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Quantum-mechanical parametric amplification

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Parametric amplification may be obtained in a harmonic oscillator if its spring constant is replaced by a time-dependent function. It is shown that the quantum transition probabilities may be expressed in a simple closed analytic form which involves only the classical amplification factor of the oscillator. The authors also compute the time evolution of the density matrix which initially describes a thermodynamic ensemble at temperature T and show that this matrix can similarly be expressed in such a closed form.

I. INTRODUCTION AND SUMMARY

The quantum-mechanical description of the parametric amplification provided by a harmonic oscillator with a time-varying "spring constant" may well be relevant to certain experimental devices. In particular, such amplification may be useful[1] in the University of Washington experiment[2] which traps a single, essentially free electron and measures its anomalous magnetic moment with unprecedented precision. Here we shall work out the quantum theory of this parametric amplification.

Let us review the basic idea of parametric amplification with an elementary and perhaps familiar example[3] in order to place the later development in a simple context. We consider an oscillating LC circuit as shown in Fig. 1. The separation of the capacitor plates d may be varied, causing the capacitance of the circuit to vary. Suppose that the plates are pulled apart when they are charged and put back to their original separation when they are discharged. Since the oppositely charged plates attract one another, this action transfers energy to the LC circuit and increases its oscillation amplitude. Moreover, since the magnitude of the oscillating charge on the plates is increased, the force between the oppositely charged plates is increased in each cycle. Hence, subsequent cycles transfer ever increasing amounts of energy to the LC circuit, and the amplitude of its oscillation increases exponentially. Since the capacitor plates are charged twice in a single cycle, this amplification occurs when the capacitance (the parameter) varies at twice the natural frequency of the oscillation. The parametric variation must have the proper phase to produce an exponential increase in the oscillator amplitude. It is easy to see from the LC circuit example that a change in this phase by 90° produces an exponential decrease in the oscillator amplitude and yields parametric deamplification.

Section II develops the general theory of parametric amplification in the classical harmonic oscillator. In the general case, the natural frequency ω of the oscillator is replaced by an arbitrary function of the time, $\omega(t)$, so that the equation of motion for the oscillator coordinate $q(t)$ reads

$$\frac{d^2 q(t)}{dt^2} + \omega(t)^2 q(t) = 0 . \tag{1.1}$$

We shall assume that the parameter $\omega(t)$ becomes constant for times in the remote past and for times in the far future,

$$t < -T: \quad \omega(t) = \omega_- , \tag{1.2a}$$

$$t > +T: \quad \omega(t) = \omega_+ . \tag{1.2b}$$

Thus, initially the oscillator is a simple harmonic oscillator with a natural frequency ω_-, while finally it is also a simple harmonic oscillator but with a frequency ω_+ that may differ from the initial frequency ω_-. The general form of the parameter $\omega(t)^2$ is sketched in Fig. 2.

Our method of analysis of the motion of the classical oscillator is motivated by the fact that Eq. (1.1) is formally identical to a one-dimensional Schrödinger equation if we make the following replacements: time $t \to$ spatial coordinate x, oscillator coordinate $q(t) \to$ wave function $\psi(x)$, parametric function $\omega(t)^2 \to 2m[E - V(x)]$, where E and $V(x)$ are the total energy and potential energy of a particle of mass m. The classical oscillator mo-

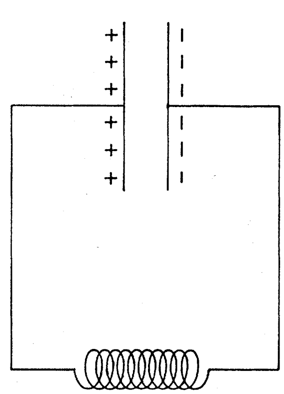

FIG. 1. Oscillating LC circuit with variable capacitance (variable plate separation d) illustrates parametric amplification.

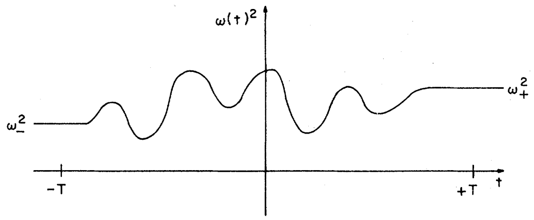

FIG. 2. The functional form of $\omega(t)^2$ is arbitrary except that it is required to approach constant values for $|t| > T$.

tion is, therefore, related to a one-dimensional, quantum-mechanical-barrier-penetration problem with a potential essentially of the form displayed in Fig. 2.

The asymptotic behavior of the two scattering wave functions in the barrier-penetration problem is described by a 2×2 scattering matrix S. Its elements give the transition amplitudes for a particle incident from the right to be reflected to the right (S_{++}) or transmitted to the left (S_{-+}) and the transition amplitudes for a particle incident from the left to the reflected to the left (S_{--}) or transmitted to the right (S_{+-}). As is well known, the particle flux $j = i\hbar[(d\psi^*/dx)\psi - \psi^*(d\psi/dx)]$ is a constant in the barrier penetration problem. This is a special case of the constancy of the Wronskian of any two solutions to Eq. (1.1). By computing the Wronskian of various pairs of solutions, one can prove easily that the scattering matrix S is unitary and symmetrical.

Any solution of the classical oscillator can be written as a linear combination of the two scattering solutions. Thus, the amplitude and phase of the final simple harmonic motion are related to the amplitude and phase of the initial harmonic motion by elements of the matrix S. As the initial phase of the oscillator is varied, the amplitude of the final oscillation varies. The initial phase can be chosen to make a maximum or a minimum final amplitude (cf. the LC circuit description above). The unitarity and symmetry of the S matrix relate these two extreme values of the final amplitude. We find that the S matrix is related to the classical amplification factor $\exp[2\chi]$ by

$$e^{2\chi} = (1 + |S_{++}|)/(1 - |S_{++}|). \quad (1.3)$$

With an optimal initial phase, an initial oscillator energy E^i is maximally amplified to a final oscillator energy $E^f_{\max}$ given by

$$E^f_{\max} = e^{2\chi}(\omega_+/\omega_-)E^i. \quad (1.4a)$$

On the other hand, an anti-optimal initial phase gives a maximal deamplification to a final energy $E^f_{\min}$ given by

$$E^f_{\min} = e^{-2\chi}(\omega_+/\omega_-)E^i. \quad (1.4b)$$

The appearance of the factor ω_+/ω_- in Eqs. (1.4) is easily understood by considering a case where $\omega(t)$ corresponds to a reflectionless scattering, for example, when the frequency $\omega(t)$ changes adiabatically from ω_- to ω_+. In such a case $e^{2\chi} = 1$ and the energy scales linearly with the frequency (in the quantum-mechanical theory the quantum number of the oscillator is not changed). It should be emphasized that the reciprocity displayed in Eqs. (1.4) is a consequence of the unitarity of the S matrix and holds for an arbitrary parametric function $\omega(t)$.

Section III applies the scattering matrix formalism to the motion of a quantum oscillator. The Wronskian of the quantum-oscillator coordinate $q(t)$ with the two scattering solutions give two constant quantum operators. Evaluating these two operator Wronskians at asymptotic times, we find that the final creation and annihilation operators of the oscillator, $a^\dagger(t_2)$ and $a(t_2)$, are linearly related by elements of the S matrix to the initial creation and annihilation operators, $a^\dagger(t_1)$ and $a(t_1)$. It is a simple matter to use these linear relations to obtain a formula for the quantum-amplification factor. We find that if the oscillator is initially in the nth energy eigenstate with an initial energy

$$E^i_n = \hbar\omega_-(n + \tfrac{1}{2}), \quad (1.5)$$

then the expectation value of the final energy (the average of the final oscillator energies) is given by

$$\langle H\rangle^f = (1 + |S_{++}|^2)/(1 - |S_{++}|^2)(\omega_+/\omega_-)E^i_n. \quad (1.6)$$

The definition (1.3) of the classical amplification factor may be used to write this as[4]

$$\langle H\rangle^f = \tfrac{1}{2}(e^{2\chi} + e^{-2\chi})(\omega_+/\omega_-)E^i_n. \quad (1.7)$$

We see that the amplification of the quantum energy eigenstate, which has a random phase, is the average of the classical maximal amplification and deamplification.

