New classical Hamilton-Lagrange mechanics for open systems

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Certain systems in physics are commonly referred to as "Hamiltonian systems"; these are all closed systems: open systems, which may undergo damping or growth phenomena, are "non-Hamiltonian." The mathematical basis of this distinction is a little-known theorem of Helmholtz, stating necessary and sufficient conditions on the functions whose vanishing expresses the equations of motion, that a Lagrange function exist in a certain sense. When the Helmholtz conditions are satisfied, the equations of motion always can be written as $G_n(x,\dot{x},\ddot{x},t) \equiv (d/dt)f_{2n}(x,\dot{x},t) - f_{1n}(x,\dot{x},t) = 0$, $n = 1$–N, but many systems of physical importance having this form fail to satisfy those conditions. The present paper is the first of a planned series presenting a new classical Hamilton-Lagrange mechanics for systems having equations of the form just stated, restricted for convenience to the case $\det \|\partial f_{2n}/\partial \dot{x}_m\| \neq 0$, but with otherwise arbitrary f_{1n}, f_{2n}. To get around the Helmholtz conditions in a way useful for application to modern theoretical physics (quantum mechanics and statistical mechanics), we have generalized the classical formalism. The new mechanics subsumes the old in a very reasonable and theoretically promising way. A formal quantization leads to a generalized Schrödinger equation of a state function whose natural physical interpretation is a density function (density "matrix"), rather than a "half density," or wave function. Another, for closed systems, gives the usual Schrödinger equation twice—once for ψ and once for ψ^*, while a third gives a classical Hamilton-Jacobi partial differential equation.

I. INTRODUCTION

In the early part of the century, the classical Rutherford atom, dynamically unstable owing to the effects of radiation damping, was salvaged by the Bohr quantum conditions. As corresponding quantum phenomena, such as excited atom decay to a ground state and level (Lamb) shift, have been well described by quantum electrodynamics (QED), a phenomenological quantum theory of damping or "friction," no doubt has seemed unnecessary. However, recent heavy-ion nuclear scattering data at high energy, for example, show effects which appear to require the operation of strong friction forces between the two nuclei at close ranges during the time of collision and while the nuclear surfaces are in contact.[1] Theoretical approaches having appropriate (and inappropriate) classical limits have been devised to treat such problems as this, and some of these have greater generality than others.[2,3] Nevertheless, a satisfactory, clearcut first-principles *phenomenological* quantum mechanics for open systems still appears lacking; the current theoretical situation seems analogous to and more reminiscent of the days of the pre-Schrödinger "old quantum theory" than those of the post-1925 era of the "new quantum theory."

While microscopic many-body-theory techniques are available for analysis of macroscopic processes, these also give redundant and physically inappropriate information; and the desired information can be difficult to extract. Indeed, quantum electrodynamics tells of many assumed subprocess contributions (Feynman diagrams) to the behavior of atomic systems, such as involve bare-mass and coupling-constant parameters, but which are observable only to unphysical, "ideal observers."[4] Here, "physically observable" seems even to imply phenomenological or "macroscopic" (specifically: renormalized).

Also, it is interesting to note, save for the Vlasov equation, the absence of a proper Liouville theorem for the single-particle "phase" space of a particle in a gas or an electron in a plasma. For, owing to collisions, a single-particle Hamiltonian does not exist. There are various generalizations of the Liouville equation; one example, based upon the theory of stochastic processes, is the Fokker-Planck equation.[5] But a statistical mechanics devised for single-particle motions and employing a Gibbsian-ensemble-type approach[6] would require generalization of the Liouville equation along other lines, e.g., to the theoretical mechanics of the single-particle motion as that of a non-Hamiltonian elementary system.

We believe that an open-system phenomenological Schrödinger theory, and a statistical mechanics grounded in generalized Liouville equations as indicated above, are both likely to find useful applications. In addition, there are broadly theoretical, fundamental issues involved. It seems evident that, apart from an intrinsic interest of its own, the classical subject deserves serious investigation.

To begin, it may be that a given system of equa-

tions is not derivable from a Lagrangian directly (see below), but that an equivalent set, with integrating factors, can be so derived. Unless one admits extra variables (about which, more later), open systems require integrating factors. But the effect of the latter appears to be to destroy the all important possibility for an interpretation of the Hamiltonian as system energy. The last crucial point has been stressed by Havas[7] and is discussed further below. Recently, H. Dekker[8] has constructed a very interesting scheme, in which this requirement is met.

We remark additionally that Santilli[9] gives a detailed exposition of mathematical features of the problem when integrating factors are admitted, and also proposes a new scheme, based upon a non-Lie algebraic (Lie-admissible) time-evolution law for the Hamiltonian variables. We will review some of Havas's early analysis next and then define the scope of the present work.

The presence of integrating factors, or multipliers, to a set of classical equations can cause trouble with physical interpretation for a corresponding quantum theory. Trivially, if the Lagrangian L does not have units of energy, neither does the Hamiltonian H, and in the Schrödinger equation

$$\left(i\hbar\frac{\partial}{\partial t}-H\right)\psi=0\,, \tag{1.1}$$

where t is the time, $\hbar$ has the wrong units. Another example is provided by the Lagrangian[10]

$$L=e^{\gamma t}(\tfrac{1}{2}m\dot{x}^2-\tfrac{1}{2}m\omega_0^2x^2) \tag{1.2}$$

for which the Euler-Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right)-\frac{\partial L}{\partial x}=e^{\gamma t}(m\ddot{x}+m\gamma\dot{x}+m\omega_0^2x)=0\,, \tag{1.3}$$

which, as $e^{\gamma t}$ cannot vanish, is *equivalent* to the equation of a one-dimensional damped oscillator. The Hamiltonian[11] (derivable from L) widely discussed in the literature of this subject,

$$H=(1/2m)p^2e^{-\gamma t}+\tfrac{1}{2}m\omega_0^2e^{+\gamma t}\,, \tag{1.4}$$

has units of energy, but not the physical significance of a system energy. And the physical interpretation of the Schrödinger equation from Eq. (1.4) has been controversial.[2]

There is a general theorem due to Helmholtz stating necessary and sufficient conditions that, given a set of functions, $G_n=G_n(x,\dot{x},\ddot{x},t)$, $n=1-N$, there exists a function $L=L(x,\dot{x},t)$ such that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}_n}\right)-\frac{\partial L}{\partial x_n}\equiv G_n,\quad n=1-N\,. \tag{1.5}$$

