Mode competition in a homogeneously broadened ring laser

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The equations previously derived for the intensity fluctuations of a two-mode ring laser are applied to a homogeneously broadened laser. It is shown that the mean light intensity of the more lossy mode passes through a maximum and then tends to zero as the excitation is increased. The probability distribution of the intensity of each mode may exhibit two peaks, and spontaneous switching between the intensities associated with the peaks may occur. This is manifest in relative intensity fluctuations of the more lossy mode that can be much greater than unity. Estimates are given for the characteristic switching time.

I. INTRODUCTION

The ring laser has been the subject of a great deal of attention in recent years, not only because of its potential application as a gyroscope, but also because it exhibits strong laser mode-competition effects. When the laser cavity is tuned close to the line center, the two counterpropagating traveling-wave modes compete partly for the same excited atomic population, in such a way that anticorrelations and several new phenomena appear. The statistical theory of the process has recently been discussed in several papers,[1-3] and a theoretical treatment of the inhomogeneously broadened two-mode ring laser has now been given[3] that is as complete as that of the usual single-mode laser. Moreover, it has also been demonstrated experimentally[4] that in an inhomogeneously broadened ring laser one mode may suppress the growth of the other one, and one mode may become highly coherent, while the other one becomes incoherent and obeys thermal statistics. Obviously, these characteristics need to be taken into account in the design of laser gyros. When the ring laser is homogeneously broadened, it is capable of exhibiting a still wider variety of unusual features, including spontaneous switching between states, as manifest, for example, in very large relative intensity fluctuations of one mode. Although the possibility of bistability and switching in such a ring laser has been noted previously by several other authors,[5] these treatments were deterministic, and therefore did not deal with the coherence and fluctuation properties of the emitted light.

II. EQUATIONS OF MOTION

The starting point for our treatment of the two-mode ring laser is the pair of coupled Langevin equations of motion for the two dimensionless, complex mode amplitudes E_1 and E_2,

$$\begin{aligned}\frac{dE_1}{dt}&=(a_1-|E_1|^2-\xi|E_2|^2)E_1+q_1(t)\\ \frac{dE_2}{dt}&=(a_2-|E_2|^2-\xi|E_1|^2)E_2+q_2(t)\,.\end{aligned}\tag{1}$$

a_1, a_2 are the so-called pump parameters of the two laser modes, which are negative below threshold and positive above threshold. $q_1(t)$, $q_2(t)$ are complex Langevin noise terms representing spontaneous emission fluctuations. They are taken to be statistically independent, δ correlated, and Gaussian, with

$$\left.\begin{aligned}&\langle q_1^*(t)q_2(t')\rangle=0\\ &\langle q_1^*(t)q_1(t')\rangle=2\delta(t-t')=\langle q_2^*(t)q_2(t')\rangle\end{aligned}\right\}\tag{2}$$

ξ is the mode coupling constant. It has been shown[5-7] that, when the ring laser is inhomogeneously broadened, ξ depends on the detuning $\Delta\omega$ of the laser cavity from the atomic line center, and is given by

$$\xi=1/[1+(\Delta\omega T_1)^2]\tag{3}$$

where T_1 is the natural lifetime of the atomic transition. The maximum coupling constant ξ is therefore unity for such a ring laser. However, it may readily be shown from the analyses of Sargent *et al.*[6] and Hambenne and Sargent[5] that for a homogeneously broadened ring laser ξ can be 2, and this larger value of the coupling has important implications for the solutions of the equations of motion.

To the set of coupled Langevin equations there corresponds a Fokker-Planck equation for the joint probability density $p(E_1, E_2, t)$ of the two-mode amplitudes E_1, E_2. If we write $E_1\equiv x_1+ix_2$, $E_2\equiv x_3+ix_4$, this takes the form

$$\frac{\partial p}{\partial t}=-\sum_{i=1}^{4}\frac{\partial}{\partial x_i}(A_ip)+\sum_i\frac{\partial^2 p}{\partial x_i{}^2}\,,\tag{4}$$

in which the drift coefficients A_1, A_2, A_3, A_4 are simple polynomials in the x's.[3]

