$.

(iv) Momentum increment: It follows from Eq. (2.15) that, to first order in $\vec{\mathfrak{v}}$,

$$t(p^2/2\mu;\vec{p}'+\mu\vec{\mathfrak{v}},\vec{p}+\mu\vec{\mathfrak{v}})-t(p^2/2\mu,\vec{p}',\vec{p})$$
$$=\int d^3p''\,t(p^2/2\mu;\vec{p}',\vec{p}'')\times\frac{\vec{\mathfrak{v}}\cdot\vec{p}''}{(p^2/2\mu-p''^2/2\mu)^2}\,t(p^2/2\mu;\vec{p}'',\vec{p}). \tag{2.19}$$

The derivation appears in Ref. 6. [The result can also be obtained by combining Eq. (2.16) with Eq. (3.37) of I.] By expansion of the left-hand side of Eq. (2.19) we obtain the equivalent form

$$\int d^3p''\,t(p^2/2\mu;\vec{p}',\vec{p}'')\times\frac{\vec{\mathfrak{v}}\cdot\vec{p}''}{(p^2/2\mu-p''^2/2\mu)^2}\,t(p^2/2\mu;\vec{p}'',\vec{p})$$
$$=\vec{\mathfrak{v}}\cdot\left(\frac{(\vec{p}+\vec{p}')}{2}\frac{\partial t}{\partial\nu}-\vec{p}\frac{\partial t}{\partial\xi}-\vec{p}'\frac{\partial t}{\partial\xi'}\right). \tag{2.20}$$

Note that $t(\bar{\nu},\bar{\tau},0,0)$ and $\partial t/\partial\nu$ can be determined from a knowledge of the on-shell scattering amplitude over a small range of energies and angles. On the other hand, one must go off the energy shell to determine $\partial t/\partial\xi$ and $\partial t/\partial\xi'$.

III. LOW-FREQUENCY APPROXIMATION

The amplitude for scattering from an initial unperturbed state $|n;\vec{p}\rangle$ to a final unperturbed state $|n';\vec{p}'\rangle$, in the presence of the radiation field, is given by[8]

$$T_{n'\vec{p}';n\vec{p}}=\langle\psi_{n'\vec{p}'}|T(E_{n\vec{p}})|\psi_{n\vec{p}}\rangle, \tag{3.1}$$

where the scattering operator T satisfies

$$T(E_{n\vec{p}})=V+VG_0(E_{n\vec{p}})T(E_{n\vec{p}}). \tag{3.2}$$

The particle-field propagator is represented by the eigenfunction expansion

$$G_0(E_{n\vec{p}})=\sum_{n''}\int d^3p''\frac{|\psi_{n''\vec{p}''}\rangle\langle\psi_{n''\vec{p}''}|}{E_{n\vec{p}}-E_{n''\vec{p}''}}. \tag{3.3}$$

Energy conservation requires that

$$p'^2/2\mu+n'\hbar\omega=p^2/2\mu+n\hbar\omega. \tag{3.4}$$

Our starting point is a low-frequency approximation for G_0. When the expansion (2.6) is inserted for the particle-field states Eq. (3.3) becomes

$$G_0(E_{n\vec{p}})=\sum_{n''}\int d^3p''\sum_{l}\sum_{l'}|\vec{p}''-(l'-n'')\hbar\vec{k}\rangle|l'\rangle\left(\frac{p^2}{2\mu}+n\hbar\omega-\frac{p''^2}{2\mu}-n''\hbar\omega\right)^{-1}\gamma_{l'-n''}\gamma^*_{l-n''}\langle l|\langle\vec{p}''-(l-n'')\hbar\vec{k}|. \tag{3.5}$$