The Helmholtz conditions are[10]

$$\frac{\partial G_n}{\partial\ddot{x}_m}\equiv\frac{\partial G_m}{\partial\ddot{x}_n}\,, \tag{1.6}$$

$$\frac{\partial G_n}{\partial\dot{x}_m}+\frac{\partial G_m}{\partial\dot{x}_n}\equiv\frac{d}{dt}\left(\frac{\partial G_n}{\partial\ddot{x}_m}+\frac{\partial G_m}{\partial\ddot{x}_n}\right)\,, \tag{1.7}$$

$$\frac{\partial G_n}{\partial x_m}-\frac{\partial G_m}{\partial x_n}\equiv\frac{1}{2}\frac{d}{dt}\left(\frac{\partial G_n}{\partial\dot{x}_m}-\frac{\partial G_m}{\partial\dot{x}_n}\right)\,. \tag{1.8}$$

Applied to the example

$$G=m\ddot{x}+m\gamma\dot{x}+m\omega_0^2x\,,\quad\gamma\neq0\,, \tag{1.9}$$

the Helmholtz conditions fail, yet while applied to $e^{\gamma t}G$, they are satisfied. The general situation is more or less typified here, that "integrating factors" such as $e^{\gamma t}$ in the example, sometimes can be found when the G_n are such that Eqs. (1.5) cannot be satisfied.

Havas has argued that some form of standardization of the classical Hamilton-Lagrange mechanics is needed for a suitable basis of canonical quantization. It should be possible to couple any system, whether open or closed, to a measuring apparatus for a physical interpretation of the quantum theory; so there should exist a joint Hamiltonian. This requires "normalization" of the system Hamiltonian function to that of the apparatus. The situation is similar even in classical statistical mechanics, since no physical interpretation seems possible if coupling to other systems cannot be effected.

In applications of physical interest, the formulations of physical laws provide bases of choice for functions G_n to express the equations of motion. Thus for the motion of a relativistic particle in an electric field, we may require

$$G_n=(1-\dot{\vec{x}}^2c^{-2})^{-1/2}\left(m\ddot{x}_n+\frac{\dot{\vec{x}}\cdot\ddot{\vec{x}}c^{-2}}{1-\dot{\vec{x}}^2c^{-2}}m\dot{x}_n\right)-qE_n(\vec{x},t)\,, \tag{1.10}$$

where E_n is the x_n component of the field, m the particle mass, q its charge, and c the speed of light *in vacuo*. The integrating factor $(1-\dot{\vec{x}}^2c^{-2})^{1/2}m^{-1}$ gives a physically uninteresting choice, however,

$$G_n=\ddot{x}_n+\frac{\dot{\vec{x}}\cdot\ddot{\vec{x}}c^{-2}}{1-\dot{\vec{x}}^2c^{-2}}\dot{x}_n-qm^{-1}(1-\dot{\vec{x}}^2c^{-2})^{1/2}E_n(\vec{x},t)\,, \tag{1.11}$$

which no longer has the form or even the units needed (i.e., force) to assure a Hamiltonian with the significance of particle energy.

Accordingly, we propose to set a standardization criterion at the stage of the equations of motion themselves, and we suppose that a set of functions $G_n(x,\dot{x},\ddot{x},t)$ has been specified on physical grounds to begin with. Furthermore, we require that a Hamiltonian formulation for a classical system have the property that the associated

Lagrangian, defined by Legendre transformation from H in the usual way, satisfy Eqs. (1.5) for the given G_n. With these restrictions the traditional domain of classical theoretical mechanics is subsumed as the class of systems satisfying the Helmholtz conditions. All of these systems are closed systems, exhibiting time-symmetric behavior, i.e., no damping or amplification, no creation or annihilation, no transient degrees of freedom (as an "acceleration" coordinate in an inelastic process), etc. Classical Hamilton-Lagrange theoretical mechanics, thus restricted, does not exist for the non-Helmholtz case.

The present paper is the first of a planned series describing a generalized Hamilton-Lagrange theory for treating systems represented by functions G_n expressible in the form

$$G_n(x,\dot{x},\ddot{x},t) = \frac{d}{dt} f_{2n}(x,\dot{x},t) - f_{1n}(x,\dot{x},t), \quad n = 1-N \tag{1.12}$$

for arbitrarily specified f_{1n}, f_{2n}. This includes, along with many more, all those cases for which the Helmholtz conditions are met, as we will show (in a later paper): save for a restriction which we have imposed for simplicity, to the so-called "standard case," namely,

$$\det\left|\left|\frac{\partial f_{2n}}{\partial \dot{x}_m}\right|\right| \neq 0, \tag{1.13}$$

which in the Helmholtz case corresponds to

$$\det\left|\left|\frac{\partial^2 L}{\partial \dot{x}_n \partial \dot{x}_m}\right|\right| \neq 0. \tag{1.14}$$

The condition (1.14) assures that the equations of the generalized momenta can be solved uniquely for the velocities so that the Hamilton function is fixed uniquely by L. The new formalism subsumes the old in a very natural way, and canonical quantization gives a surprising, and gratifying, preliminary result (see below).

For orientation to the formalism we will present, we make a few initial remarks. One of the main mathematical instruments of our approach is the Dirac formalism of weak and strong equality.[12] In the Dirac formalism, if one is given a Lagrangian, the Hamiltonian is *not uniquely fixed* if primary constraints are present, but is fixed only up to strong equality (i.e., only as to value and gradient in the constraint hypersurface[13]). In this paper we will find a Lagrangian that is also not uniquely fixed, but only to a strong equality defined over the space of coordinates plus velocities. The resulting Hamilton-(Dirac)-Lagrange theory is consistent and exhibits remarkable symmetries, along with a new universal bisymmetric structure reminiscent of, but distinct from that of the symplectic bisymmetry of the ordinary canonical formalism. The new bisymmetry appears in the corresponding quantum theory, for the Helmholtz case, in the guise of antiunitarily equivalent representations of a system by ψ and ψ^*. (See Sec. IV for details.)

In the present scheme a universal constraint hypersurface, which is $2N$-dimensional and whose manifold corresponds to the usual phase space, is embedded in a doubled, $4N$-dimensional phase space. Together with the constraints, the doubling gives rise to identical duplicate classical representations of system motion [by generalized Hamiltonian equations (3.19) or (3.20)], and this is the source of the duplicate representation by ψ and ψ^* in the quantum theory. But that feature now is present in the classical theory even for open systems. Finally, as canonical covariance over the doubled phase space will control the general transformation properties of Eqs. (3.19) and (3.20), a generalized canonical covariance structure also is implied.