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III. STEADY-STATE SOLUTION

Although the general time-dependent solution of Eq. (4) is complicated,[3] the steady-state solution is very simple and easily obtained. Because of the assumed nature of the Langevin forces, the phases of $E_1(t)$ and $E_2(t)$ are random and uncorrelated. If we denote the instantaneous intensities of the two laser modes by $I_1 \equiv |E_1|^2$ and $I_2 = |E_2|^2$, then the joint probability density $\mathcal{P}(I_1, I_2)$ of I_1 and I_2 in the steady state is found to be [2,3]

$$\mathcal{P}(I_1, I_2) = Q^{-1}\exp(\tfrac{1}{2}a_1 I_1 + \tfrac{1}{2}a_2 I_2 - \tfrac{1}{4}I_1^2 - \tfrac{1}{4}I_2^2 - \tfrac{1}{2}\xi I_1 I_2)\,, \qquad (5)$$

where Q is a normalizing constant. By integrating this with respect to I_2, say, we immediately obtain the probability distribution $\mathcal{P}_1(I_1)$ of the other light intensity I_1,

$$\mathcal{P}_1(I_1) = \sqrt{\pi}\,Q^{-1}\exp[\tfrac{1}{4}(\xi^2 - 1)I_1^2 - \tfrac{1}{2}(a_2\xi - a_1)I_1 + \tfrac{1}{4}a_2^2] \times [1 - \mathrm{erf}(\tfrac{1}{2}\xi I_1 - \tfrac{1}{2}a_2)]\,. \qquad (6)$$

It is already apparent by inspection of this equation that $\mathcal{P}_1(I_1)$ may display quite different behavior when $\xi < 1$ and when $\xi > 1$. If $\xi = 2$, as for a homogeneously broadened ring laser, it is possible for $\mathcal{P}_1(I_1)$ to exhibit two peaks. For small $I_1 \ll 1$ and for approximately equal a_1, $a_2 > 0$, $\mathcal{P}_1(I_1)$ tends to become a falling exponential function of I_1 given by

$$\mathcal{P}_1(I_1) \simeq 2\sqrt{\pi}\,Q^{-1}\exp[-\tfrac{1}{2}(a_2\xi - a_1)I_1 + \tfrac{1}{4}a_2^2]\,. \qquad (7)$$

On the other hand, when I_1 is sufficiently large that $\xi I_1 - a_2 > 3$, we can use the asymptotic form[8] of the error function to show that

$$\mathcal{P}_1(I_1) \simeq Q^{-1}\exp[-\tfrac{1}{4}(I_1 - a_1)^2 + \tfrac{1}{4}a_1^2]/\tfrac{1}{2}(\xi I_1 - a_2)\,, \qquad (8)$$

which may exhibit a peak at $I_1 \simeq a_1$ when $\xi \approx 2$. The behavior of $\mathcal{P}_1(I_1)$ given by Eq. (6) is illustrated in Fig. 1 for $a_1 = 15$ and for several different values of $\Delta a \equiv a_1 - a_2$. It will be seen that $\mathcal{P}_1(I_1)$ has peaks both at zero and at nonzero light intensities. This behavior is sometimes associated with an internal phase transition.[7,9] The double peaks in the probability distribution suggest the possibility of spontaneous on-off switching, with resulting large relative intensity fluctuations. For the less lossy mode, nonzero values of the light intensity are very much more probable than near zero values, but the opposite may be true for the more lossy mode, so much so that the light in this mode turns on only momentarily.

General expressions derivable from Eq. (5) for the mean light intensities $\langle I_1\rangle$, $\langle I_2\rangle$, the second moments $\langle(\Delta I_1)^2\rangle$, $\langle(\Delta I_2)^2\rangle$ and the cross correlation $\langle \Delta I_1 \Delta I_2\rangle$ in the steady state have already been given,[2,3] and we shall not repeat them here. In