The linear relationships between the creation and annihilation operators in the far future and remote past are employed in Sec. III to compute the transformation function for "coherent states," which are eigenstates of the annihilation operator.[5] The ground-state-transformation function (the "vacuum-vacuum amplitude") appears as an integration constant in this computation. In Sec. IV this amplitude is expressed as a Fredholm determinant which is evaluated using methods[6] developed for scattering theory. The coherent-state-transformation function is a generating function for energy eigenstate-transition amplitudes. These are calculated in Sec. V. The probability of observing a final-energy eigenstate when the system is initially in a thermodynamic, canonical ensemble at temperature T is also calculated in Sec.

V.

Our major results are the following. Suppose that initially, before $\omega(t)$ has started to vary, the oscillator is in the nth energy eigenstate with energy $\hbar\omega_-(n+\frac{1}{2})$ while finally, after $\omega(t)$ has ceased to vary, the oscillator is in the n'th energy eigenstate with energy $\hbar\omega_+(n'+\frac{1}{2})$. There is no transition unless n' and n differ by an even integer. The probability for the latter is given by

$$|\langle n'+|n-\rangle|^2 = |S_{+-}|\,\frac{n'!}{n!}\,|P^{(1/2)(n-n')}_{(1/2)(n+n')}(|S_{+-}|)|^2\,, \tag{1.8}$$

where $P_l^m(z)$ is the associated Legendre polynomial.[7] It follows from the definition (1.3) and the unitarity of the S matrix that the modulus of the element S_{+-} is given by

$$|S_{+-}| = (\cosh\chi)^{-1}\,. \tag{1.9}$$

We see that the quantum-transition probabilities are determined by the classical amplification factor $e^{2\chi}$. Thus, the classical solution of the parametric amplification completely specifies the quantum solution. Since the Legendre polynomials obey

$$P_l^{-m}(z) = [(l-m)!\,/(l+m)!]\,P_l^m(z)\,, \tag{1.10}$$

the probability of making a transition from n' to n is the same as the probability of making a transition from n to n',

$$|\langle n+|n'-\rangle|^2 = |\langle n'+|n-\rangle|^2\,. \tag{1.11}$$

This symmetry holds even though the parametric function $\omega(t)$ is, in general, not an even function of the time t and the oscillation is not time-reflection invariant.

The transition probabilities from the lowest two initial levels illustrate the character of our general result. They can be written in an explicit form by using

$$P_l^{-l}(z) = z^{-1}P_{l+1}^{-l}(z) = (z^2-1)^{l/2}(1/2^l l!) \tag{1.12}$$

in conjunction with Eq. (1.8). We get, for m an integer,

$$|\langle 2m+|0-\rangle|^2 = \frac{(2m)!}{4^m(m!)^2}\,\frac{\tanh^{2m}\chi}{\cosh\chi}\,, \tag{1.13}$$

and

$$|\langle 2m+1+|1-\rangle|^2 = \frac{(2m+1)!}{4^m(m!)^2}\,\frac{\tanh^{2m}\chi}{\cosh^3\chi}\,. \tag{1.14}$$

In the limit of large amplfication, $\chi \gg 1$, transitions to individual energy eigenstates are vanishingly small; only sums over large numbers of final states are significant. Hence in this limit we may use Stirling's approximation for the factorials and approximate $(1\pm e^{-2\chi})^{2m}$ by $\exp(\pm 2me^{-2\chi})$ to obtain

$$|\langle 2m+|0-\rangle|^2 = [2/(\pi m)^{1/2}]\,e^{-\chi}\exp(-4me^{-2\chi})\,, \tag{1.15}$$

and

$$|\langle 2m+1+|1-\rangle|^2 = 16(m/\pi)^{1/2}e^{-3\chi}\exp(-4me^{-2\chi})\,. \tag{1.16}$$

Suppose now that we have initially a thermal ensemble of oscillators at temperature T. The ensemble is described by the density operator

$$\rho_-(\beta) = 2\sinh\tfrac{1}{2}\beta\hbar\omega_-\exp(-\beta H_-)\,, \tag{1.17}$$

where

$$\beta = 1/kT\,, \tag{1.18}$$

and H_- is the initial Hamiltonian operator corresponding to the natural frequency ω_-. The density operator is normalized,

$$\mathrm{Tr}\rho_-(\beta) = \sum_{n=0}^{\infty}\langle n|\rho_-(\beta)|n\rangle = 1\,. \tag{1.19}$$

The probability of finding an oscillator in the n'th final energy eigenstate is defined by

$$P(n';\beta) = \langle n'+|\rho_-(\beta)|n'+\rangle\,. \tag{1.20}$$

The computation of Sec. V gives

$$P(n';\beta) = 2\sinh\tfrac{1}{2}\beta\hbar\omega_-(\cosh 2\chi\sinh\beta\hbar\omega_- + \cosh\beta\hbar\omega_-)^{-n'-1/2}(1-\sinh^2 2\chi\sinh^2\beta\hbar\omega_-)^{n'/2}$$
$$\times P_{n'}((1-\sinh^2 2\chi\sinh^2\beta\hbar\omega_-)^{-1/2})\,, \tag{1.21}$$

where $P_{n'}(z)$ is the Legendre polynomial. Since $x^{n'/2}P_{n'}(1/\sqrt{x})$ is a finite polynomial in x, there is no difficulty when the argument of the square root in Eq. (1.21) becomes negative. We should note that the initial thermal ensemble evolves into a final ensemble which does not have a thermal character. In particular, the density operator is not diagonal in the final-energy eigenstate basis. The general matrix element $\langle n'+|\rho_-(T)|n''+\rangle$ is also computed in Sec. V.

The generating function for Legendre polynomials

$$\sum_{n'=0}^{\infty} t^{n'}P_{n'}(z) = (1-2zt+t^2)^{-1/2} \tag{1.22}$$

is a tool for checking the validity of our result. A little calculation using this generating function and the result (1.21) shows that

$$\sum_{n'=0}^{\infty} P(n';\beta)=1. \tag{1.23}$$

The probabilities $P(n;\beta)$ are, therefore, correctly normalized. If we apply the derivative operator $t^{1/2}(d/dt)t^{1/2}$ to the generating function (1.22) we get a sum involving $(n'+1/2)P_{n'}(z)$. This sum can be used to compute the average final energy,

$$\bar{E}^f_\beta=\sum_{n'=0}^{\infty}\hbar\omega_+(n'+\tfrac{1}{2})P(n';\beta)$$
$$=\hbar\omega_+\cosh 2\chi\,[\tfrac{1}{2}+1/(e^{\beta\hbar\omega_-}-1)]. \tag{1.24}$$

Since the average energy of the initial thermal ensemble $\bar{E}^i_\beta$ is given by

$$\bar{E}^i_\beta=\hbar\omega_-[\tfrac{1}{2}+1/(e^{\beta\hbar\omega_-}-1)]. \tag{1.25}$$

we have

$$\bar{E}^f_\beta=\tfrac{1}{2}(e^{2\chi}+e^{-2\chi})(\omega_+/\omega_-)\bar{E}^i_\beta. \tag{1.26}$$

This agrees with the previous result for the quantum-mechanical energy amplification, Eq. (1.7).

