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Effect of cooperative atomic interactions on photon statistics in a single-mode laser

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The effect of cooperative atomic interactions on the photon statistics in a single-mode laser is studied on the basis of an equation of motion for the reduced-density operator of the field that was recently derived by Huang and Mandel. The corresponding antinormal ordering distribution function is shown to satisfy a Fokker-Planck equation. The steady-state solution of this equation is used to determine the photon-number distribution, the average intensity, and the intensity fluctuation.

I. INTRODUCTION

The study of the cooperative effects in a laser has been the subject of several recent investigations.[1,2] Huang and Mandel[1] studied the effects of cooperative atomic interactions on photon statistics in a single-mode laser on the basis of a model similar to the laser model of Scully and Lamb.[3] Unlike Scully and Lamb,[3] who made use of the Weisskopf-Wigner method, Huang and Mandel[1] employed a perturbation expansion[4] to derive an equation of motion for the reduced-density operator $\hat{\rho}_F$ of the field. They evaluated the expectation values of the light intensity $\langle\hat{I}(\vec{r})\rangle$ and the intensity fluctuation $\langle(\Delta\hat{I})^2\rangle/\langle\hat{I}\rangle^2$ in the steady state by first solving the equation for the diagonal element $p(n)=\langle n|\hat{\rho}_F|n\rangle$ of the field-density operator. Because of the presence of the terms representing the cooperative effects, the equation for $p(n)$ in the steady state does not satisfy the condition for detailed balance, and it is difficult to obtain exact solution of this equation. Huang and Mandel[1] made some approximations to simplify the equation for $p(n)$ and reduced it to the form of the Scully-Lamb equation of motion.

In this paper we evaluate the quantities $p(n)$, $\langle\hat{I}(\vec{r})\rangle$, and $\langle(\Delta\hat{I})^2\rangle/\langle\hat{I}\rangle^2$ by the use of the equation of motion for the reduced-density operator ρ_F [Eq. (23), Ref. 1] without any approximations. In Sec. II we derive an equation for the antinormal ordering distribution function.[5,6] The advantage of using the distribution function associated with the antinormal ordering rather than the more commonly used P representation[7] is that the resulting equation has in the present case the form of a Fokker-Planck equation. We solve this equation under steady-state conditions. In Sec. III the probability $p(n)$ for n photons in the field is determined and the expectation values of the light intensity and the intensity fluctuation are evaluated by taking the appropriate moments with respect to the antinormal ordering distribution function. In Sec. IV the photon statistics are studied on the basis of the equation of motion for $p(n)$.

II. ANTINORMAL ORDERING DISTRIBUTION FUNCTION

The laser model employed in Ref. 1 consists of a coupled system of a field and identical two-level atoms. It is assumed that N_2 atoms pumped to upper level at a time t are removed from the system at time $t+T_2$ after interacting with the field during an effective atomic lifetime T_2; during this lifetime they give their total contribution to the field. As in the Scully-Lamb theory, the cavity losses are simulated in a corresponding manner by assuming that groups of N_1 atoms are injected at random times in the lower level so as to absorb energy from the laser field making a transition to upper level during another atomic lifetime T_1. A course-grained rate of change of the reduced-density operator $\hat{\rho}_F$ is found by multiplying the change due to N_2 (N_1) atoms introduced in the upper (lower) level by the rate R_2 (R_1) at which the atoms are introduced in the upper (lower) level and by adding the terms associated with gains and losses. The gain term is calculated to the fourth order in the coupling constant, so as to retain the essential nonlinearity required for a steady state, at least for one not too far above the laser threshold. The loss term is calculated to the second order in the coupling constant, so as to simulate losses proportional to the light intensity. The resulting master equation [Eq. (23), Ref. 1] is