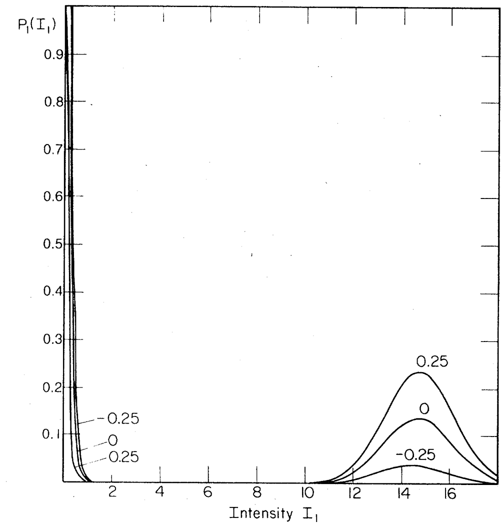

FIG. 1. Examples of the probability distribution $\mathcal{P}_1(I_1)$ for $a_1 = 15$ and for values $\Delta a \equiv a_1 - a_2 = -0.25$, 0, 0.25.

the following, we wish to focus on the behavior of the ring laser when it is homogeneously broadened and the coupling $\xi = 2$. We may readily show from the general expressions for large a_1, with $\Delta a \equiv a_1 - a_2 > 0$ held constant, that

$$\langle I_1\rangle \to a_1 - 4/(a_1 + \Delta a) + O(1/a_1)^3\,, \qquad (9)$$

$$\langle I_2\rangle \to 2/(a_1 + \Delta a) + O(1/a_1)^3\,, \qquad (10)$$

$$\langle(\Delta I_1)^2\rangle/\langle I_1\rangle^2 \to 2/a_1^2 + O(1/a_1)^3\,, \qquad (11)$$

$$\langle(\Delta I_2)^2\rangle/\langle I_2\rangle^2 \to 1 + 12/(a_1 + \Delta a)^2 + O(1/a_1)^4\,, \qquad (12)$$

$$\langle \Delta I_1 \Delta I_2\rangle/\langle I_1\rangle\langle I_2\rangle \to -4/a_1(a_1 + \Delta a) - O(1/a_1)^4\,. \qquad (13)$$

These expressions may be compared with the asymptotic values given by M-Tehrani and Mandel[3] for the inhomogeneously broadened ring laser at line center, for which

$$\langle I_1\rangle \to a_1 - 2/\Delta a\,, \qquad (14)$$

$$\langle I_2\rangle \to 2/\Delta a\,, \qquad (15)$$

$$\langle(\Delta I_1)^2\rangle/\langle I_1\rangle^2 \to 2/a_1^2 + 4/a_1^2(\Delta a)^2\,, \qquad (16)$$

$$\langle(\Delta I_2)^2\rangle/\langle I_2\rangle^2 \to 1\,, \qquad (17)$$

$$\langle \Delta I_1 \Delta I_2\rangle/\langle I_1\rangle\langle I_2\rangle \to -2/a_1\Delta a\,. \qquad (18)$$

One major difference is that in the present case, when $\Delta a \equiv a_1 - a_2 > 0$, $\langle I_2\rangle$ tends asymptotically to zero rather than to a nonzero value. The mode having the slightly greater loss is therefore suppressed altogether in this limit, rather than being merely prevented from growing with increasing excitation. Although the fluctuations of this mode tend to the thermal limit $\langle(\Delta I_2)^2\rangle/\langle I_2\rangle^2 \to 1$ in both cases, they may approach this limit from above in the case of the homogeneously broadened

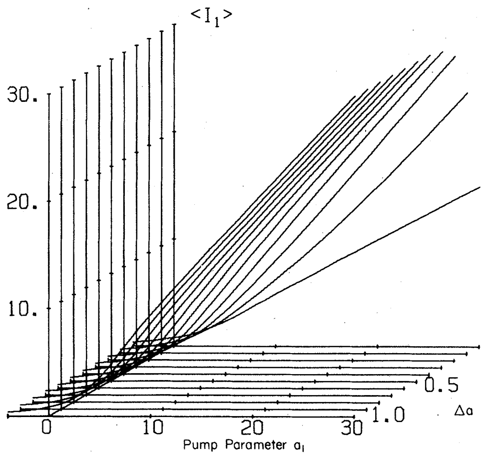

FIG. 2. The variation of the mean light intensity $\langle I_1\rangle$ with pump parameter a_1, for several different values of $\Delta a \equiv a_1 - a_2$.