On physical grounds we rejected earlier the approach through Lagrangian theory which requires use of integrating factors to the $G_n(x,\dot{x},\ddot{x},t)$. There is a general theorem of Hamiltonian theory due to Lie and Koenigs[14] which states that any system of first-order equations can be given a Hamiltonian form, by addition of extra coordinates, and an analogous result was proved by Havas for first-order equations for Lagrangian theory.[7] Results such as these seem too general to be of much use and, without some sort of bounding consideration problems of physical interpretation, seem likely to get out of hand. Moreover, dynamical laws of particle motions are normally framed as second-order equations, and this further reduces the convenience of theorems of this sort.[15] The matter was made somewhat less vague, nevertheless, by G. D. Birkhoff,[16] who wrote down a Hamiltonian, also in a doubled space, that does the job for a given first-order set of equations. Despite certain similarities of the Birkhoff Hamiltonian to our Eq. (2.13) below, the scheme we are developing is a departure from the usual mechanics, as noted earlier. We will discuss the Birkhoff Hamiltonian specifically in Sec. II.

Our original motivation in the present undertaking was to explore the consequence of quantization from the equations of motion of a classically radiating charge, with the Abraham–von Laue four vector of radiation reaction,[17] or a "moral equivalent" of that,[18] included rather than omitted. It is clear to us now, as it was not when we first began, that a consequence of the absence of a

suitable classical formalism has been that theories of radiation and particle creation and annihilation, methods of statistical mechanics and kinetic theory, and even the measurement theory of quantum mechanics, have not had opportunity to develop along lines *other* than those rooted in *closed-system limited* Hamiltonian theory. It is difficult to refrain from repeating the observation made earlier, that in the case of quantum electrodynamics the simple "microscopic" (unrenormalized) description is really supposed to be unphysical, i.e., the bare masses and charges are not the observed quantities, while the renormalized theory seems to correspond to a kind of coarse-graining description, with physically observable corresponding to macroscopic. These remarks, of course, can be applied as well to theories of non-Abelian gauge-field interactions.

The formalism begun herein is restricted to systems for which the $G_n(x,\dot{x},\ddot{x},t)$ are linear in the $\ddot{x}_n$, however, which can be seen from Eqs. (1.12). This prevents us from considering the problem of radiation at this stage since the accelerations necessarily enter quadratically into the equations of motion. So for those applications it is desirable to generalize the present formalism to include the case of G_n nonlinear in the $\ddot{x}_n$.

In later papers we will establish formal results concerning the relation of the new mechanics to the old, construct the generalized covariance theory for the new scheme, and extend our theory to the case of nonlinear G_n. The plan of the present paper is as follows: In Sec. II we begin by producing universal Hamiltonian and Lagrangian theories. In Sec. III we introduce the formalism of the bisymmetric structure for the Lagrangian and Hamiltonian theories, along with the generalized Hamiltonian and Lagrangian equations. In Sec. IV we provide an interpretation of the new scheme and explore briefly its relationship to quantum theory. In Sec. V we conclude the paper.

II. UNIVERSAL HAMILTONIAN AND LAGRANGIAN

A. Hamiltonian

The Euler-Lagrange equations produced from the Lagrangian

$$\mathcal{L}_0(x,z,\dot{x},\dot{z},t) = \sum_n f_{1n}(x,\dot{x},t)z_n + \sum_n f_{2n}(x,\dot{x},t)\dot{z}_n, \tag{2.1}$$

consist of the N equations,

$$G_n(x,\dot{x},\ddot{x},t) = \frac{d}{dt}f_{2n}(x,\dot{x},t) - f_{1n}(x,\dot{x},t) = 0, \quad n = 1-N, \tag{2.2}$$

together with another N, with which the subsidiary conditions,

$$z_n = 0, \quad n = 1-N, \tag{2.3}$$

are always consistent. To obtain Eqs. (2.3) themselves we introduce N more auxiliary coordinates, $y_1,\ldots,y_n$, and a slightly different Lagrangian

$$\mathcal{L}_0 \to \mathcal{L}_1 = \mathcal{L}_1(x,z,\dot{x},\dot{z},t;y) = \mathcal{L}_0 + \sum_n \frac{1}{2} y_n z_n^2. \tag{2.4}$$

The new Euler-Lagrange equations now uniquely imply Eqs. (2.2) and (2.3), and allow arbitrary time dependence for the auxiliary coordinates.

In passing to the Hamiltonian formalism one encounters N first-class constraints,

$$\frac{\partial \mathcal{L}}{\partial \dot{y}_n} = p_{y_n} \approx 0, \quad n = 1-N, \tag{2.5}$$

the wavy equal sign denoting weak equality after Dirac.[12,13,19] We need only the N relations,

$$p_{z_n} = f_{2n}(x,\dot{x},t), \quad n = 1-N, \tag{2.6}$$

assumed soluble for the $\dot{x}_n$, to obtain the total Hamiltonian

$$H_T \simeq -\sum_n f_{1n}(x,\dot{x},t)z_n + \sum_n \dot{x}_n p_{x_n} - \sum_n \frac{1}{2} y_n z_n^2 + \sum_n v_n p_{y_n}, \tag{2.7}$$

where the v_n are arbitrary functions of the time. We have used the Dirac sign $\simeq$ to denote strong equality.

The consistency conditions to Eqs. (2.5) are

$$0 \approx \dot{p}_{y_n} \approx \{p_{y_n}, H_T\} \approx \tfrac{1}{2} z_n^2, \quad n = 1-N, \tag{2.8}$$

so that[20]

$$z_n \approx 0, \quad n = 1-N; \tag{2.9}$$

also,

$$z_n^2 \simeq 0, \quad n = 1-N. \tag{2.10}$$

Equation (2.9) are secondary constraints and the consistency conditions to these may be taken to be

$$p_{x_n} \approx 0, \quad n = 1-N, \tag{2.11}$$

provided

$$\det \left\| \frac{\partial f_{2n}}{\partial \dot{x}_m} \right\| \neq 0, \tag{2.12}$$

which is guaranteed by the assumed uniqueness of the solutions to Eqs. (2.6).