Suppose that for each value of n'' the integration variable $\vec{p}''$ is shifted to $\vec{p}''-n''\hbar\vec{k}$. This has the effect of removing the index n'' from the momentum states in Eq. (3.5). The coefficients $\gamma_{l'-n''}$ and $\gamma^*_{l-n''}$ are unchanged. This can be verified by observing the effect on the variables defined in Eqs. (2.8)–(2.10) resulting from a shift in the momentum $\vec{p}$ by a multiple of $\hbar\vec{k}$. In Eq. (2.8) we may thus neglect corrections of order $(\hbar\vec{k})^2/\mu=\hbar\omega(\hbar\omega/\mu c^2)$ in the nonrelativistic limit considered here. Furthermore, $\hat{\rho}$ in Eq. (2.9) is unaffected by such a shift since $\vec{k}\cdot\vec{\lambda}=0$. The shifted value of the energy denominator in Eq. (3.5) is, with $\eta''\equiv\hbar\omega-\vec{p}''\cdot\hbar\vec{k}/\mu$,

$$D=p^2/2\mu-p''^2/2\mu+n\hbar\omega-n''\hbar\eta''. \tag{3.6}$$

We introduce the low-frequency approximation by writing

$$D^{-1}\cong\frac{1}{p^2/2\mu-p''^2/2\mu}+\frac{n''\hbar\eta''-n\hbar\omega}{(p^2/2\mu-p''^2/2\mu)^2}, \tag{3.7}$$

neglecting terms of order $(\hbar\omega)^2$. If we retain only the first term on the right-hand side of Eq. (3.7) we are left with the approximate Green's function

$$\bar{G}_0=\sum_l\int d^3p''\,|l\rangle|\vec{p}''-l\hbar\vec{k}\rangle\times\frac{1}{p^2/2\mu-p''^2/2\mu}\langle\vec{p}''-l\hbar\vec{k}|\langle l|. \tag{3.8}$$

The summation over n'' has been performed using the addition formula (2.13) in the form

$$\sum_{n''}\gamma^*_{l-n''}(\hat\rho'',\alpha'')\gamma_{l'-n''}(\hat\rho'',\alpha'')=\gamma_{l'-l}(0,0)$$
$$=\delta_{ll'}, \qquad (3.9)$$

the second equality following directly from the integral representation (2.7). The scattering operator associated with $\overline{G}_0$ is defined by

$$\overline{T}=V+V\overline{G}_0\overline{T}. \qquad (3.10)$$

From the momentum-space representation of Eq. (3.10) we find, by comparison with Eq. (2.15),

$$\langle l'|\langle\vec{p}'|\overline{T}|\vec{p}\rangle|l\rangle$$
$$=\delta_{l'l}t(p^2/2\mu;\vec{p}'+l\hbar\vec{k},\vec{p}+l\hbar\vec{k}). \qquad (3.11)$$

This result enables us to evaluate $\overline{T}_{n'\vec{p}';n\vec{p}}\equiv\langle\psi_{n'\vec{p}'}|\overline{T}|\psi_{n\vec{p}}\rangle$. Using Eq. (2.13) to carry out the summation over photon states, we obtain

$$\overline{T}_{n'\vec{p}';n\vec{p}}=\gamma_{n'-n}(\hat\rho-\hat\rho',\alpha-\alpha')$$
$$\times t(p^2/2\mu;\vec{p}'+n'\hbar\vec{k},\vec{p}+n\hbar\vec{k}). \qquad (3.12)$$

With the t matrix expanded in terms of the scalar variables introduced in Sec. II, and with terms of order $(\hbar\omega)^2$ dropped, Eq. (3.12) becomes

$$\overline{T}_{n'\vec{p}';n\vec{p}}=\gamma_{n'-n}(\hat\rho-\hat\rho',\alpha-\alpha')$$
$$\times\left[t+\tfrac{1}{2}\left(\frac{n'\vec{p}'\cdot\hbar\vec{k}}{\mu}+\frac{n\vec{p}\cdot\hbar\vec{k}}{\mu}\right)\frac{\partial t}{\partial\nu}-\frac{n\vec{p}\cdot\hbar\vec{k}}{\mu}\frac{\partial t}{\partial\xi}\right.$$
$$\left.+\left((n'-n)\hbar\omega-\frac{n'\vec{p}'\cdot\hbar\vec{k}}{\mu}\right)\frac{\partial t}{\partial\xi'}\right]. \qquad (3.13)$$

Here and in the following the t matrix and its derivatives are evaluated with the scalar variables taking on the values $\nu=\frac{1}{2}(p^2/2\mu+p'^2/2\mu)$, $\tau=(\vec{p}'+n'\hbar\vec{k}-\vec{p}-n\hbar\vec{k})^2$, $\xi=0$, $\xi'=0$.