$$\frac{d\hat{\rho}_F}{dt}=-\tfrac{1}{2}A[\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)+\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger-2\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}]-\tfrac{1}{2}C[\hat{a}^\dagger\hat{a}\hat{\rho}_F(t)+\hat{\rho}_F(t)\hat{a}^\dagger\hat{a}-2\hat{a}\hat{\rho}_F(t)\hat{a}^\dagger]$$
$$+\tfrac{1}{8}B[\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)+\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger+6\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger-4\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger\hat{a}-4\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}]$$

$$+\tfrac{1}{8}D[\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)+\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger+6\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger-4\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger\hat{a}-4\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}-2\hat{a}^2\hat{a}^{\dagger 2}\hat{\rho}_F(t)+2\hat{\rho}_F(t)\hat{a}^2\hat{a}^{\dagger 2}$$
$$+12\hat{a}^{\dagger 2}\rho_F(t)\hat{a}^2-8\hat{a}^\dagger\rho_F(t)\hat{a}^2\hat{a}^\dagger-8\hat{a}\hat{a}^{\dagger 2}\hat{\rho}_F(t)\hat{a}], \quad (1)$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the destruction and creation operators for the field and

$$A=(fT_2)^2R_2\sum_{i=1}^{N_2}|U(\vec{r}_i)|^2, \quad (2a)$$

$$B=\tfrac{1}{3}(fT_2)^4R_2\sum_{i=1}^{N_2}|U(\vec{r}_i)|^4, \quad (2b)$$

$$C=(fT_1)^4R_1\sum_{i=1}^{N_1}|U(\vec{r}_i)|^2, \quad (2c)$$

$$D=\tfrac{1}{3}(fT_2)^4R_2\sum_{i=1,\, i\neq k}^{N_2}\sum_{k=1}^{N_2}|U(\vec{r}_i)|^2|U(r_k)|^2. \quad (2d)$$

In Eqs. (2a)–(2d), $U(\vec{r}_i)$ is the mode function of the field at the position of the ith atom and

$$f=-\vec{\mu}\cdot\vec{\epsilon}\,(\omega/2\hbar\epsilon_0)^{1/2}, \quad (3)$$

where $\vec{\mu}$ is the transition dipole moment of the atom, $\vec{\epsilon}$ is the unit polarization vector of the field mode, ω is the frequency of the fundamental cavity mode, and ϵ_0 is the electric permittivity.

The coefficients A, B, and C play the roles of gain, saturation, and loss parameters, respectively. C is related to the cavity loss parameter $\mathcal{L}$, which represents the fractional loss per cavity transit by the equation

$$C=c\mathcal{L}/2\pi l. \quad (4)$$

Here l is the cavity length and c is the speed of light *in vacuo*. The parameter D is associated with the cooperative atomic interactions.

We now derive an equation of motion for the antinormal ordering distribution function, which may be defined by the formula[5,6]

$$\Phi(v,v^*)=(1/\pi)\langle v|\hat{\rho}_F|v\rangle. \quad (5)$$

Here $|v\rangle$ is a coherent state.[7] We recall some properties of the coherent states which we will need later:

$$|v\rangle=e^{(-1/2)|v|^2}e^{va^\dagger}|0\rangle, \quad (6)$$

$$\hat{a}|v\rangle=v|v\rangle, \quad \langle v|\hat{a}^\dagger=v^*\langle v|, \quad (7)$$

For any operator $\hat{O}$

$$\langle v|\hat{O}\hat{a}^\dagger|v\rangle=e^{-|v|^2}\langle 0|e^{v^*\hat{a}}\hat{O}\,\hat{a}^\dagger e^{va^\dagger}|0\rangle$$
$$=e^{-|v|^2}\frac{\partial}{\partial v}\langle 0|e^{v^*\hat{a}}\hat{O}\,e^{v\hat{a}^\dagger}|0\rangle$$
$$=\left(v^*+\frac{\partial}{\partial v}\right)e^{-|v|^2}\langle 0|e^{v^*\hat{a}}\hat{O}\,e^{v\hat{a}^\dagger}|0\rangle$$
$$=\left(v^*+\frac{\partial}{\partial v}\right)\langle v|\hat{O}|v\rangle; \quad (8a)$$

similarly

$$\langle v|\hat{a}\hat{O}|v\rangle=\left(v+\frac{\partial}{\partial v^*}\right)\langle v|\hat{O}|v\rangle. \quad (8b)$$