The auxiliary coordinates may be discarded from the total Hamiltonian. The third term in Eq. (2.7) actually vanishes by Eq. (2.10), and the sole effect of the last term in the canonical equations is to give back the physically irrelevant equations

of the y_n and p_{y_n}. So we have[21]

$$H_T - \mathcal{H} \simeq \sum_n F_{1n}(x, p_z, t) z_n + \sum_n F_{2n}(x, p_z, t) p_{x_n}, \quad (2.13)$$

where we have written

$$-f_{1n}(x, \dot{x}, t) = F_{1n}(x, p_z, t), \quad (2.14a)$$

$$\dot{x}_n = F_{2n}(x, p_z, t). \quad (2.14b)$$

As f_{1n}, f_{2n} are arbitrary, Eq. (2.13) holds universally; so we refer to $\mathcal{H}$ as the universal Hamiltonian. Similarly, the $2N$ first-class constraints, Eqs. (2.9) and (2.11), are the same for all systems; they will be called the universal constraints, and the $2N$-dimensional hypersurface they define, the universal hypersurface.

B. Lagrangian

If two Lagrangians, $\mathcal{L}$ and $\bar{\mathcal{L}}$ differ by an "O_2 term," by which is meant a function of $(x, \dot{x}, z, \dot{z}, t)$ vanishing quadratically in the z_n and $\dot{z}_n$, for small values of these, the universal Hamiltonian is unaffected. We show this as follows. Equations (2.9) will still hold as they are consistency conditions to Eqs. (2.5); while the consistency conditions giving Eqs. (2.11) are more conveniently for present purposes expressed as

$$\dot{z}_n \approx 0, \quad n = 1 - N. \quad (2.15)$$

Given

$$\bar{\mathcal{L}} = \mathcal{L} + O_2, \quad (2.16)$$

we have

$$\bar{p}_{x_n} = \frac{\partial \bar{\mathcal{L}}}{\partial \dot{x}_n} + O_2 \simeq p_{x_n}, \quad (2.17a)$$

$$\bar{p}_{z_n} = \frac{\partial \bar{\mathcal{L}}}{\partial \dot{z}_n} + \frac{\partial}{\partial \dot{z}_n} O_2 \approx p_{z_n}, \quad (2.17b)$$

whence, supposing the appropriate eliminations to have been performed for the computation of $\bar{\mathcal{H}}$, the uniqueness of which is assured by Eq. (2.12),

$$\bar{\mathcal{H}} = \sum_n (\bar{p}_{x_n} \dot{x}_n + \bar{p}_{z_n} \dot{z}_n) - \bar{\mathcal{L}} \simeq \mathcal{H}; \quad (2.18)$$

for Eq. (2.16), combined with Eqs. (2.9) and (2.15) guarantees $\bar{\mathcal{L}} \simeq \mathcal{L}$, while Eqs. (2.15), (2.17a) and (2.17b) give the rest.[22]

From the point of view of the canonical formalism it is therefore unnecessary to specify $\mathcal{L}$ to more than a strong equality on the "expanded configuration space," of coordinates plus velocities, with the hypersurface defined by Eqs. (2.9) and (2.15) being used to define weak and strong equality on this space. If we regard Eq. (2.4) as a strong equation, the last term vanishes and the auxiliary coordinates, with this, disappear from the Lagrangian theory now also. We really do not need the auxiliary coordinates at all. They have been an aid to constructing $\mathcal{H}$ and the constraint formalism, and now can be dismissed entirely. The strongly specified Lagrangian

$$\mathcal{L} \simeq \sum_n f_{1n}(x, \dot{x}, t) z_n + \sum_n f_{2n}(x, \dot{x}, t) \dot{z}_n \quad (2.19)$$

is the universal Lagrangian for systems defined by Eqs. (2.2). Equations (2.3) are now to be reappended to the equations of motion as subsidiary conditions, but no arbitrary (gauge) functions are introduced into $\mathcal{H}$ as a consequence.

C. Simple example

For a single particle experiencing linear damping one has the equations of motion

$$\frac{d}{dt} m\dot{\vec{x}} = -m\gamma\dot{\vec{x}} + \vec{\mathcal{F}}(\vec{x}, t), \quad (2.20)$$

for which

$$\mathcal{L} \simeq [-m\gamma\dot{\vec{x}} + \vec{\mathcal{F}}(\vec{x}, t)] \cdot \vec{z} + m\dot{\vec{x}} \cdot \dot{\vec{z}}, \quad (2.21)$$

so that

$$\vec{p}_z = m\dot{\vec{x}} \quad (2.22)$$

and

$$\mathcal{H} \simeq [\gamma \vec{p}_z - \vec{\mathcal{F}}(\vec{x}, t)] \cdot \vec{z} + m^{-1} \vec{p}_z \cdot \vec{p}_x. \quad (2.23)$$

Evidently,

$$\vec{p}_x = -\gamma m \vec{z} + m\dot{\vec{z}} \approx 0. \quad (2.24)$$

We note the crossed associations $\vec{p}_z \to \dot{\vec{x}}$ and $\vec{p}_x \to \dot{\vec{z}}$.

D. Birkhoff's Hamiltonian

Birkhoff observed[16] that the equations, for arbitrary functions X_i,

$$\frac{dq_i}{dt} = X_i(q, t), \quad i = 1 - n, \quad (2.25)$$

are produced from the Hamiltonian

$$H_B = -\sum_j X_j(q, t) p_j, \quad (2.26)$$

where the p_j are n additional coordinates conjugate to the q_j. The equations of the p_i, concomitant under Eq. (2.26), are

$$\frac{dp_i}{dt} = -\sum_j \frac{\partial X_j}{\partial q_i} p_j, \quad i = 1 - N. \quad (2.27)$$

If we apply a Legendre transformation to H_B with the p_i as active variables, we find the Lagrangian

$$L_B = \sum_j \dot{q}_j p_j - H_B \equiv 0, \quad (2.28)$$

since the defining equations for the transformation are

$$\dot{q}_j = \frac{\partial H_B}{\partial p_j} = -X_j(q,t), \quad j = 1-n . \tag{2.29}$$

On the other hand, a Legendre transformation of Eq. (2.13) for $\mathfrak{K}$, with p_x, p_z active, gives Eq. (2.19). Despite the striking difference, the function H_B will be the same as the function appearing on the right-hand side of Eq. (2.13) for $\mathfrak{K}$, provided $n = 2N$ and we make the identifications,

$$q_1 \cdots q_N \to x_1 \cdots x_N; \quad -p_{N+1} \cdots -p_{2N} \to z_1 \cdots z_N;$$
$$p_1 \cdots p_N \to p_{x_1} \cdots p_{x_N}; \quad q_{N+1} \cdots q_{2N} \to p_{z_1} \cdots p_{z_N};$$
$$X_{N+1} \cdots X_{2N} \to F_{11} \cdots F_{1N}; \quad -X_1 \cdots -X_N \to F_{21} \cdots F_{2N} . \tag{2.30}$$

The staggering of the correspondences reflects the crossed associations of conjugate variables to velocities noted for the example of Sec. IIC.