In the limit of large amplification where $\chi\gg 1$ only high-lying final-energy levels are significantly populated, $n'\gg 1$. In this limit we find in Sec. V that

$$P(n';\beta)\simeq\left(\frac{2}{\pi n'}\right)^{1/2}e^{-\chi}\left(\frac{e^{\beta\hbar\omega_-}-1}{e^{\beta\hbar\omega_-}+1}\right)^{1/2}\left[\left(1-2e^{-2\chi}\frac{e^{\beta\hbar\omega_-}-1}{e^{\beta\hbar\omega_-}+1}\right)^{n'+1/2}+(-1)^{n'}\left(1-2e^{-2\chi}\frac{e^{\beta\hbar\omega_-}+1}{e^{\beta\hbar\omega_-}-1}\right)^{n'+1/2}\right]. \tag{1.27}$$

With n' large, we are essentially in the classical limit in the final states. Experiments will include a range of n' values, and only an average value of $P(n';\beta)$ will be measured; hence only an average probability is needed,

$$\bar{P}(n';\beta)=\tfrac{1}{2}[P(n'+1;\beta)+P(n';\beta)]\simeq\left(\frac{2}{\pi n'}\right)^{1/2}e^{-\chi}\left(\frac{e^{\beta\hbar\omega_-}-1}{e^{\beta\hbar\omega_-}+1}\right)^{1/2}\exp\left(-2(n'+\tfrac{1}{2})e^{-2\chi}\frac{e^{\beta\hbar\omega_-}-1}{e^{\beta\hbar\omega_-}+1}\right). \tag{1.28}$$

Here a power has been written as an exponential, $(1-e^{-2\chi}\,\mathrm{const})^{n'+1/2}\simeq\exp[-(n'+\tfrac{1}{2})e^{-2\chi}\,\mathrm{const}]$, since $e^{-2\chi}$ is small. In the classical limit it is more realistic to examine the probability $\mathcal{P}(E';\beta)dE'$ for observing the oscillator to have a final energy in an interval $E'-E'+dE'$ rather than the probability to be in a definite final quantum state. Since $E'=\hbar\omega_+(n'+\tfrac{1}{2})$, the energy probability density is related to the number probability by

$$\mathcal{P}(E';\beta)=(1/\hbar\omega_+)\bar{P}(n';\beta). \tag{1.29}$$

The energy probability density is put in simple, physical terms by using Eqs. (1.25) and (1.26) in conjunction with Eqs. (1.28) and (1.29):

$$\mathcal{P}(E';\beta)\simeq[1/(2\pi E'\bar{E}^f_\beta)^{1/2}]\exp(-E'/2\bar{E}^f_\beta). \tag{1.30}$$

Thus in the limit of large parametric amplification of an initial thermodynamic ensemble, the probability of finding an energy E' in the final ensemble has a simple exponential character. To check the approximation, we observe that Eq. (1.30) implies the correct normalization,

$$\int_0^\infty dE'\,\mathcal{P}(E';\beta)=1, \tag{1.31}$$

and the correct average final energy,

$$\int_0^\infty dE'\,E'\,\mathcal{P}(E';\beta)=\bar{E}^f_\beta. \tag{1.32}$$

We now present the derivation of the results which we just have described.

II. CLASSICAL PARAMETRIC AMPLIFICATION AND QUANTUM SCATTERING

As we remarked before, the equation of motion for the parametrically amplified oscillator, Eq. (1.1), is akin to the one-dimensional Schrödinger equation, and the oscillator solutions can be obtained from the analogous scattering solutions. For this purpose we introduce the comparison function

$$\phi(t)=[2\omega(t)]^{-1/2}\exp\left(i\int_0^t dt'\,\omega(t')\right). \tag{2.1}$$

Recalling that $\omega(t)$ corresponds to $[2m(E-V)]^{1/2}$ in the scattering analogy, we see that $\phi(t)$ and its complex conjugate $\phi^*(t)$ are the familiar WKB approximate solutions to Eq. (1.1). They are exact solutions to a comparison "Schrödinger equation,"

$$\left(\frac{d^2}{dt^2}+u(t)\right)\phi(t)=0, \tag{2.2}$$

with

$$u(t)=\omega(t)^2+\tfrac{1}{2}[\ddot{\omega}(t)/\omega(t)]-\tfrac{3}{4}[\dot{\omega}(t)^2/\omega(t)^2], \tag{2.3}$$

where a dot denotes a time derivative. The "potential" $u(t)$ has the same asymptotic limits as does $\omega(t)^2$,

$$t<-T:\ u(t)=\omega(t)^2=\omega_-^2, \tag{2.4a}$$

$$t>+T:\ u(t)=\omega(t)^2=\omega_+^2. \tag{2.4b}$$

Thus Eq. (2.2) gives smoothly interpolating, reflectionless solutions ϕ, ϕ^* which have the asymptotic behavior of the solutions $\psi_\pm(t)$ to the original scattering problem,

$$\left(\frac{d^2}{dt^2}+\omega(t)^2\right)\psi_\pm(t)=0\,. \tag{2.5}$$

The solutions to this second-order differential equation are specified by two integration constants. We define one solution, $\psi_-(t)$, by the conditions that it contain a unit component of $\phi^*(t)$ when $t\to-\infty$ and no component of $\phi(t)$ when $t\to+\infty$. Hence

$$\psi_-(t)=\begin{cases}\phi^*(t)+S_{--}\phi(t)\,, & t\to-\infty\,,\\ S_{+-}\phi^*(t)\,, & t\to+\infty\,.\end{cases} \tag{2.6a}$$

The other solution, $\psi_+(t)$, is defined by the conditions that it contain no component of $\phi^*(t)$ when $t\to-\infty$ and a unit component of $\phi(t)$ when $t\to+\infty$. Hence,

$$\psi_+(t)=\begin{cases}S_{-+}\phi(t)\,, & t\to-\infty\,,\\ \phi(t)+S_{++}\phi^*(t)\,, & t\to+\infty\,.\end{cases} \tag{2.6b}$$

The four complex elements of the scattering matrix

$$S=\begin{pmatrix}S_{++} & S_{+-}\\ S_{-+} & S_{--}\end{pmatrix} \tag{2.7}$$

are determined by the differential equation (2.5). Not all of these elements are independent, however, because the Wronskian of any pair of solutions (ψ_1,ψ_2) to Eq. (2.5) is a constant,

$$W[\psi_1,\psi_2]=\psi_1\overleftrightarrow{\frac{d}{dt}}\psi_2=\psi_1\frac{d\psi_2}{dt}-\frac{d\psi_1}{dt}\psi_2=\text{const}\,. \tag{2.8}$$

Computing the following Wronskians at $t\to-\infty$ and $t\to+\infty$ and equating the two asymptotic values leads to the condition shown below,

$$W[\psi_-^*,\psi_-]:\;\; 1-|S_{--}|^2=|S_{+-}|^2\,, \tag{2.9a}$$

$$W[\psi_+^*,\psi_+]:\;\; |S_{-+}|^2=1-|S_{++}|^2\,, \tag{2.9b}$$

$$W[\psi_-^*,\psi_+]:\;\; S_{--}^*S_{-+}=-S_{+-}^*S_{++}\,, \tag{2.9c}$$

$$W[\psi_+^*,\psi_-]:\;\; S_{--}S_{-+}^*=-S_{+-}S_{++}^*\,. \tag{2.9d}$$

These equations simply state that S is a unitary matrix,

$$S^\dagger S=1\,. \tag{2.10}$$

A final Wronskian condition remains,

$$W[\psi_-,\psi_+]:\;\; S_{-+}=S_{+-}\,. \tag{2.11}$$

Hence S is a symmetrical, unitary matrix. The symmetry (2.11) together with the unitarity conditions (2.9a) and (2.9b) imply that $|S_{--}|=|S_{++}|$, but in general the phases of S_{--} and S_{++} differ. If, however, $\omega(t)^2$ is an even function of the time, $\psi_-(-t)$ solves the differential equation (2.5) with the boundary conditions of $\psi_+(t)$. Hence in this case $\psi_-(-t)=\psi_+(t)$ and $S_{--}=S_{++}$. Note that with our conventions the absence of scattering corresponds to the limit

$$S\to\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{2.12}$$

For a first application of these results, we consider a classical oscillator with the initial motion

$$t\to-\infty:\;\; q(t)=A\,\mathrm{Re}\{\exp[i(\omega_-t+\theta)]\}\,. \tag{2.13}$$