The expectation value of any antinormally ordered function $F_A(\hat{a},\hat{a}^\dagger)$ of $\hat{a}$ and $\hat{a}^\dagger$ may be determined from $\Phi(v,v^*)$ via the relation[5,6]

$$\langle F_A(\hat{a},\hat{a}^\dagger)\rangle=\int F_A(v,v^*)\Phi(v,v^*)d^2v. \quad (9)$$

In particular

$$\langle\hat{a}\hat{a}^\dagger\rangle=\int|v|^2\Phi(v,v^*)d^2v, \quad (10)$$

$$\langle\hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger\rangle=\int|v|^4\Phi(v,v^*)d^2v. \quad (11)$$

The function $\Phi(v,v^*)$ thus makes it possible to evaluate quantum-mechanical expectation values of the antinormally ordered functions using the methods of classical statistical mechanics. Furthermore it can be shown by using Eqs. (5) and (6) that the probability

$$p(n)=\langle n|\hat{\rho}_F|n\rangle \quad (12)$$

that there are n photons in the field can be determined from a knowledge of $\Phi(v,v^*)$ by the relation

$$p(n)=\frac{\pi}{n!}\left(\frac{\partial^{2n}}{\partial v^n\partial v^{*n}}[\Phi(v,v^*)e^{|v|^2}]\right)_{v=v^*=0}. \quad (13)$$

We now take the expectation values of Eq. (1) with respect to coherent states. It follows that, in view of Eqs. (5), (7), and (8), $\Phi(v,v^*)$ obeys the equation

$$\frac{\partial\Phi}{\partial t}=\frac{\partial}{\partial v}\left((-4A+4C-B-D+4B|v|^2)\frac{v}{8}\Phi\right)$$
$$+\frac{\partial^2}{\partial v^2}\left((B+3D)\frac{v^2}{8}\Phi\right)$$
$$+\frac{\partial^2}{\partial v\partial v^*}\left[\left((B+D)\frac{3|v|^2}{8}+\frac{C}{2}\right)\Phi\right]+\text{c.c.} \quad (14)$$

Next we express v in terms of its real and imaginary parts, i.e.,

$$v = x_1 + ix_2,\ v^* = x_1 - ix_2. \tag{15}$$

We then have

$$\frac{\partial}{\partial v} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right),\ \frac{\partial}{\partial v^*} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right). \tag{16}$$

We will denote by $F(x_1, x_2)$ the function $\Phi(v, v^*)$ when expressed in terms of the real variables x_1 and x_2. It then follows from Eq. (14) that $F(x_1, x_2)$ satisfies the following Fokker-Planck equation:

$$\frac{\partial F}{\partial t} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(A_i x_i + \sum_{j=1}^{2} B_{ij}\frac{\partial}{\partial x_j}\right)F, \tag{17}$$

where

$$A_{1,2} = \tfrac{1}{2}[-A + C + B + 2D + B(x_1^2 + x_2^2)], \tag{18a}$$

$$B_{11} = \tfrac{1}{16}[(B + 3D)(x_1^2 - x_2^2) + 3(B + D)(x_1^2 + x_2^2) + 4C], \tag{18b}$$

$$B_{12} = B_{21} = \tfrac{1}{8}(B + 3D)x_1 x_2, \tag{18c}$$

$$B_{22} = \tfrac{1}{16}[(B + 3D)(x_2^2 - x_1^2) + 3(B + D)(x_1^2 + x_2^2) + 4C]. \tag{18d}$$

In the steady state, $\partial F/\partial t = 0$ and it follows from Eq. (17) that under these circumstances

$$\left(A_i x_i + \sum_{j=1}^{2} B_{ij}\frac{\partial}{\partial x_j}\right)F = 0, \quad i = 1, 2. \tag{19}$$