Two features of the present formalism distinguish it from the usual mechanics. One is the occurrence of tetrads of crossed pairs, (x,p_z) and (z,p_x), half of which are first-class (and therefore crossed-pair) constraints. A second is the fact that $\mathfrak{K}$ and $\mathfrak{L}$ are specified only to strong equality, and hence do not represent functions, two members of a given class being allowed to differ by a strongly vanishing function. Thus it is meaningless to infer from Eq. (2.30) that $H_B = \mathfrak{K}$. Moreover, regarded as a transformation (the inverse of), Eq. (2.30) will not be available to us, even though it is canonical, as its effect would be to destroy the crossed association structure defined by the universal constraints. So H_B also cannot be equivalent to (any of the representatives of) $\mathfrak{K}$ in this formalism.

We will give substance to these remarks through the developments of Sec. III, where we deal with the constraints.

III. GENERALIZED MECHANICS: CENTRAL COORDINATES AND PRINCIPAL COORDINATES

For convenience we rewrite the expressions here for the universal Lagrangian and Hamiltonian, Eqs. (2.13) and (2.19):

$$\mathfrak{L} \simeq \sum_n f_{1n}(x,\dot{x},t)\dot{z}_n + \sum_n f_{2n}(x,\dot{x},t)\dot{z}_n \tag{3.1}$$

$$\mathfrak{K} \simeq \sum_n F_{1n}(x,p_z,t)\dot{z}_n + \sum_n F_{2n}(x,p_z,t)\dot{p}_{x_n} . \tag{3.2}$$

The formalism will contain two classes of coordinate system, the one used in Eqs. (3.1) and (3.2) and consisting of the (x,z,p_x,p_z)—these will be called central coordinates—and another, introduced below, to be called principal coordinates.

Before introducing these we note a gauge feature of the present scheme.

A. Addition of a time derivative to $\mathfrak{L}$

The most general time derivative of a function of the coordinates and the time can be written

$$\Delta\mathfrak{L} \simeq \frac{d}{dt}\sum_n z_n \mathfrak{a}_n(x,t) , \tag{3.3}$$

where the $\mathfrak{a}_n$ are arbitrary, so that under

$$\mathfrak{L} \to \mathfrak{L} + \Delta\mathfrak{L} \tag{3.4}$$

we find the generalized gauge transformation

$$f_{1n} \to f'_{1n} = f_{1n} + \frac{d}{dt}\mathfrak{a}_n , \tag{3.5}$$

$$f_{2n} \to f'_{2n} = f_{2n} + \mathfrak{a}_n . \tag{3.6}$$

The transformations (3.5) and (3.6) leave Eqs. (2.2) invariant,

$$\frac{d}{dt} f'_{2n} - f'_{1n} = 0 . \tag{3.7}$$

Conversely, the transformations (3.5) and (3.6) of the f_{1n}, f_{2n}, induce (3.4) via Eq. (3.1). Some of the significance of the foregoing will be apparent presently.

B. Principal coordinates

We perform a point transformation of the (x_n,z_n) to a new set of coordinates (ξ_n, ξ'_n) which are the principal coordinates for the problem

$$\xi_n = x_n + \tfrac{1}{2} z_n , \tag{3.8a}$$

$$\xi'_n = x_n - \tfrac{1}{2} z_n . \tag{3.8b}$$

The transformation of Eqs. (3.8), induces a canonical transformation in the phase space, the other half of which is determined by

$$p_n = p_{z_n} + \tfrac{1}{2} p_{x_n} , \tag{3.9a}$$

$$-p'_n = p_{z_n} - \tfrac{1}{2} p_{x_n} . \tag{3.9b}$$

Notice $(\xi_n, \xi'_n, p_n, p'_n)$ is canonical, while $(\xi_n, \xi'_n, p_n, -p'_n)$ is not. The universal constraints are given by

$$z_n = \xi_n - \xi'_n \approx 0 , \tag{3.10a}$$

$$p_{x_n} = p_n + p'_n \approx 0 . \tag{3.10b}$$

We write $\mathfrak{L}$ and $\mathfrak{K}$ in an alternative form,

$$\mathfrak{L} \simeq L(\xi,\dot{\xi},t) - L(\xi',\dot{\xi}',t) + \delta\mathfrak{L}(\xi,\dot{\xi};\xi',\dot{\xi}',t) , \tag{3.11}$$

$$\mathfrak{K} \simeq H(\xi,p,t) - H(\xi',-p',t) + \delta\mathfrak{K}(\xi,p;\xi',-p',t) , \tag{3.12}$$

where the function pairs $(L, \delta\mathfrak{L})$ and $(H, \delta\mathfrak{K})$ may be chosen arbitrarily. Equations (3.11) and (3.12) will be called "principal decompositions" of $\mathfrak{L}$ and $\mathfrak{K}$; L and H are "principal part" functions; and $\delta\mathfrak{L}$ and $\delta\mathfrak{K}$ "residual part" functions of the respective decompositions. We substitute Eqs. (3.8) and (3.9) into Eqs. (3.11) and (3.12) and expand in Taylor series about vanishing $(z, \dot{z})$ and (z, p_x), respectively, retaining only first-order terms since higher-order contributions all vanish strongly. The resulting expressions, upon comparison with Eqs. (3.1) and (3.2), yield

$$f_{1n}(x,\dot{x},t)=\frac{\partial L(x,\dot{x},t)}{\partial x_n}+\delta f_{1n}(x,\dot{x},t)\,, \tag{3.13a}$$

$$f_{2n}(x,\dot{x},t)=\frac{\partial L(x,\dot{x},t)}{\partial \dot{x}_n}+\delta f_{2n}(x,\dot{x},t)\,, \tag{3.13b}$$