We shall absorb factors of the square root of the oscillator mass into the definition of the oscillator coordinate $q(t)$ and momentum $p(t)$ so that the Hamiltonian appears in the form

$$H=\tfrac{1}{2}[p(t)^2+\omega(t)^2q(t)^2]\,. \tag{2.14}$$

With this convention the initial energy is given by

$$E^i=\tfrac{1}{2}[\dot{q}(t)^2+\omega^2q(t)^2]=\tfrac{1}{2}A^2\omega_-^2\,. \tag{2.15}$$

Except for constant factors, the scattering solution $\psi_+(t)$ obeys the initial conditions of Eq. (2.13). Hence for general times Eqs. (2.1), (2.6b) and (2.13) give

$$q(t)=A\,\mathrm{Re}\{[(2\omega_-)^{1/2}/S_{-+}]e^{i\theta}\psi_+(t)\}\,, \tag{2.16}$$

and in particular for $t\to+\infty$

$$q(t)=A\,\mathrm{Re}\left(\frac{(2\omega_-)^{1/2}}{S_{-+}}e^{i\theta}\phi(t)+\frac{(2\omega_-)^{1/2}}{S_{-+}}e^{i\theta}S_{++}\phi^*\right)=\left(\frac{\omega_-}{\omega_+}\right)^{1/2}A\,\mathrm{Re}\left[\left(\frac{1}{S_{-+}}e^{i\theta}+\frac{S_{++}^*}{S_{-+}^*}e^{-i\theta}\right)\exp[i(\omega_+t+\delta)]\right], \tag{2.17}$$

where δ is the asymptotic constant phase accumulated in the definition (2.1) of $\phi(t)$. Accordingly, the final and initial energies are related by

$$E^f=\frac{1}{|S_{-+}|^2}\left|1+S_{++}^*\frac{S_{-+}}{S_{-+}^*}e^{-2i\theta}\right|^2\frac{\omega_+}{\omega_-}E^i\,. \tag{2.18}$$

As the initial phase θ varies, the magnitude of the final energy E^f varies between the maximum and minimum values

$$E^f_{\max}=[(1+|S_{++}|)^2/|S_{-+}|^2](\omega_+/\omega_-)E^i\,, \tag{2.19a}$$

and

$$E^f_{\min}=[(1-|S_{++}|)^2/|S_{-+}|^2](\omega_+/\omega_-)E^i\,. \tag{2.19b}$$

In view of the unitarity condition (2.9b), we may write these maximum and minimum final energy values as

$$E^f_{\rm max} = e^{2\chi}(\omega_+/\omega_-)E^i\,, \tag{2.20a}$$

and

$$E^f_{\rm min} = e^{-2\chi}(\omega_+/\omega_-)E^i\,, \tag{2.20b}$$

where the classical amplification factor is defined by

$$e^{2\chi} = (1+|S_{++}|)/(1-|S_{++}|)\,. \tag{2.21}$$

Thus, we have established the results quoted in Sec. I. Note incidently that the definition (2.21) and the unitarity conditions (2.9) give

$$|S_{--}| = |S_{++}| = \tanh\chi \tag{2.22a}$$

and

$$|S_{+-}| = (\cosh\chi)^{-1}\,. \tag{2.22b}$$

It was remarked before that the wave function $\phi(t)$ solves a comparison "potential scattering" problem. We conclude this section by amplifying this remark and deriving a formal expression for the scattering matrix element $S_{+-}=S_{-+}$ which will prove useful for our later work. Recalling the definition (2.3) of $u(t)$, we see that

$$\omega(t)^2 = u(t) + v(t)\,, \tag{2.23}$$

where

$$v(t) = -\tfrac{1}{2}[\ddot{\omega}(t)/\omega(t)] + \tfrac{3}{4}[\dot{\omega}(t)^2/\omega(t)^2]\,. \tag{2.24}$$

We need to solve the Schrödinger equation

$$\left(\frac{d^2}{dt^2} + u(t) + v(t)\right)\Phi(t) = 0\,. \tag{2.25}$$

This differential equation can be cast into an integral equation by means of the retarded Green's function

$$G_r(t,t') = \theta(t-t')\,i[\phi(t)\,\phi^*(t') - \phi^*(t)\,\phi(t')] \tag{2.26}$$

which obeys

$$\left(-\frac{d^2}{dt^2} - u(t)\right)G_r(t,t') = \delta(t-t')\,. \tag{2.27}$$

Thus

$$\Phi(t) = \phi(t) + \int dt'\, G_r(t,t')\, v(t')\,\Phi(t') \tag{2.28}$$

obeys the differential equation (2.25). It is convenient to introduce the scalar product notation

$$(\phi,\psi) = \int dt\;\phi(t)\,\psi(t)\,. \tag{2.29}$$

The boundary conditions implied by the integral equation (2.28)

$$t\to-\infty:\;\; \Phi(t)\to\phi(t)\,, \tag{2.30a}$$

$$t\to+\infty:\;\; \Phi(t)\to\phi(t)[1+i(\phi^*, v\Phi)] - \phi^*(t)\,i(\phi, v\Phi)\,, \tag{2.30b}$$

identify [cf. Eq. (2.6b)]

$$\Phi(t) = S^{-1}_{-+}\psi_+(t) \tag{2.31}$$

and yield

$$S^{-1}_{-+} = 1 + i(\phi^*, v\Phi)\,. \tag{2.32}$$

By writing the integral equation (2.28) in an operator notation, we secure the formal solution

$$\Phi = [1/(1-G_r v)]\phi \tag{2.33}$$

which gives the desired formula:

$$S^{-1}_{-+} = 1 + i\left(\phi^*, v\left(\frac{1}{1-G_r v}\right)\phi\right). \tag{2.34}$$

III. COHERENT-STATE AMPLITUDES

We turn now to the quantum oscillator. We work in the Heisenberg picture and define creation and annihilation operators by

$$a^\dagger(t) = \left(\frac{\omega(t)}{2}\right)^{1/2} q(t) - i\left(\frac{1}{2\omega(t)}\right)^{1/2} p(t)\,, \tag{3.1a}$$

and

$$a(t) = \left(\frac{\omega(t)}{2}\right)^{1/2} q(t) + i\left(\frac{1}{2\omega(t)}\right)^{1/2} p(t)\,. \tag{3.1b}$$

They obey the commutation relation

$$[a(t), a^\dagger(t)] = 1 \tag{3.2}$$

in the natural units with $\hbar=1$ which we henceforth employ. We shall use the operators for times $t_1<-T$, where $\omega(t)=\omega_-$, or for times $t_2>+T$, where $\omega(t)=\omega_+$. At these remote times $a^\dagger(t)$ and $a(t)$ are the usual creation and annihilation operators for simple-harmonic oscillators. The operators at t_1 and t_2 are related linearly to one another since the Wronskian of two solutions to the equation of motion is a constant. Let us define

$$\Omega(t) = \int_0^t dt'\,\omega(t') \tag{3.3}$$

so that the comparison function (2.1) can be written as

$$\phi(t) = \frac{1}{[2\omega(t)]^{1/2}} e^{i\Omega(t)}\,. \tag{3.4}$$

We use the asymptotic limits for $\psi_-(t)$ given by Eq. (2.6a) to evaluate the constant Wronskian

$$iW[q,\psi_-] = q(t)\, i\,\frac{\overleftrightarrow{d}}{dt}\,\psi_-(t) \tag{3.5a}$$

at times $t_1<-T$ and $t_2>+T$ and thereby establish that

$$\exp[-i\Omega(t_1)]\,a^\dagger(t_1) - S_{--}\exp[i\Omega(t_1)]\,a(t_1) = S_{+-}\exp[-i\Omega(t_2)]\,a^\dagger(t_2)\,. \tag{3.6a}$$