This pair of partial differential equations is equivalent to the pair of equations

$$\frac{\partial F}{\partial x_1} = \left(\frac{A_1 B_{22} x_1 - A_2 B_{12} x_2}{B_{12}B_{21} - B_{22}B_{11}}\right)F, \tag{20a}$$

$$\frac{\partial F}{\partial x_2} = \left(\frac{A_2 B_{11} x_2 - A_1 B_{21} x_1}{B_{21}B_{12} - B_{11}B_{22}}\right)F. \tag{20b}$$

On substituting for A_i and B_{ij} from Eqs. (18a)–(18d), we find that

$$\frac{\partial F}{\partial x_i} = 4\left(\frac{A - C - B - 2D - B(x_1^2 + x_2^2)}{2C + (2B + 3D)(x_1^2 + x_2^2)}\right)x_i F, \quad i = 1, 2. \tag{21}$$

The solution of Eq. (21) is given by

$$F(x_1, x_2) = \text{const}\,(\gamma + x_1^2 + x_2^2)^{\beta + \alpha\gamma - 1} e^{-\alpha(x_1^2 + x_2^2)}, \tag{22}$$

where

$$\alpha = \frac{2B}{2B + 3D}, \tag{23a}$$

$$\beta = \frac{2A - 2C - D}{2B + 3D}, \tag{23b}$$

$$\gamma = \frac{2C}{2B + 3D}. \tag{23c}$$

It is convenient to introduce a new variable q defined by

$$q = |v| = (x_1^2 + x_2^2)^{1/2}. \tag{24}$$

We will denote by $P(q)$ the distribution function $F(x_1, x_2)$ when expressed as a function of q rather than of x_1 and x_2, i.e., $P(q) \equiv F(x_1, x_2)$. Solution (22) then becomes

$$P(q) = (1/2\pi N)(\gamma + q^2)^{\beta + \alpha\gamma - 1} e^{-\alpha q^2}, \tag{25}$$

where

$$N = \int_0^\infty (\gamma + q^2)^{\beta + \alpha\gamma - 1} e^{-\alpha q^2} q\, dq = (e^{\alpha\gamma}/2\alpha^{\beta + \alpha\gamma})\Gamma(\beta + \alpha\gamma, \alpha\gamma) \tag{26}$$

is the normalization constant. The truncated Γ function, which appears in Eq. (26), viz.,

$$\Gamma(b, z) = \int_z^\infty p^{b-1} e^{-p}\, dp, \tag{27}$$

has the following properties[8]:

$$\frac{\partial \Gamma(b, z)}{\partial z} = -z^{b-1} e^{-z}, \tag{28a}$$

$$\Gamma(b + 1, z) = b\Gamma(b, z) + z^b e^{-z}, \tag{28b}$$

$$\Gamma(b, z) = z^b e^{-z} \psi(1, 1 + b, z), \tag{28c}$$

where

$$\psi(1, 1 + b, z) = \int_0^\infty e^{-zt}(1 + t)^{b-1}\, dt \tag{29}$$

is the degenerate hypergeometric function.

III. PHOTON STATISTICS

The probability $p(n)$ for n laser photons in the cavity can now be determined from Eq. (13). It follows that, on substituting from Eqs. (24), (25), (26), and (28c) into Eq. (13),

$$p(n) = \frac{1}{n!\,\gamma^{\beta + \alpha\gamma}\psi(1, \beta + \alpha\gamma + 1, \alpha\gamma)}\left(\frac{\partial^{2n}}{\partial v^n \partial v^{*n}}[(\gamma + vv^*)^{\beta + \alpha\gamma - 1} e^{(1-\alpha)vv^*}]\right)_{v = v^* = 0} = \frac{1}{\gamma\psi(1, \beta + \alpha\gamma + 1, \alpha\gamma)} \sum_{m=0}^{n} \binom{n}{m} (1 - \alpha)^{n-m} \prod_{r=1}^{m} \frac{\beta + \alpha\gamma - r}{\gamma}. \tag{30}$$

An equivalent expression for $p(n)$ can be obtained directly from the equation of motion for $p(n)$. This will be shown in Sec. IV. It is clear from Eq. (30) that $p(n)$ becomes negative for large values of n, namely, for $n > \beta + \alpha\gamma$. This is a consequence of using the perturbation expansion in the equation of motion for the reduced-density operator $\hat{\rho}_F$. This difficulty can be avoided by letting $\beta + \alpha\gamma$ have an integer value.