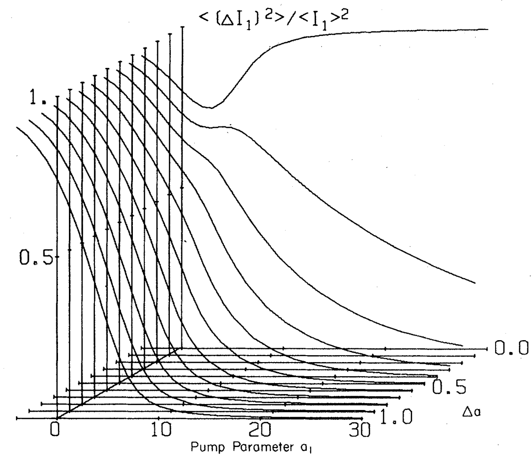

FIG. 4. The variation of the relative mean-squared intensity fluctuation $\langle(\Delta I_1)^2\rangle/\langle I_1\rangle^2$ with pump parameter a_1, for several different values of $\Delta a \equiv a_1 - a_2$.

ring laser, and the fluctuations can be much greater than thermal.

The main features of the behavior of the homogeneously broadened ring laser are illustrated graphically in Figs. 2–6 in terms of the moments of the two light intensities calculated with the help of the steady-state probability distribution given by Eq. (5). Each curve corresponds to a fixed difference $\Delta a \equiv a_1 - a_2$ between the two pump parameters. In practice, the diffraction losses of the two counterpropagating ring-laser modes tend to be slightly different, and the difference between the pump parameters remains approximately constant as the excitation is varied. It will be seen that the less lossy mode grows with increasing excitation, whereas the more lossy mode reaches a maximum intensity and is then gradually suppressed, no matter how small the difference Δa may be. This asymmetry in the behavior of the two modes is even more pronounced than for the inhomogeneously broadened ring laser. The relative intensity fluctuations exhibit another strong asymmetry. Whereas those of the less lossy mode die away to zero with increasing excitation, and the light becomes increasingly coherent, the relative fluctuations of the more lossy mode may pass through a maximum and exceed the value unity that is characteristic of thermal light. Moderately large excitations can produce very large relative intensity fluctuations, although the asymptotic value of $\langle(\Delta I_2)^2\rangle/\langle I_2\rangle^2$ is unity. These large fluctuations reflect the switching[5] that occurs be-

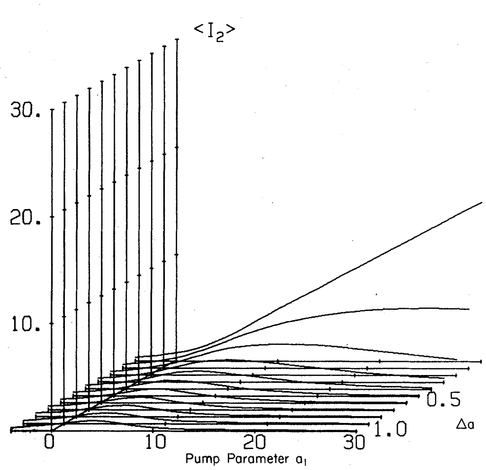

FIG. 3. The variation of the mean light intensity $\langle I_2\rangle$ with pump parameter a_1, for several different values of $\Delta a \equiv a_1 - a_2$.

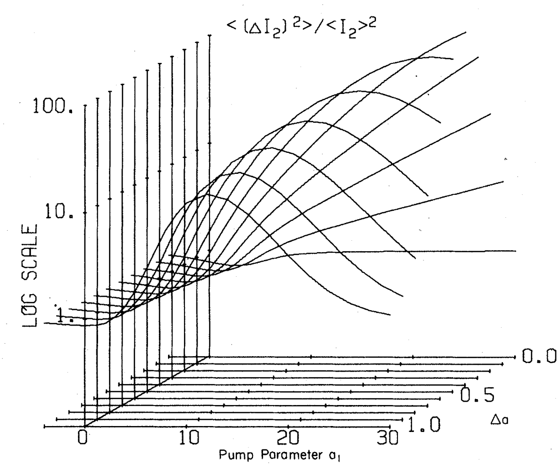

FIG. 5. The variation of the relative mean-squared intensity fluctuation $\langle(\Delta I_2)^2\rangle/\langle I_2\rangle^2$ with pump parameter a_1, for several different values of $\Delta a \equiv a_1 - a_2$.