Similarly, the asymptotic limits for $\psi_+(t)$ given by Eq. (2.6b) and the constant Wronskian

$$iW[q,\psi_+]=q(t)\,i\,\overleftrightarrow{\frac{d}{dt}}\,\psi_+(t) \tag{3.5b}$$

yield

$$S_{-+}\exp[i\Omega(t_1)]\,a(t_1) =\exp[i\Omega(t_2)]\,a(t_2)-S_{++}\exp[-i\Omega(t_2)]\,a^\dagger(t_2)\,. \tag{3.6b}$$

The Hamiltonian of the oscillator may be expressed in terms of the creation and annihilation operators,

$$H(t)=\tfrac{1}{2}\omega(t)[a^\dagger(t)\,a(t)+a(t)\,a^\dagger(t)]\,. \tag{3.7}$$

The linear relation (3.6a) implies that

$$\begin{aligned}|S_{+-}|^2[a^\dagger(t_2)\,a(t_2)+a(t_2)\,a^\dagger(t_2)] &=(1+|S_{--}|^2)[a^\dagger(t_1)\,a(t_1)+a(t_1)\,a^\dagger(t_1)]\\ &\quad-2S_{--}^*\exp[-2i\Omega(t_1)]\,a^\dagger(t_1)^2\\ &\quad-2S_{--}\exp[2i\Omega(t_1)]\,a(t_1)^2\,.\end{aligned} \tag{3.8}$$

Suppose that the oscillator is initially in an energy eigenstate $|nt_1\rangle$,

$$H(t_1)|nt_1\rangle=|nt_1\rangle E_n^i=|nt_1\rangle\,\omega_-(n+\tfrac{1}{2})\,. \tag{3.9}$$

Then in view of Eqs. (3.7) and (3.8), the final average energy observed will be given by

$$\langle H\rangle^f=\langle nt_1|H(t_2)|nt_1\rangle=\frac{1+|S_{--}|^2}{|S_{+-}|^2}\,\frac{\omega_+}{\omega_-}E_n^i\,, \tag{3.10}$$

since $a^\dagger(t_1)$ and $a(t_1)$ raise and lower the initial levels and thus have no diagonal matrix element,

$$\langle nt_1|a^\dagger(t_1)^2|nt_1\rangle=0=\langle nt_1|a(t_1)^2|nt_1\rangle\,. \tag{3.11}$$

Equations (2.22) can be used to express this result as

$$\langle H\rangle^f=\tfrac{1}{2}(e^{2\chi}+e^{-2\chi})(\omega_+/\omega_-)E_n^i\,, \tag{3.12}$$

the form quoted in Sec. I.[8]

The major purpose of this section is the construction of the coherent-state transformation function $\langle z_2^*t_2|z_1t_1\rangle$. As will be shown later, it is a generating function for the energy eigenstate-transition amplitudes. The initial and final ground or "vacuum" states satisfy

$$a(t_1)|0t_1\rangle=0\,, \tag{3.13a}$$

and

$$\langle 0t_2|a^\dagger(t_2)=0\,. \tag{3.13b}$$

Coherent states are built from these vacuum states according to

$$|zt_1\rangle=\exp[za^\dagger(t_1)]|0t_1\rangle\,, \tag{3.14a}$$

and

$$\langle z^*t_2|=\langle 0t_2|\exp[z^*a(t_2)]\,. \tag{3.14b}$$

By virtue of the commutation relation (3.2) of $a^\dagger$, a and the properties (3.13) of the vacuum, the coherent states are eigenstates of the annihilation and creation operators,

$$a(t_1)|zt_1\rangle=|zt_1\rangle z\,, \tag{3.15a}$$

and[9]

$$\langle z^*t_2|a^\dagger(t_2)=z^*\langle z^*t_2|\,. \tag{3.15b}$$

On the other hand, it follows directly from the definitions (3.14) that

$$a^\dagger(t_1)|zt_1\rangle=\frac{\partial}{\partial z}|zt_1\rangle\,, \tag{3.16a}$$

and

$$\langle z^*t_2|a(t_2)=\frac{\partial}{\partial z^*}\langle z^*t_2|\,. \tag{3.16b}$$

We now have all the tools in hand for a rapid construction of the coherent-state-transformation function. We take $\langle z_2^*t_2|\ldots|z_1t_1\rangle$ matrix elements of the linear relations (3.6) amongst the creation and annihilation operators and use Eqs. (3.15) and (3.16) to derive differential equations for the transformation function:

$$\left(\exp[-i\Omega(t_1)]\,\frac{\partial}{\partial z_1}-S_{--}\exp[i\Omega(t_1)]\,z_1-S_{+-}\exp[-i\Omega(t_2)]\,z_2^*\right)\langle z_2^*t_2|z_1t_1\rangle=0\,, \tag{3.17a}$$

$$\left(\exp[i\Omega(t_2)]\,\frac{\partial}{\partial z_2^*}-S_{++}\exp[-i\Omega(t_2)]\,z_2^*-S_{-+}\exp[i\Omega(t_1)]\,z_1\right)\langle z_2^*t_2|z_1t_1\rangle=0\,. \tag{3.17b}$$

The two differential equations are compatible because of the symmetry $S_{+-}=S_{-+}$, and they have the solution

$$\langle z_2^*t_2|z_1t_1\rangle=\langle 0t_1|0t_2\rangle\exp(\tfrac{1}{2}Z^TSZ)\,, \tag{3.18}$$

where Z is the column vector

$$Z=\begin{pmatrix}\exp[-i\Omega(t_2)]\,z_2^*\\ \exp[i\Omega(t_1)]\,z_1\end{pmatrix}\,. \tag{3.19}$$

Here the integration constant has been identified with the vacuum amplitude $\langle 0t_2|0t_1\rangle$ since $\langle z_2^*t_2|z_1t_1\rangle$ has this limit when $z_2^*=0=z_1$.

IV. VACUUM AMPLITUDE

The vacuum-vacuum transformation function can be expressed in terms of a functional integral

$$\langle 0t_2|0t_1\rangle = \int [dq]e^{iW} \tag{4.1}$$

with action

$$W = \frac{1}{2}\int_{t_1}^{t_2} dt\, q(t)\left[-\frac{d^2}{dt^2} - \omega(t)^2\right]q(t)\,. \tag{4.2}$$

The functional integral of a quadratic form produces a Fredholm determinant and so

$$\langle 0t_2|0t_1\rangle = \det{}^{-1/2}\left[-\frac{d^2}{dt^2} - \omega^2\right]. \tag{4.3}$$

To evaluate the determinant, we recall Eq. (2.23) with $u(t)$ giving rise to a simply solvable comparison Green's function,

$$\left[-\frac{d^2}{dt^2} - u(t)\right]G(t,t') = \delta(t-t')\,. \tag{4.4}$$

This Green's function has positive-frequency boundary conditions in accordance with the $i\epsilon$ prescription implicit in the functional integral (4.1). Hence, remembering Eqs. (2.1), (2.2), and (3.3), we see that

$$G(t,t') = -\tfrac{1}{2}i[\omega(t)\,\omega(t')]^{-1/2} \times \exp[-i|\Omega(t) - \Omega(t')|]\,. \tag{4.5}$$