In the special case when $D = 0$ we have, from Eqs. (23a)–(23c), $\alpha = 1$, $\beta = (A - C)/B$, and $\gamma = C/B$. According to Eq. (30), $p(n)$ is then given by

$$p(n) = \frac{1}{(C/B)\psi(1, 1 + A/B, C/B)} \prod_{r=1}^{n} \frac{(A/B - r)}{C/B}, \tag{31}$$

which is in agreement with the result obtained by Scully and Lamb.[3]

We now evaluate the expectation values of the optical intensity and the intensity fluctuation. With the help of Eq. (25) it can be shown that

$$\begin{aligned}\langle aa^\dagger\rangle &= 2\pi \int_0^\infty q^2 P(q) q\, dq \\ &= \frac{1}{N}\int_0^\infty (\gamma + q^2)^{\beta+\alpha\gamma-1} e^{-\alpha q^2} q^3\, dq \\ &= \frac{\beta}{\alpha} + \frac{1}{\alpha\psi(1, \beta+\alpha\gamma+1, \alpha\gamma)},\end{aligned} \tag{32}$$

$$\begin{aligned}\langle aaa^\dagger a^\dagger\rangle &= 2\pi \int_0^\infty q^4 p(q) q\, dq \\ &= \frac{1}{N}\int_0^\infty (\gamma + q^2)^{\beta+\alpha\gamma-1} e^{-\alpha q^2} q^5\, dq \\ &= \frac{1}{\alpha}\left(\beta^2 + \beta + \alpha\gamma + \frac{\beta+1}{\psi(1, \beta+\alpha\gamma+1, \alpha\gamma)}\right).\end{aligned} \tag{33}$$

In deriving Eqs. (32) and (33) use has been made of Eqs. (26), (28), and (29). For the single-mode laser field we therefore obtain

$$\langle \hat{I}(\vec{r})\rangle = (\hbar\omega/2\epsilon_0)|U(\vec{r})|^2\langle \hat{a}^\dagger\hat{a}\rangle = (\hbar\omega/2\epsilon_0)|U(\vec{r})|^2(\langle\hat{a}\hat{a}^\dagger\rangle - 1) = (\hbar\omega/2\alpha\epsilon_0)|U(\vec{r})|^2\{\beta - \alpha + [\psi(1, \beta+\alpha\gamma+1, \alpha\gamma)]^{-1}\}, \tag{34}$$

$$\begin{aligned}\frac{\langle(\Delta\hat{I}(\vec{r}))^2\rangle}{\langle\hat{I}(\vec{r})\rangle^2} &= \frac{\langle(\Delta\hat{n})^2\rangle}{\langle\hat{n}\rangle^2} = \frac{\langle\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle^2} - 1 = \frac{1}{(\langle\hat{a}\hat{a}^\dagger\rangle - 1)^2}(\langle\hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger\rangle - \langle\hat{a}\hat{a}^\dagger\rangle^2 - \langle\hat{a}\hat{a}^\dagger\rangle) \\ &= |\beta - \alpha + [\psi(1, \beta+\alpha\gamma+1, \alpha\gamma)^{-1}]^{-2}[\beta + \alpha\gamma - \alpha\beta - (\alpha+\beta-1) \\ &\quad\times[\psi(1, \beta+\alpha\gamma+1, \alpha\gamma)]^{-1} - [\psi(1, \beta+\alpha\gamma+1, \alpha\gamma)]^{-2}].\end{aligned} \tag{35}$$