tween the peaks in the probability distribution of the light intensity at zero and nonzero values. Much of the time, the light intensity of this mode is near zero, but it switches on occasionally. As a result, the light in the more lossy mode fluctuates far more than a thermal light beam. We also observe that the normalized cross correlation of the light intensities is always negative and also passes through a peak, after which it tends to zero with increasing excitation. The numerical peak value can be close to unity, compared with a maximum of $\frac{1}{3}$ for the inhomogeneously broadened ring laser. Negative cross correlations, of course, reflect the competition of the two modes for the same population of excited laser atoms.

Some of the curves corresponding to exactly equal pump parameters, or $\Delta a = 0$, are singular, and are probably unrealizable in practice. In the case of exact symmetry the intensities of both modes grow with increasing excitation, but the fluctuations of both modes revert to the thermal state sufficiently far above threshold, with correlation coefficient -1. More precisely, we find the following asymptotic expressions for the first and second moments of the light intensities when $a_1 = a_2 = a$,

$$\langle I_s\rangle \to \tfrac{1}{2}a - 1/a - O(1/a)^3\,, \quad s = 1,2 \tag{19}$$

$$\langle(\Delta I_s)^2\rangle/\langle I_s\rangle^2 \to 1 - 4/a^2 + O(1/a)^4\,, \quad s = 1,2 \tag{20}$$

$$\langle \Delta I_1 \Delta I_2\rangle/\langle I_1\rangle\langle I_2\rangle \to -1 + 8/a^2 + O(1/a)^4\,, \tag{21}$$

which may be compared with the corresponding limiting values $\frac{1}{2}a$, $\frac{1}{3}$, $-\frac{1}{3}$ for the inhomogeneously broadened, symmetric ring laser on resonance.[3]

IV. THE TIME SCALE OF THE FLUCTUATIONS

Without solving the general time-dependent Fokker-Planck equation, we can make an estimate of the characteristic switching time of the light intensity from its most probable nonzero value to zero. For this purpose we shall make use of the formalism for solving first passage time problems in one dimension.[10] It can be shown that if I is a general random process obeying a Fokker-Planck equation with diffusion rate $D(I)$, and if the steady-state probability distribution $\mathcal{P}(I)$ can be written in the form

$$\mathcal{P}(I) = K\exp[-U(I)]/D(I)\,, \tag{22}$$

in which the so-called "potential" $U(I)$ of the random process has minimum values at $I=0$ and $I=I_{\min}$, with a maximum in between at $I=I_{\max}$, then we can express the average first passage time $T(I)$ for a transition from any value I near $I_{\min}$ to the region associated with the other minimum as a combination of simple integrals. This takes the general form[10]

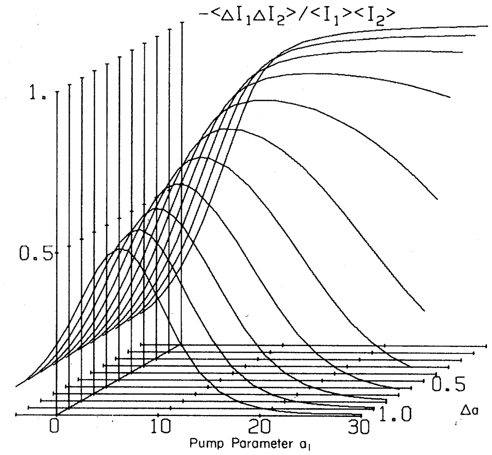

FIG. 6. The variation of the normalized cross correlation $\langle \Delta I_1 \Delta I_2\rangle/\langle I_1\rangle\langle I_2\rangle$ with pump parameter a_1, for several different values of $\Delta a \equiv a_1 - a_2$.