Since the determinant of a product of operators is a product of determinants, we may now write

$$\det\left[-\frac{d^2}{dt^2} - \omega^2\right] = \det\left[-\frac{d^2}{dt^2} - u\right]\det[1 - Gv]\,. \tag{4.6}$$

The variational formula

$$\delta \ln\det X = \mathrm{Tr} X^{-1}\delta X \tag{4.7}$$

can be employed to compute

$$\delta\ln\det\left[-\frac{d^2}{dt^2} - u\right] = -\int_{t_1}^{t_2} dt\, G(t,t)\,\delta u(t)\,, \tag{4.8}$$

and with Eqs. (2.3) and (4.5) we get

$$\delta\ln\det\left[-\frac{d^2}{dt^2} - u\right] = i\,\delta\int_{t_1}^{t_2} dt\,\omega(t) + \frac{i}{4}\int_{t_1}^{t_2} dt\,\frac{d}{dt}\left[\frac{\delta\dot\omega(t)}{\omega(t)^2} - \frac{\dot\omega(t)\,\delta\omega(t)}{\omega(t)^3}\right]. \tag{4.9}$$

The second integral gives no contribution since $\dot\omega(t)$ vanishes at the integration limits. Hence,

$$\det{}^{-1/2}\left[-\frac{d^2}{dt^2} - u\right] = \exp\left(-\frac{i}{2}\int_{t_1}^{t_2} dt\,\omega(t)\right), \tag{4.10}$$

where the overall constant is determined by the requirement that Eq. (4.10) reduce to the familiar simple oscillator amplitude in the limit where $\omega(t)$ becomes a constant.

The remaining determinant in Eq. (4.6) is easily calculated if we note that the Green's function (4.5) with positive frequency boundary conditions differs from the retarded Green's function (2.26) by the addition of a separable form,

$$G = G_r - i\phi\phi^*\,. \tag{4.11}$$

Again making use of the fact that the determinant of an operator product is the product of the determinants of the operators we obtain

$$\det[1 - Gv] = \det[1 - G_r v]\det\left[1 + i\,\frac{1}{1 - G_r v}\,\phi\phi^* v\right]. \tag{4.12}$$

Since the Green's function $G_r(t,t')$ is retarded and vanishes for equal times, the first determinantal factor on the right-hand side of Eq. (4.12) involves a triangular matrix with unit diagonal entries. Accordingly, we have

$$\det[1 - G_r v] = 1\,. \tag{4.13}$$

The second determinantal factor on the right-hand side of Eq. (4.12) involves a separable kernel and thus can be written in terms of a single matrix element,

$$\det\left[1 + i\,\frac{1}{1 - G_r v}\,\phi\phi^* v\right] = 1 + i\left(\phi^*, v\,\frac{1}{1 - G_r v}\,\phi\right) = S_{-+}^{-1}\,, \tag{4.14}$$

with the second equality following from Eq. (2.34).

Putting the pieces together yields the amplitude for the ground state to remain the ground state after parametric amplification:

$$\langle 0t_2|0t_1\rangle = (S_{-+})^{1/2}\exp\left(-\frac{i}{2}\int_{t_1}^{t_2} dt\,\omega(t)\right). \tag{4.15}$$

V. ENERGY-TRANSITION PROBABILITIES

Recalling Eqs. (3.18) and (3.19), the full coherent-state-transformation function is now seen to be expressed as

$$\langle z_2^* t_2|z_1 t_1\rangle = (S_{-+})^{1/2}\exp\left(-\frac{i}{2}\int_{t_1}^{t_2} dt\,\omega(t)\right)\exp(\tfrac{1}{2}Z^T S Z)\,, \tag{5.1}$$

in which

$$\tfrac{1}{2}Z^TSZ = \tfrac{1}{2}S_{++}\exp[-2i\Omega(t_2)]z_2^{*2} + \tfrac{1}{2}S_{--}\exp[2i\Omega(t_1)]z_1^2 + S_{-+}\exp\{-i[\Omega(t_2)-\Omega(t_1)]\}z_2^*z_1 . \quad (5.2)$$

Let us consider first the limit in which the parametric function $\omega(t)$ becomes the constant natural frequency ω of a simple harmonic oscillator. In this limit, $S_{-+}=1$, $S_{++}=0=S_{--}$, and $\Omega(t)=\omega t$, giving

$$\langle z_2^*t_2|z_1t_1\rangle = \exp[-\tfrac{1}{2}i\omega(t_2-t_1)] \times \exp\{z_2^*\exp[-i\omega(t_2-t_1)]z_1\}. \quad (5.3)$$

On the other hand, introducing a complete set of intermediate energy eigenstates gives

$$\langle z_2^*t_2|z_1t_1\rangle = \sum_{n=0}^{\infty}\langle z_2^*|n\rangle\exp[-iE_n(t_2-t_1)]\langle n|z_1\rangle , \quad (5.4)$$

which, by comparison with the expansion of the second exponential in Eq. (5.3), yields the familiar energy eigenvalue formula $E_n=\omega(n+\frac{1}{2})$ and in addition defines (up to a conventional phase choice) the wave functions[10]

$$\langle z_2^*|n\rangle = z_2^{*n}/(n!)^{1/2} , \quad (5.5a)$$

and

$$\langle n|z_1\rangle = z_1^n/(n!)^{1/2} . \quad (5.5b)$$

With nontrivial parametric amplification, the amplitude $\langle n'+|n-\rangle$ for the transition from the nth initial energy level to the n'th final energy level can be identified from the expansion

$$\langle z_2^*t_2|z_1t_1\rangle = \sum_{n'n}\langle z_2^*|n'\rangle\exp[-i(n'+\tfrac{1}{2})\Omega(t_2)] \times\langle n'+|n-\rangle\exp[i(n+\tfrac{1}{2})\Omega(t_1)]\langle n|z_1\rangle \quad (5.6)$$

by using the wave functions of Eqs. (5.5). The factors involving $\Omega(t_2)$ and $\Omega(t_1)$ have been introduced in Eq. (5.6) so as to make the transition amplitude $\langle n'+|n-\rangle$ independent of the times t_1 and t_2.

Inserting Eqs. (5.1) and (5.2) into Eq. (5.6), expanding the resulting product of three exponentials, and identifying the coefficient of the wave functions (5.5) we secure

$$\langle n'+|n-\rangle = (n'!\,n!\,S_{-+})^{1/2}\sum_{klm}\times\frac{1}{k!}(\tfrac{1}{2}S_{++})^k\frac{1}{l!}(\tfrac{1}{2}S_{--})^l\frac{1}{m!}(S_{-+})^m \times\delta_{n',2k+m}\delta_{n,2l+m} . \quad (5.7)$$

This transition amplitude vanishes unless $n'-n$ is an even integer,

$$n'-n=2t, \quad t=0,\pm1,\pm2,\ldots . \quad (5.8)$$

Using the unitarity condition

$$S_{--}S_{+-}^* = -S_{-+}S_{++}^* \quad (5.9)$$

and eliminating the sums over k and m by the Kronecker deltas in Eq. (5.7), we obtain

$$\langle n'+|n-\rangle = (S_{-+})^{1/2}|S_{-+}|^n|\tfrac{1}{2}S_{++}|^t\left(\frac{S_{-+}}{S_{-+}^*}\right)^{(n+n')/4}\left(-\frac{S_{++}}{S_{--}}\right)^{(n'-n)/4} \times(n'!\,n!)^{1/2}\sum_l\frac{1}{l!}\frac{1}{(l+t)!}\frac{1}{(n-2l)!}\left(-\frac{1}{4}\frac{|S_{++}|^2}{|S_{-+}|^2}\right)^l . \quad (5.10)$$