Now from Eq. (30) it follows that

$$\psi(1, \beta+\alpha\gamma+1, \alpha\gamma) = 1/\gamma p(0), \tag{36}$$

and hence we can write Eqs. (34) and (35) in the following alternative form:

$$\langle\hat{I}(\vec{r})\rangle = \frac{\hbar\omega}{2\alpha\epsilon_0}|U(\vec{r})|^2[\beta - \alpha + \gamma p(0)], \tag{37}$$

$$\begin{aligned}\frac{\langle[\Delta\hat{I}(\vec{r})]^2\rangle}{\langle\hat{I}(\vec{r})\rangle^2} &= [\beta - \alpha + \gamma p(0)]^{-2} \\ &\quad\times[\beta + \alpha\gamma - \alpha\beta - \gamma(\alpha+\beta-1)p(0) - \gamma^2 p(0)].\end{aligned} \tag{38}$$

These expressions for $\langle\hat{I}(\vec{r})\rangle$ and $\langle[\Delta\hat{I}(\vec{r})]^2\rangle/\langle\hat{I}(\vec{r})\rangle^2$ are direct consequences of Eq. (1).

It follows from Eqs. (30), (37), and (38) that the quantities $p(n)$, $\langle\hat{I}(\vec{r})\rangle$, and $\langle[\Delta\hat{I}(\vec{r})]^2\rangle/\langle\hat{I}(\vec{r})\rangle^2$ depend on A, B, C, and D through the parameters α, β, and γ. It is thus apparent from Eqs. (23a)–(23c) that the effect of the cooperative atomic interactions associated with the parameter D on photon statistics in a single-mode laser can be neglected within the accuracy of Eq. (1) if $D \ll B$.

When the laser is operating sufficiently high above threshold, $p(0) \to 0$. Furthermore $\beta \gg \alpha$. We then obtain

$$\begin{aligned}\langle\hat{I}(\vec{r})\rangle &\simeq \frac{\hbar\omega}{2\alpha\epsilon_0}|U(\vec{r})|^2\beta \\ &= \frac{\hbar\omega}{2\epsilon_0}|U(\vec{r})|^2\left(\frac{2A - 2C - D}{2B}\right),\end{aligned} \tag{39}$$

$$\begin{aligned}\frac{\langle(\Delta\hat{I}(\vec{r}))^2\rangle}{\langle\hat{I}(\vec{r})\rangle^2} &\simeq \frac{\beta + \alpha\gamma - \alpha\beta}{\beta^2} \\ &= \frac{3D}{2A - 2C - D} + \frac{4BC}{(2A - 2C - D)^2}.\end{aligned} \tag{40}$$

It is evident from Ref. 1 that for a small inhomogeneously broadened He : Ne laser the following inequality holds:

$$A \gg D. \tag{41}$$

Under this condition, Eqs. (39) and (40) further simplify and we obtain

$$\langle\hat{I}(\vec{r})\rangle \simeq \frac{\hbar\omega}{2\epsilon_0}|U(\vec{r})|^2\left(\frac{A - C}{B}\right), \tag{42}$$

$$\frac{\langle[\Delta \hat{I}(\vec{r})]^2\rangle}{\langle \hat{I}(\vec{r})\rangle^2} \simeq \frac{3D}{2(A-C)} + \frac{BC}{(A-C)^2} . \tag{43}$$

It is clear from these equations that there is no significant effect of cooperative atomic interactions on the form of the average intensity of the laser operating sufficiently high above threshold. However, when $D \gtrsim BC/(A-C)$, the intensity fluctuations depend upon the cooperative effect parameter D.