with

$$\delta f_{1n}(x,\dot{x},t) = \frac{\partial}{\partial \xi_n}\delta\mathfrak{L}(\xi,\dot{\xi};\,\xi',\dot{\xi}',t)\Big|_{\xi'=\xi=x,\,\dot{\xi}'=\dot{\xi}=\dot{x}}\,, \tag{3.14a}$$

$$\delta f_{2n}(x,\dot{x},t) = \frac{\partial}{\partial \dot{\xi}_n}\delta\mathfrak{L}(\xi,\dot{\xi};\,\xi',\dot{\xi}',t)\Big|_{\xi'=\xi=x,\,\dot{\xi}'=\dot{\xi}=\dot{x}}\,; \tag{3.14b}$$

and with analogous equations for the Hamiltonian quantities,

$$F_{1n}(x,p_z,t)=\frac{\partial H(x,p_z,t)}{\partial x_n}+\delta F_{1n}(x,p_z,t)\,, \tag{3.15a}$$

$$F_{2n}(x,p_z,t)=\frac{\partial H(x,p_z,t)}{\partial p_{z_n}}+\delta F_{2n}(x,\dot{p}_z,t)\,, \tag{3.15b}$$

where

$$\delta F_{1n}(x,p_z,t) = \frac{\partial}{\partial \xi_n}\delta\mathfrak{K}(\xi,p;\,\xi',-p',t)\Big|_{\xi'=\xi=x,\,-p'=p=p_z}\,, \tag{3.16a}$$

$$\delta F_{2n}(x,p_z,t) = \frac{\partial}{\partial p_n}\delta\mathfrak{K}(\xi,p;\,\xi',-p',t)\Big|_{\xi'=\xi=x,\,-p'=p=p_z}\,. \tag{3.16b}$$

We have written Eqs. (3.13)–(3.16) as strict equations since they involve only coordinates of points in the universal hypersurface and are independent of the constraints.

The Euler-Lagrange equations from Eq. (3.1) are Eqs. (2.2). If a principal decomposition of $\mathfrak{L}$ exists for which $\delta\mathfrak{L}\simeq 0$, then Eqs. (3.13) and (3.14) give

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_n}-\frac{\partial L}{\partial x_n}=0\,. \tag{3.17}$$

It may be that $\delta\mathfrak{L}\not\simeq 0$, but a gauge transformation (3.5) and (3.6) exists, sending $\mathfrak{L}\to\mathfrak{L}+\Delta\mathfrak{L}$, and where $\mathfrak{L}+\Delta\mathfrak{L}$ has a principal decomposition with vanishing residual part. In this case Eqs. (3.17) are again recovered. In the most general case $\delta\mathfrak{L}$ cannot be made to vanish in this way, and this is the non-Helmholtz case, where a "Lagrange potential" does not exist.

Generalized Hamiltonian equations in the universal hypersurface now follow from the canonical equations for $\mathfrak{K}$, via Eqs. (3.2) and (3.15) and (3.16); these are

$$\dot{x}_n=\frac{\partial H}{\partial p_{z_n}}+\delta F_{2n}\,, \tag{3.18a}$$

$$\dot{p}_{z_n}=-\frac{\partial H}{\partial x_n}-\delta F_{1n}\,. \tag{3.18b}$$

In the Helmholtz case where $\delta\mathfrak{L}$ vanishes (having been brought to zero by a gauge transformation, let us say), the Legendre transformation to $\mathfrak{K}$ factors, and Eq. (3.11) induces a natural principal decomposition of $\mathfrak{K}$ for which $\delta\mathfrak{K}$ vanishes. Equations (3.18) then reduce to the ordinary Hamilton equations. In the general case δF_{1n}, δF_{2n} cannot be brought to zero in this way.

There is a difficulty with Eqs. (3.18), namely that the x_n and p_{z_n} all have vanishing Poisson bracket with one another. However, if we start from Eq. (3.12) we find

$$\dot{\xi}_n\approx\frac{\partial H}{\partial p_n}+\delta F_{2n}(\xi,p,t)\,, \tag{3.19a}$$

$$\dot{p}_n\approx-\frac{\partial H}{\partial \xi_n}-\delta F_{1n}(\xi,p,t)\,, \tag{3.19b}$$

$$\dot{\xi}'_n\approx\frac{\partial H'}{\partial(-p'_n)}+\delta F_{2n}(\xi',-p',t)\,, \tag{3.20a}$$

$$-\dot{p}_n\approx-\frac{\partial H'}{\partial \xi'_n}-\delta F_{1n}(\xi',-p',t)\,, \tag{3.20b}$$

where $H'\equiv H(\xi',-p',t)$ and we have used

$$\delta H(\xi',-p';\,\xi,p,t)\simeq-\delta H(\xi,p;\,\xi',-p',t)\,, \tag{3.21}$$

which follows from Eqs. (3.1), (3.10), and (3.12); Eq. (3.21) is needed to obtain the identity of the residual term functions in Eqs. (3.20) to those in Eqs. (3.19). Since the (ξ,p) variables satisfy the usual fundamental Poisson-bracket relations, as do the $(\xi',-p')$ apart from a sign, the previous difficulty with Eqs. (3.18) is now removed. Thus the formalism has produced *duplicate* sets of generalized Hamiltonian equations, in terms of the unprimed variables and the primed variables, separately [as well as Eqs. (3.18)].

From Eqs. (3.19) the divergence of the phase fluid velocity vector $(\dot{\xi},\dot{p})$ no longer vanishes unless the δF terms do; so the Liouville theorem

will be affected. Thus, for instance, in the general case $\delta\mathcal{K}$ will not vanish, but it can always be sent to zero by canonical transformation, and with it the residual (δF) terms in the generalized Hamiltonian equations. However, such a transformation need not be canonical with respect to the universal hypersurface manifold; also, canonical invariance in the full space will induce a significant transformation structure in the manifold, an intricate matter whose consideration we defer to a later paper. We consider some examples now.