The finite sum which appears here may be expressed in terms of the hypergeometric function $F(a,b;c;z)$. This function is encountered if the Legendre duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\tfrac{1}{2}) \quad (5.11)$$

and the functional relation

$$\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z \quad (5.12)$$

are used to write

$$\left(\frac{1}{4}\right)^l\frac{n!}{(n-2l)!} = \frac{\Gamma(l-\frac{1}{2}n)\Gamma(l-\frac{1}{2}n+\frac{1}{2})}{\Gamma(-\frac{1}{2}n)\Gamma(-\frac{1}{2}n+\frac{1}{2})} , \quad (5.13)$$

where n is temporarily assigned a nonintegral value so as to avoid divergences. Therefore

$$\langle n'+|n-\rangle = (S_{-+})^{1/2}|S_{-+}|^n|\tfrac{1}{2}S_{++}|^t\left(\frac{S_{-+}}{S_{-+}^*}\right)^{(n+n')/4}\left(-\frac{S_{++}}{S_{--}}\right)^{(n'-n)/4} \times\left(\frac{n'!}{n!}\right)^{1/2}\frac{1}{t!}\sum_l\frac{\Gamma(l-\frac{1}{2}n)\Gamma(l-\frac{1}{2}n+\frac{1}{2})\Gamma(t+1)}{\Gamma(-\frac{1}{2}n)\Gamma(-\frac{1}{2}n+\frac{1}{2})\Gamma(l+t+1)}\frac{1}{l!}\left(-\frac{|S_{++}|^2}{|S_{-+}|^2}\right)^l$$

$$= (S_{-+})^{1/2}|S_{-+}|^n|\tfrac{1}{2}S_{++}|^t\left(\frac{S_{-+}}{S_{-+}^*}\right)^{(n+n')/4}\left(-\frac{S_{++}}{S_{--}}\right)^{(n'-n)/4}\left(\frac{n'!}{n!}\right)^{1/2}\frac{1}{t!}F\left(-\tfrac{1}{2}n,-\tfrac{1}{2}n+\tfrac{1}{2};t+1;-\frac{|S_{++}|^2}{|S_{-+}|^2}\right). \quad (5.14)$$

The hypergeometric function in Eq. (5.14) is related to an associated Legendre polynomial because[11]

$$P_\nu^\mu(z) = 2^\mu (z^2 - 1)^{-\mu/2} z^{\nu+\mu}[1/\Gamma(1-\mu)] \times F(-\tfrac{1}{2}\nu - \tfrac{1}{2}\mu, \tfrac{1}{2} - \tfrac{1}{2}\nu - \tfrac{1}{2}\mu; 1-\mu; 1 - 1/z^2). \quad (5.15)$$

Accordingly, with the help of the unitarity condition

$$|S_{++}|^2 + |S_{-+}|^2 = 1 \quad (5.16)$$

we arrive at our final form:

$$\langle n' + | n - \rangle = (S_{-+})^{1/2} \left(\frac{S_{-+}}{S_{-+}^*}\right)^{(n+n')/4} \left(\frac{S_{++}}{S_{--}}\right)^{(n'-n)/4} \times \left(\frac{n'!}{n!}\right)^{1/2} P_{(n+n')/2}^{(n-n')/2}(|S_{+-}|). \quad (5.17)$$

Since $|S_{++}| = |S_{--}|$, this gives the result (1.8) quoted in Sec. I. In view of the symmetry (1.10) of the Legendre polynomial, the transition amplitude (5.17) is symmetrical in n' and n except for a phase,

$$\langle n + | n' - \rangle = (S_{++}/S_{--})^{(n-n')/2} \langle n' + | n - \rangle . \quad (5.18)$$

As shown in Sec. II, if $\omega(t)$ is an even function of the time, the parametrically amplified oscillator is time-reversal invariant with $S_{++} = S_{--}$. In this case the transition amplitude is perfectly symmetrical in n' and n.

We turn now to consider the parametric amplification of an oscillator system which is initially in a thermal, canonical ensemble at temperature $T = (k\beta)^{-1}$ described by the initial density operator

$$\rho_-(\beta) = 2\sinh\tfrac{1}{2}\beta\omega_- \exp[-\beta H(t_1)] . \quad (5.19)$$

Here the normalization is chosen to give

$$\mathrm{Tr}\rho_-(\beta) = 1 . \quad (5.20)$$

We shall obtain the matrix element $\langle n' + | \rho_-(\beta) | n'' + \rangle$ by computing its generating function $\langle z_2^* t_2 | \rho_-(\beta) | z_1 t_2 \rangle$. To do this, we first need to derive a convenient resolution of the identity operator. Since the equal-time, coherent-state-transformation function

$$\langle z_2^* | z_1 \rangle = \exp(z_2^* z_1) \quad (5.21)$$

obeys

$$\int \frac{dx\,dy}{\pi} \langle z_2^* | z \rangle e^{-|z|^2} \langle z^* | z_1 \rangle = \langle z_2^* | z_1 \rangle \quad (5.22)$$

where

$$z = x + iy , \quad (5.23)$$

we conclude that

$$\int \frac{dx\,dy}{\pi} | z t \rangle e^{-|z|^2} \langle z^* t | = 1 . \quad (5.24)$$

It is easy to prove that

$$e^{-\beta H(t_1)} a^\dagger(t_1) e^{\beta H(t_1)} = e^{-\beta\omega_-} a^\dagger(t_1) \quad (5.25)$$

by differentiating with respect to β and evaluating the resulting commutator. Hence [cf. Eq. (3.14a)]

$$e^{-\beta H(t_1)} | z t_1 \rangle = \exp[z e^{-\beta H(t_1)} a^\dagger(t_1) e^{\beta H(t_1)}] \times e^{-\beta H(t_1)} | 0 t_1 \rangle = | e^{-\beta\omega_-} z t_1 \rangle e^{-\beta\omega_-/2} , \quad (5.26)$$

and therefore

$$e^{-\beta H(t_1)} = \int \frac{dx\,dy}{\pi} | e^{-\beta\omega_-} z t_1 \rangle e^{-\beta\omega_-/2} \times e^{-|z|^2} \langle z^* t_1 | . \quad (5.27)$$

This expression puts the final coherent-state matrix element of the initial density operator (5.19) into the form

$$\langle z_2^* t_2 | \rho_-(\beta) | z_1 t_2 \rangle = 2\sinh\tfrac{1}{2}\beta\omega_- e^{-\beta\omega_-/2} \times \int \frac{dx\,dy}{\pi} \langle z_2^* t_2 | e^{-\beta\omega_-} z t_1 \rangle \times e^{-|z|^2} \langle z_1^* t_2 | z t_1 \rangle^* . \quad (5.28)$$

The coherent-state-transformation functions which appear in the integrand involve the simple Gaussian function displayed in Eqs. (5.1) and (5.2). Hence the integral in Eq. (5.28) can be computed by the usual technique of completing the square and after a little calculation we find the result

$$\langle z_2^* t_2 | \rho_-(\beta) | z_1 t_2 \rangle = 2\sinh\tfrac{1}{2}\beta\omega_- c^{1/2} \times \exp(c z_2^* z_1 + \tfrac{1}{2} b z_1^2 + \tfrac{1}{2} b^* z_2^{*2}) , \quad (5.29)$$

where the unitarity of the S matrix can be used to put the coefficients c and b in relatively simple forms,

$$c = |S_{-+}|^2 \frac{e^{-\beta\omega_-}}{1 - e^{-2\beta\omega_-} |S_{--}|^2} , \quad (5.30)$$

and

$$b = S_{++}^* e^{2i\Omega(t_2)} \frac{1 - e^{-2\beta\omega_-}}{1 - e^{-2\beta\omega_-} |S_{--}|^2} . \quad (5.31)$$

In terms of its dependence on the variables z_2^*, z_1, the result (5.29) has the general structure of the original coherent-state-transformation function given in Eq. (5.1). Hence its expansion into final energy eigenstates is essentially identical to our previous work, and with simple substitutions we derive

$$\langle n'+|\rho_-(\beta)|n''+\rangle = 2\sinh\tfrac{1}{2}\beta\omega_-\left(\frac{S_{++}}{S_{++}^*}\right)^{(n'-n'')/4} \times c^{(n'+n''+1)/2}\left(1-\left|\frac{b}{c}\right|^2\right)^{(n'+n'')/4}\left(\frac{n'!}{n''!}\right)^{1/2} P^{(n''-n')/2}_{(n''+n')/2}\left[\left(1-\left|\frac{b}{c}\right|^2\right)^{-1/2}\right]. \quad (5.32)$$

Here n' and n'' must differ by an even integer; if $n'-n''$ is odd, the matrix element vanishes.