IV. PHOTON STATISTICS FROM EQUATION OF MOTION FOR $p(n)$

In this section we derive the expression for $p(n)$ on the basis of the equation of motion for $p(n)$. If we calculate the matrix element between Fock states $\langle n|$ and $|n\rangle$ of each term in Eq. (1) and make use of the fact that $p(n) = \langle n|\hat{\rho}_F|n\rangle$, we obtain the equation

$$\frac{\partial p(n)}{\partial t} = c_{n+1} p(n+1) + a_n p(n) + b_{n-1} p(n-1) + d_{n-2} p(n-2) , \tag{44}$$

where

$$a_n = -A(n+1) + B(n+1)^2 - Cn + \tfrac{1}{2} D[2(n+1)^2 + (n+1)(n+2)] , \tag{45a}$$

$$b_n = A(n+1) - B(n+1)^2 - D[(n+1)^2 + (n+1)(n+2)] , \tag{45b}$$

$$c_n = Cn , \tag{45c}$$

$$d_n = 3D(n+1)(n+2) . \tag{45d}$$

In matrix notation the steady-state equation (with $\partial p_n/\partial t = 0$) may be written in the form

$$\begin{pmatrix} a_0 & c_1 & 0 & & \\ b_0 & a_1 & c_2 & & \\ d_0 & b_1 & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \\ 0 & \cdot & \cdot & & \end{pmatrix} \begin{pmatrix} p(0) \\ p(1) \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = 0 . \tag{46}$$

It can be easily proved by the method of induction that the solution of this equation is

$$p(n) = \frac{(-1)^n p(0)}{c_1 \times \cdots \times c_n} M(n) , \tag{47}$$

where

$$M(n) = \det \begin{pmatrix} a_0 & c_1 & & & 0 \\ b_0 & \cdot & \cdot & & \\ d_0 & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & c_{n-1} \\ 0 & & d_{n-3} & b_{n-2} & a_{n-1} \end{pmatrix} . \tag{48}$$

$p(0)$ is determined by the normalization condition. By using the properties of the determinant, this solution can be reduced to the same form as Eq. (30).

It is worthwhile to mention that in Eq. (44) $p(n)$ is related to $p(n-1)$ and $p(n-2)$, which represent one-photon and two-photon emission processes. If an exact equation for $p(n)$ is derived, using the Wigner-Weisskopf procedure, then we would expect $p(n)$ to be related to $p(n-1), p(n-2), \ldots, p(n-N_2)$. The equation of motion for $p(n)$ would then be of the form

$$\frac{\partial p(n)}{\partial t} = c_{n+1} p(n+1) + \sum_{m=0}^{N_2} d_{n-m}^{(m)} p(n-m) , \tag{49}$$

with $d_{-m}^{(m)} = d_{-m+1}^{(m)} = \cdots = d_{-1}^{(m)} = 0, m \geq 1$. The formal steady-state solution of this equation is

$$p(n) = \frac{(-1)^n p(0)}{c_1 \times \cdots \times c_n} M'(n) , \tag{50}$$

where

$$M'(n) = \det \begin{pmatrix} d_0^{(0)} & c_1 & & & 0 \\ d_0^{(1)} & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ \vdots & \cdot & & & \\ d_0^{(N_2)} & & & & \\ & \cdot & & & c_{n-1} \\ 0 & & d_{n-N_2-1}^{(N_2)} & d_{n-2}^{(1)} & d_{n-1}^{(0)} \end{pmatrix} . \tag{51}$$

The usefulness of this solution may be limited in many practical situations in which one is interested in the evaluation of $p(n)$ for large values of n. Some approximations to expression (50) may then be needed to simplify it.

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[1] C.-Y. Huang and L. Mandel, Phys. Rev. A 18, 644 (1978).
[2] F. Casagrande and R. Cordoni, Phys. Rev. A 18, 1628 (1978).
[3] M. O. Scully and W. E. Lamb, Jr., Phys. Rev. 159, 208 (1967).
[4] M. O. Scully, in *Quantum Optics* (Italian Physical Society: Course 42), edited by R. J. Glauber, (Academic, New York, 1970), p. 586.
[5] Y. Kano, J. Math. Phys. 6, 1913 (1965).
[6] C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. 138, B274 (1965).
[7] R. J. Glauber, Phys. Rev. 131, 2766 (1963).
[8] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1965), pp. 934, 942, and 1058.