When the particle-field Green's function is corrected by inclusion of the second term on the right-hand side of Eq. (3.7) we are led, with the aid of an analog of Eq. (2.16), to a correction term C to be added on to $\overline{T}$. It takes the form

$$C_{n'\vec{p}';n\vec{p}}=\int d^3p''\,t(p^2/2\mu;\vec{p}',\vec{p}'')$$
$$\times(p^2/2\mu-p''^2/2\mu)^{-2}t(p^2/2\mu;\vec{p}'',\vec{p})\Gamma, \qquad (3.14)$$

with

$$\Gamma=\sum_{n''}(n''\eta''-n\hbar\omega)\gamma_{n'-n''}(\hat\rho''-\hat\rho',\alpha''-\alpha')$$
$$\times\gamma_{n''-n}(\hat\rho-\hat\rho'',\alpha-\alpha''). \qquad (3.15)$$

Here we have used Eq. (3.12), but with momentum shifts of order $\hbar\vec{k}$ dropped in the argument of the t matrix since Γ is itself of order $\hbar\omega$. In order to carry out the summation in Eq. (3.15) we write

$$n''\eta''-n\hbar\omega=(n''-n)\eta''-n\vec{p}''\cdot\hbar\vec{k}/\mu, \qquad (3.16)$$

and consider the two terms separately. Since the second term is independent of n'', the summation can be performed using the addition formula after first making the replacement

$$\gamma_{n'-n''}(\hat\rho''-\hat\rho',\alpha''-\alpha')$$
$$=\gamma^*_{n''-n'}(\hat\rho'-\hat\rho'',\alpha'-\alpha'') \qquad (3.17)$$

in Eq. (3.15); this can be justified from the representation (2.7). Taking into account the first term in Eq. (3.16), we use the recursion relation (2.12) in the form

$$(n''-n)\gamma_{n''-n}$$
$$=-[\tfrac{1}{2}(\hat\rho-\hat\rho'')\gamma_{n''-n+1}+\tfrac{1}{2}(\hat\rho-\hat\rho'')^*\gamma_{n''-n-1}$$
$$+(\alpha-\alpha'')(\gamma_{n''-n+2}+\gamma_{n''-n-2})]. \qquad (3.18)$$

According to the definition (2.9) we have

$$\tfrac{1}{2}(\hat\rho-\hat\rho'')=\left(\frac{2\pi\hbar e^2n}{\mu^2\omega L^3}\right)^{1/2}\left(\frac{\vec{p}}{\eta}-\frac{\vec{p}''}{\eta''}\right)\cdot\vec\lambda. \qquad (3.19)$$

In the following we use the approximation

$$\eta''/\eta\cong1+\vec{p}\cdot\hat{k}/\mu c-\vec{p}''\cdot\hat{k}/\mu c, \qquad (3.20)$$

valid in the nonrelativistic limit. Equation (3.15) can now be put in the form

$$\Gamma=\mathcal{E}+\vec{\upsilon}\cdot\vec{p}'', \qquad (3.21)$$

where

$$\mathcal{E}=-\{(2\pi\hbar e^2n/\mu^2\omega L^3)^{1/2}$$
$$\times[(1+\vec{p}\cdot\hat{k}/\mu c)(\vec{p}\cdot\vec\lambda\gamma_{n'-n+1}+\vec{p}\cdot\vec\lambda^*\gamma_{n'-n-1})]$$
$$+\tfrac{1}{2}\Delta\cos2\chi(\vec{p}\cdot\hat{k}/\mu c)(\gamma_{n'-n+2}+\gamma_{n'-n-2})\} \qquad (3.22)$$