The symmetry (1.10) of the Legendre polynomial shows that, except for the phase factor $(S_{++}/S_{++}^*)^{(n'-n'')/4}$, the right-hand side of Eq. (5.32) is not altered if n' and n'' are interchanged. This symmetry implies that Eq. (5.32) defines the matrix element of a Hermitian operator,

$$\langle n'+|\rho_-(\beta)|n''+\rangle = \langle n''+|\rho_-(\beta)|n'+\rangle^*, \quad (5.33)$$

as of course it must. The result (5.32), in conjunction with the previous formula (5.17) for the energy-transition amplitude, implies an addition formula for the associated Legendre polynomials. This addition formula follows from substituting Eqs. (5.32) and (5.17) into the completeness relation

$$\langle n'+|\rho_-(\beta)|n''+\rangle = 2\sinh\tfrac{1}{2}\beta\omega_-\sum_n \langle n'+|n-\rangle \times\exp[-\beta\omega_-(n+\tfrac{1}{2})]\langle n-|n''+\rangle. \quad (5.34)$$

Since an infinite sum appears here, one might expect that this addition formula is related to an infinite-dimensional representation of the group multiplication law for some noncompact group. This is indeed true. The addition formula implied by Eq. (5.34) is a special case of the addition formula[12] for certain hypergeometric functions which follows from their role as the representation functions for the open group SL(2C).

The coefficients c and $|b|$ defined by Eqs. (5.30) and (5.31) may be expressed in terms of the classical amplification factor $e^{2\chi}$ through Eqs. (2.22):

$$c = (\cosh 2\chi \sinh\beta\omega_- + \cosh\beta\omega_-)^{-1}, \quad (5.35)$$

and

$$|b|/c = \sinh 2\chi \sinh\beta\omega_-. \quad (5.36)$$

Accordingly, the diagonal matrix element of Eq. (5.32) gives

$$\langle n'+|\rho_-(\beta)|n'+\rangle = 2\sinh\tfrac{1}{2}\beta\omega_-(\cosh 2\chi\sinh\beta\omega_- + \cosh\beta\omega_-)^{-n'-1/2} \times(1-\sinh^2 2\chi\sinh^2\beta\omega_-)^{n'/2}P_{n'}[(1-\sinh^2 2\chi\sinh^2\beta\omega_-)^{-1/2}], \quad (5.37)$$

which is the result (1.21) quoted in Sec. I. To obtain a form which is useful for large amplification where only the final levels with $n'\gg 1$ are significantly occupied, we use a standard formula[13] to express the Legendre function in terms of a pair of hypergeometric functions and obtain

$$P(n';\beta) = \frac{2\sinh(\frac{1}{2}\beta\omega_-)}{(2\pi\sinh 2\chi\sinh\beta\omega_-)^{1/2}}\frac{\Gamma(n'+1)}{\Gamma(n'+\frac{3}{2})}(\cosh 2\chi\sinh\beta\omega_- + \cosh\beta\omega_-)^{-n'-1/2} \times\left[(\sinh 2\chi\sinh\beta\omega_- + 1)^{n'+1/2}F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2}+n';\frac{1}{2}+\frac{1}{2\sinh 2\chi\sinh\beta\omega_-}\right) + (-1)^{n'}(\sinh 2\chi\sinh\beta\omega_- - 1)^{n'+1/2}F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2}+n';\frac{1}{2}-\frac{1}{2\sinh 2\chi\sinh\beta\omega_-}\right)\right]. \quad (5.38)$$

In the limit $n'\to\infty$, $F(\frac{1}{2},\frac{1}{2};\frac{3}{2}+n';z)\to 1$ and Eq. (5.38) produces the result (1.27) quoted in Sec. I.

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[1]H. G. Dehmelt (private communication); H. G. Dehmelt and G. Gabrielse, Bull. Am. Phys. Soc. 24, 5 (1979).
[2]This experiment is described by R. S. Van Dyck, Jr., P. B. Schwinberg, and H. G. Dehmelt in *New Frontiers in High-Energy Physics*, edited by B. Kursunoglu, A. Perlmutter, and L. F. Scott (Plenum, New York, 1978).
[3]Less elementary accounts appear in C. Kittel, W. D.

Knight, and M. A. Ruderman, *Mechanics*, *Berkeley Physics Course* (McGraw-Hill, New York, 1965), Vol. 1, p. 229; L. D. Landau and E. M. Lifshitz, *Mechanics*, *Course of Theoretical Physics*, translated by J. B. Sykes and J. S. Bell (Pergamon, Oxford, 1960), Vol. 1, Sec. 27.

[4]We first learned of a special case of Eq. (1.7) from H. G. Dehmelt (private communication).

[5]The techniques we use are described in J. Schwinger, *Quantum Kinematics* (Benjamin, New York, 1970), pp. 125–140. See also, R. J. Glauber, Phys. Rev. 131, 2766 (1963).

[6]L. S. Brown, D. I. Fivel, B. W. Lee, and R. F. Sawyer, Ann. Phys. (N.Y.) 23, 187 (1963).

[7]Properties of the Legendre functions are discussed, for example, in *Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1, Chap. III.

[8]Note that Eq. (3.8) may be employed to compute expectation values of energy fluctuations. In particular, with an initial energy eigenstate we obtain

$$(\Delta E^2)^f = \langle H^2\rangle^f - (\langle H\rangle^f)^2$$
$$= (\omega_+^2/\omega_-^2)[\tfrac{1}{2}(E_n^i)^2 + \tfrac{3}{8}\omega_-^2]\sinh^2 2\chi \,.$$

A similar result holds for the ease of an initial thermal ensemble. There the squared energy fluctuation is the sum of two terms. One is the energy dispersion given above with $(E_n^i)^2$ replaced by the ensemble average $\overline{(E_\beta^i)^2}$. The other is the energy dispersion of the original ensemble, ΔE_β^2, multiplied by the amplification factor $\omega_+^2/\omega_-^2 \cosh^2 2\chi$.

[9]Equations (3.15) may be used in conjunction with Eq. (3.8) to compute the final expectation value of the energy when there is an initial coherent state. The quantum expectation value, $\langle z^* t_1 | H(t_2) | z t_1\rangle / \langle z^* t_1 | z t_1\rangle$, differs only slightly from the classical result given in Eq. (2.18), with one factor of E^i being replaced by $E^i - \frac{1}{2}\hbar\omega_-$. Here we have displayed Planck's constant $\hbar$ so as to make it manifest that the coherent-state quantum expectation value approaches the classical result when $\hbar \to 0$.

[10]The method which we have used here is that of J. Schwinger, Ref. 5.

[11]See Ref. 7, p. 129, Eq. (24).

[12]W. Miller, Jr., *Lie Theory and Special Functions* (Academic, New York, 1968), Chap. 5. We thank B. Simon for bringing this monograph to our attention.

[13]See Ref. 7, p. 142, Eq. (21).