C. Examples

1. *Charged particle in a Lorentz force field*

The equation of motion is

$$\vec{f}_2-\vec{f}_1=0\,, \tag{3.22}$$

with

$$\vec{f}_1=q(\vec{E}+c^{-1}\vec{v}\times\vec{B})\,, \tag{3.23a}$$

$$\vec{f}_2=m\vec{v}(1-c^{-2}\vec{v}^2)^{-1/2}\,, \tag{3.23b}$$

where q is the charge on the particle, c is the speed of light *in vacuo*, $\vec{v}\equiv\dot{\vec{x}}$, $\vec{E}$ and $\vec{B}$ are the field strengths, and m is the mass of the particle. The choice, Eqs. (3.23), leads to a universal Lagrangian for which $\delta\mathcal{L}$ cannot be made to vanish. But if we write

$$\vec{E}=-\nabla\phi-c^{-1}\frac{\partial\vec{A}}{\partial t}\,,\quad \vec{B}=\nabla\times\vec{A}\,, \tag{3.24}$$

the generalized gauge transformation of Sec. III A, with $\vec{\alpha}\equiv qc^{-1}\vec{A}$, leads to an $\vec{f}_1'$, $\vec{f}_2'$ choice for which a principal decomposition of $\mathcal{L}$ having $\delta\mathcal{L}\simeq 0$ can be found. One has

$$\vec{f}_1\to\vec{f}_1'=-q\nabla_{\vec{x}}\phi+qc^{-1}\left(-\frac{\partial\vec{A}}{\partial t}+\vec{v}\times(\nabla_{\vec{x}}\times\vec{A})+\frac{d\vec{A}}{dt}\right)$$
$$=\nabla_{\vec{x}}(-q\phi+qc^{-1}\vec{v}\cdot\vec{A})\,, \tag{3.25}$$

$$\vec{f}_2\to\vec{f}_2'=m\vec{v}(1-c^{-2}\vec{v}^2)^{-1/2}+qc^{-1}\vec{A}$$
$$=\nabla_{\vec{v}}[-mc^2(1-c^{-2}\vec{v}^2)^{1/2}+qc^{-1}\vec{v}\cdot\vec{A}]\,, \tag{3.26}$$

so that $\mathcal{L}$ is given by Eq. (3.11) with $\delta\mathcal{L}\simeq 0$ if we take

$$L=-mc^2(1-c^{-2}\vec{v}^2)^{1/2}-q\phi+qc^{-1}\vec{v}\cdot\vec{A}\,. \tag{3.27}$$

2. *Linear velocity damping of an oscillator*

From the equation of motion

$$m\ddot{x}+m\gamma\dot{x}+m\omega_0^2x=0\,,\quad \gamma\neq 0\,, \tag{3.28}$$

we take

$$f_1=-m\gamma\dot{x}-m\omega_0^2x\,, \tag{3.29a}$$

$$f_2=m\dot{x}\,. \tag{3.29b}$$

If we substitute for x,z in Eq. (3.1) in favor of ξ,ξ', the choice, Eqs. (3.29), gives

$$\mathcal{L}\simeq(\tfrac{1}{2}m\dot{\xi}^2-\tfrac{1}{2}m\omega_0^2\xi^2)-(\tfrac{1}{2}m\dot{\xi}'^2-\tfrac{1}{2}m\omega_0^2\xi'^2)$$
$$-\tfrac{1}{2}m\gamma(\dot{\xi}+\dot{\xi}')(\xi-\xi')\,. \tag{3.30}$$

If we drop the $\xi\dot{\xi}$ and $\xi'\dot{\xi}'$ terms, as they are time derivatives, then

$$L=\tfrac{1}{2}m\dot{\xi}^2-\tfrac{1}{2}m\omega_0^2\xi^2 \tag{3.31a}$$

gives

$$\delta\mathcal{L}\simeq-\tfrac{1}{2}m\gamma(\xi\dot{\xi}'-\xi'\dot{\xi})\,, \tag{3.31b}$$

which does not vanish and cannot be gauge transformed away.

Using the form (3.11) with $(L,\delta\mathcal{L})$ given in Eqs. (3.31) we have

$$p=m\dot{\xi}+\tfrac{1}{2}m\gamma\xi'\,, \tag{3.32a}$$

$$-p'=m\dot{\xi}'+\tfrac{1}{2}m\gamma\xi\,. \tag{3.32b}$$

The universal Hamiltonian is

$$\mathcal{K}\simeq\frac{1}{2m}p^2+\tfrac{1}{2}m\omega_0^2\xi^2+\tfrac{1}{2}\gamma\xi p-[\frac{1}{2m}(-p')^2+\tfrac{1}{2}m\omega_0^2\xi'^2$$
$$+\tfrac{1}{2}\gamma\xi\cdot(-p')]-\tfrac{1}{2}\gamma[\xi' p-\xi\cdot(-p')]\,, \tag{3.33}$$

where we have grouped the terms to suggest the (most natural) identification of $(H,\delta\mathcal{K})$, viz.

$$H=\frac{1}{2m}p^2+\tfrac{1}{2}m\omega_0^2\xi^2+\tfrac{1}{2}\gamma\xi p\,, \tag{3.34a}$$

$$\delta\mathcal{K}\simeq-\tfrac{1}{2}\gamma[\xi' p-\xi\cdot(-p')]\,. \tag{3.34b}$$

The principal part function H is the Hamiltonian of an *undamped, resonance-frequency-shifted* oscillator. As before, $\delta\mathcal{K}$ does not admit a principal decomposition with vanishing residual part, and it can be shown that there is no gauge transformation which will induce a vanishing $\delta\mathcal{K}$ in the canonical formalism.

The principal part functions L and H possess a uniqueness only when $\delta\mathcal{L}\simeq 0$ and $\delta\mathcal{K}\simeq 0$. In the general case, however, there is (as yet) no apparent basis for fixing either L or H. Nevertheless, Eqs. (3.31a) and (3.34a) are very natural choices for the present problem, and the results are interesting and suggestive. The uniqueness problem for H seems important from a physical point of view. It corresponds to the specification of the system (or, putting it another way, to the identification of the Heisenberg picture).

IV. DISCUSSION OF FORMALISM

We provide a conceptual picture of the structure of the new formalism, and then turn to the problem of quantization.

A. Interpretation of formalism

The present formalism requires an interpretation. It is different from the Hamilton-Lagrange theory in essential respects. Thus although we began our analysis by representing an N-dimensional system by a point x_n in an N-dimensional configuration space, as in Eqs. (2.2), we have produced, instead, two different methods for representing a classical system in a point space. In the first, which is the central-coordinate method, the system is represented *once*, by a point whose motions are confined to a hypersurface in the space. In the second, which is the principal-coordinate method, the system is represented *twice*, by points which execute coordinated motions. The two modes for representation of system behavior are connected through the equation defining the transformations between central and principal coordinates. We depict the approach schematically in Fig. 1. Thus the "extra" coordinates z_n really provide only apparent additional degrees of freedom; parallelling this, the equations of primed and unprimed coordinates form *duplicate* sets.