and

$$\vec{\upsilon}=-(n\hbar\vec{k}/\mu)\gamma_{n'-n}+(2\pi\hbar e^2n/\mu^2\omega L^3)^{1/2}[\hat{k}/\mu c(\vec{p}\cdot\vec\lambda\gamma_{n'-n+1}+\vec{p}\cdot\vec\lambda^*\gamma_{n'-n-1})+\vec\lambda\gamma_{n'-n+1}+\vec\lambda^*\gamma_{n'-n-1}]$$
$$+\tfrac{1}{2}\Delta\cos2\chi(\hat{k}/\mu c)(\gamma_{n'-n+2}+\gamma_{n'-n-2}). \qquad (3.23)$$

Here and in the following the arguments of the γ coefficients are understood to be $\{\rho - \rho', \alpha - \alpha'\}$. With the aid of Eqs. (2.18) and (2.20) the correction term in Eq. (3.14) is evaluated as

$$C_{n'\vec{p}';n\vec{p}} = \vec{\upsilon}\cdot\frac{\vec{p}+\vec{p}'}{2}\frac{\partial t}{\partial \nu} - (\mathcal{E}+\vec{p}\cdot\vec{\upsilon})\frac{\partial t}{\partial \xi} - (\mathcal{E}+\vec{p}'\cdot\vec{\upsilon})\frac{\partial t}{\partial \xi'}\,. \tag{3.24}$$

Combining Eqs. (3.13) and (3.24), we find, after some simple algebra, the low-frequency approximation

$$T_{n'\vec{p}';n\vec{p}} \cong \gamma_{n'-n}\, t + \frac{1}{2}\frac{\partial t}{\partial \nu}\left\{\left(\frac{2\pi\hbar e^2 n}{\mu^2\omega L^3}\right)^{1/2}\left[\left(\vec{p}\cdot\vec{\lambda}\frac{\hbar\omega}{\eta} + \vec{p}'\cdot\vec{\lambda}\frac{\hbar\omega}{\eta'}\right)\gamma_{n'-n+1} + \left(\vec{p}\cdot\vec{\lambda}^*\frac{\hbar\omega}{\eta} + \vec{p}'\cdot\vec{\lambda}^*\frac{\hbar\omega}{\eta'}\right)\gamma_{n'-n-1}\right] + \tfrac{1}{2}\Delta\cos 2\chi[(\vec{p}+\vec{p}')\cdot\hat{k}/\mu c](\gamma_{n'-n+2}+\gamma_{n'-n-2})\right\}. \tag{3.25}$$

Note that the off-shell derivatives $\partial t/\partial\xi$ and $\partial t/\partial\xi'$ have cancelled in this final result.

I conclude by commenting on several limiting cases obtained from Eq. (3.25). First, considering the static limit, in which terms of order $\vec{p}/\mu c$ and $\vec{p}'/\mu c$ are dropped, we find

$$T_{n'\vec{p}';n,\vec{p}} \cong \gamma_{n'-n} t - \vec{X}\cdot\left(\frac{\vec{p}+\vec{p}'}{2}\right)\frac{\partial t}{\partial\nu}\,, \tag{3.26a}$$

with

$$\vec{X} = -\left(\frac{2\pi\hbar e^2 n}{\mu^2\omega L^3}\right)^{1/2}(\vec{\lambda}\gamma_{n'-n+1} + \vec{\lambda}^*\gamma_{n'-n-1})\,. \tag{3.26b}$$

Since $\alpha - \alpha'$ vanishes in this limit, each of the γ coefficients in Eq. (3.26) is a phase factor multiplying a Bessel function, as shown in Eq. (2.14). Equation (3.26) is in agreement with the result which was obtained in I within the dipole approximation. If we further specialize to the case of linear polarization we find, using the recursion relation (2.12) with α and θ set equal to zero,

$$\vec{X} = [(n'-n)\hbar\omega/(\vec{p}-\vec{p}')\cdot\vec{\lambda}]\gamma_{n'-n}\vec{\lambda}\,, \tag{3.27}$$

so that

$$T_{n'\vec{p}';n\vec{p}} \cong J_{n-n'}(\rho)\times\left(t - \frac{(n'-n)\hbar\omega}{(\vec{p}-\vec{p}')\cdot\vec{\lambda}}\frac{(\vec{p}+\vec{p}')}{2}\cdot\vec{\lambda}\frac{\partial t}{\partial\nu}\right). \tag{3.28}$$