$$T(I) = -2\int_0^I dI' \frac{1}{\mathcal{P}(I')D(I')}\int_0^{I'} dI''\mathcal{P}(I'') + C_1\int_0^I dI' \exp[U(I')] + C_2\,, \tag{23}$$

in which C_1, C_2 are constants to be determined by the boundary conditions. If we suppose that the value $I = I_{\max}$ represents the crossover point separating the two regions, so that T vanishes when $I = I_{\max}$, then

$$0 = -2\int_0^{I_{\max}} dI' \frac{1}{\mathcal{P}(I')D(I')}\int_0^{I'} dI''\mathcal{P}(I'') + C_1\int_0^{I_{\max}} dI' \exp[U(I')] + C_2\,. \tag{24}$$

On the other hand, as I in Eq. (23) gets larger and larger, we expect the first passage time T to increase also, but more and more slowly, such that $dT(I)/dI \to 0$ as $I \to \infty$. This leads to the condition

$$C_1 = \frac{2}{K}\int_0^\infty dI''\mathcal{P}(I'') = \frac{2}{K}\,. \tag{25}$$

If we use this value of C_1 in Eq. (24) to determine C_2, and then substitute for both C_1 and C_2 in Eq. (23), we obtain the following expression for the first passage time from $I = I_{\min}$,

$$T = 2\int_{I_{\max}}^{I_{\min}} dI' \frac{1}{\mathcal{P}(I')D(I')}\int_{I'}^{\infty} dI''\mathcal{P}(I'')\,. \tag{26}$$

We can make an order-of-magnitude estimate of the diffusion rate $D(I)$ in the following way. We first transform the original Fokker-Planck equation (4) to polar coordinates by putting $x_1+ix_2 \equiv\sqrt{I_1}\exp i\phi_1$, and $x_3+ix_4\equiv\sqrt{I_2}\exp i\phi_2$, and then integrate over ϕ_1, ϕ_2, I_2. This leads to the following equation of motion for the probability distribution $\mathcal{P}_1(I_1)$ of I_1,

$$\frac{\partial\mathcal{P}_1(I_1)}{\partial t}=-\frac{\partial}{\partial I_1}\left[\left(a_1-I_1+\frac{2}{I_1}\right)2I_1\mathcal{P}_1(I_1)\right]-4\frac{\partial}{\partial I_1}\left[I_1\langle I_2\rangle_{I_1}\mathcal{P}_1(I_1)\right]+\frac{1}{2}\frac{\partial^2}{\partial I_1^2}\left[8I_1\mathcal{P}_1(I_1)\right]. \tag{27}$$

Here

$$\langle I_2\rangle_{I_1}\equiv\int_0^\infty\frac{\mathcal{P}(I_1,I_2)I_2\,dI_2}{\mathcal{P}_1(I_1)}$$

is the conditional mean of I_2 for a given I_1. If we approximate this term by replacing it by the unconditional mean $\langle I_2\rangle$, the equation of motion has the form of a Fokker-Planck equation for $\mathcal{P}_1(I_1)$, with diffusion rate

$$D(I_1)=8I_1 \tag{28}$$

in our dimensionless units. We can now use this form of $D(I_1)$ together with the steady-state solution for $\mathcal{P}_1(I_1)$ given by Eq. (6) to substitute in Eq. (26). To a reasonable approximation when $a_1\simeq a_2$, $I_{\min}\simeq a_1$, and $I_{\max}\simeq\frac{1}{3}a_1$, so that we obtain from Eq. (26) for the average first passage time

$$T\simeq\int_{a_1/3}^{a_1}\int_{I'}^{\infty}dI''\,\frac{\exp\frac{3}{4}(I''^2-I'^2)-(a_2-\frac{1}{2}a_1)(I''-I')}{4I'}\,\frac{1-\operatorname{erf}(I''-\frac{1}{2}a_2)}{1-\operatorname{erf}(I'-\frac{1}{2}a_2)}. \tag{29}$$

For approximately equal pump parameters, the time T is found to vary between 1.5 natural units of time when $a_1\simeq6$, to 70 units when $a_1\simeq10$, to 9×10^5 units when $a_1\simeq15$. The natural units of time are probably in the range 10–100 μsec for a homogeneously broadened dye laser. Evidently the crossover time becomes very long when the laser is operated sufficiently far above threshold, where the instability effectively disappears. However, closer to the region of threshold, the switching effects should be both rapid and pronounced.

ACKNOWLEDGMENT

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