This is equivalent to the result of Kroll and Watson.[2]

Returning now to Eq. (3.25), we consider the weak-coupling limit in which a single photon is created out of the vacuum (the standard bremsstrahlung process). Working to first order in the electronic charge, and using the small argument approximation for the Bessel function, we find

$$T_{1\vec{p}';0\vec{p}} \cong \frac{|e|}{\mu}\left(\frac{2\pi\hbar}{\omega L^3}\right)^{1/2}\vec{\lambda}^*\cdot\left[\left(\frac{\vec{p}'}{\eta'} - \frac{\vec{p}}{\eta}\right)t + \frac{1}{2}\left(\vec{p}\frac{\hbar\omega}{\eta} + \vec{p}'\frac{\hbar\omega}{\eta'}\right)\frac{\partial t}{\partial\nu}\right], \tag{3.29}$$

in agreement with Heller's result.[6] The static limit of Eq. (3.29) reproduces the form originally obtained by Low.

The presence of the A^2 term in the particle-field interaction is reflected in the form of the γ coefficients and in the appearance of the last term in Eq. (3.25), the term proportional to Δ. To see this more clearly, suppose we drop the $\vec{p}\cdot\vec{A}$ contribution and take $H' = (e^2/2\mu c^2)A^2$ as the interaction. Consider now the bremsstrahlung amplitude for two-photon creation from the vacuum, to lowest order in H'. The low-frequency approximation for this process can be determined directly by using a simple modification of the methods of Ref. 6. We find the amplitude to be

$$T_{2\vec{p}';0\vec{p}} \cong \frac{e^2}{\mu}\left(\frac{\pi\hbar}{\omega L^3}\right)\cos 2\chi \times\left[\left(\frac{1}{\eta'} - \frac{1}{\eta}\right)t + \frac{(\vec{p}+\vec{p}')\cdot\hbar\vec{k}}{\hbar\omega\mu}\frac{\partial t}{\partial\nu}\right]. \tag{3.30}$$

This is precisely the form obtained from the weak-coupling limit of Eq. (3.25), after dropping terms arising from the $\vec{p}\cdot\vec{A}$ contribution to the interaction. In arriving at this result we use $\gamma_0(0,\alpha) \cong 1$ and $\gamma_2(0,\alpha) \cong -\frac{1}{2}\alpha$, valid to lowest order in α; these approximations are readily obtained from the integral representation (2.7).

ACKNOWLEDGMENTS

This work was supported in part by the NSF under Grant No. PHY-77-10131 and by the Office of Naval Research under Contract No. N00014-76-C-0317. The hospitality of JILA personnel is gratefully acknowledged.

*1978–79 JILA Visiting Fellow. Permanent address: Dept. of Physics, New York University, New York, N. Y. 10003.

[1]E. J. Kelsey and L. Rosenberg, Phys. Rev. A 19, 756 (1979).

[2]N. M. Kroll and K. M. Watson, Phys. Rev. A 8, 804 (1973).

[3]See, for example, A. Weingartshofer, J. Holmes, G. Caudle, E. Clarke, and H. Krüger, Phys. Rev. Lett. 39, 269 (1977).

[4]The result is quoted in F. E. Low, Phys. Rev. 110, 974 (1958).

[5]H. Feshbach and D. R. Yennie, Nucl. Phys. 37, 150 (1962).

[6]L. Heller, Phys. Rev. 174, 1580 (1968).

[7]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (U. S. GPO, Washington, D. C. 1964), Nat. Bur. Stand. (U.S.) Appl. Math. Ser. 55, p. 363.

[8]L. Rosenberg, Phys. Rev. A 16, 1941 (1977).