Correspondingly, $\mathfrak{L}$ and $\mathfrak{K}$ are not just the system Lagrangian and Hamiltonian. Equally appropriately, we may put the matter this way: that the quantities (f_{1n}, f_{2n}) and (F_{1n}, F_{2n}) completely characterize the system behavior from the central-coordinate point of view; while the principal-coordinate method introduces principal decomposition elements, viz., $(L, \delta\mathfrak{L})$ and $(H, \delta\mathfrak{K})$, for the purpose. The traditional Hamilton-Lagrange theory does not have this structure, but only possesses objects to which the principal part functions L and H correspond.

Finally, as we saw in Sec. III, these structural features over the big $4N$-dimensional phase space induce a generalized Hamiltonian formalism over the $2N$-dimensional constraint manifold, in which the latter plays the role of the usual phase space.

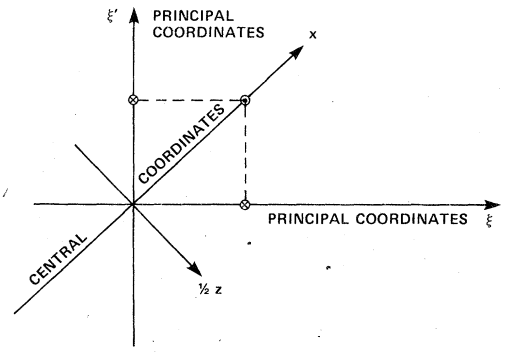

FIG. 1. In central coordinates the system is represented by a point: ONCE, keeping $z(t) = 0$. In principal coordinates the system is represented by a point: but now TWICE, keeping $\xi(t) - \xi'(t) = 0$.

B. Quantization

Formal quantization from Eq. (3.12) with $\delta\mathfrak{K} \simeq 0$ leads to a partial differential equation for a universal Schrödinger function $\Psi(\xi, \xi', t)$ satisfying

$$i\hbar\frac{\partial\bar{\Psi}}{\partial t} = \left[H\left(\xi, -i\hbar\frac{\partial}{\partial\xi}, t\right) - H\left(\xi', +i\hbar\frac{\partial}{\partial\xi'}, t\right)\right]\bar{\Psi}. \quad (4.1)$$

Consider, on the other hand, a system represented by a density matrix given by

$$\rho(\xi, \xi', t) = \sum_k \pi_k \psi_k(\xi, t)\psi_k(\xi', t)^*, \quad (4.2)$$

where the ψ_k are a set of Schrödinger amplitude functions governed by the time-dependent wave equations

$$i\hbar\frac{\partial\psi_k}{\partial t} = H\left(\xi, -i\hbar\frac{\partial}{\partial\xi}, t\right)\bar{\Psi}_k, \quad (4.3)$$

and where the real coefficients π_k sum to unity. The state corresponding to ρ is a mixed state unless all the π_k, save one, are zero; in either case $\rho(\xi, \xi', t)$ satisfies Eq. (4.1). This suggests a physical interpretation of $\Psi(\xi\xi', t)$ as a density, in Eq. (4.1).

In the presence of damping, however, the situation is a little less direct. We have seen that $\delta\mathfrak{K} \simeq 0$ is not always possible. In the general case, typified in the example of linear damping, Eq. (4.1) is replaced by

$$i\hbar\frac{\partial\bar{\Psi}}{\partial t} = \left[H\left(\xi, -i\hbar\frac{\partial}{\partial\xi}, t\right) - H\left(\xi', +i\hbar\frac{\partial}{\partial\xi'}, t\right)\right]\bar{\Psi} + \delta\mathfrak{K}\left(\xi, -i\hbar\frac{\partial}{\partial\xi}\,;\, \xi', +i\hbar\frac{\partial}{\partial\xi'}, t\right)\bar{\Psi}. \quad (4.4)$$

On the other hand, in usual quantum damping theory the density matrix for a dissipative system typically satisfies an equation of the form

$$i\hbar\frac{\partial\rho}{\partial t} = \left[H\left(\xi, -i\hbar\frac{\partial}{\partial\xi}, t\right) - H\left(\xi', +i\hbar\frac{\partial}{\partial\xi'}, t\right)\right]\rho - \tfrac{1}{2}i\hbar\langle\xi|\{\Gamma, \hat{\rho}\}|\xi'\rangle, \quad (4.5)$$

where $\hat{\rho}$ denotes the density operator for which $\rho(\xi', \xi', t)$ is the representative, viz.,

$$\rho(\xi, \xi', t) = \langle\xi|\hat{\rho}(t)|\xi'\rangle, \quad (4.6)$$

and where Γ is an operator associated with the damping. Evidently, the last terms in Eqs. (4.4) and (4.5) should correspond, but this correspondence is not straightforward. In Eq. (4.4) the primed and unprimed variables are eigenvalues of distinct operators, while in any expansion of the matrix element of $\Gamma\hat{\rho}$, appearing in Eq. (4.5),

primed and unprimed variables instead are distinct values of the same operator, viz.,

$$\xi_{op}\,|\,\xi\rangle = \xi\,|\,\xi\rangle \tag{4.7a}$$

$$\xi_{op}\,|\,\xi'\rangle = \xi'\,|\,\xi'\rangle\ . \tag{4.7b}$$

The root of the difference is the corresponding classical theories, of course. In the case of Eq. (4.4) the underlying classical formalism is realized on a 4-(or $4N$-) dimensional phase space, while in the case of Eq. (4.5) the classical roots of the description involve the usual 2-(or $2N$-) dimensional phase space and only this.

On the other hand, if we start directly from Eqs. (3.19) and (3.20), corresponding to $\delta\mathcal{K}\simeq 0$, we recover the usual Schrödinger equation, in duplicate, of course, and gratifyingly so (that for the unprimed coordinates being for ψ and for the primed coordinates ψ^*). But if $\delta\mathcal{K}\not\simeq 0$, we are forced to consider again some kind of generalization and we cannot escape implications of the remarks made in the closing paragraph of Sec. III B. Evidently, the quantum mechanics deriving from the big $4N$-dimensional phase-space picture and from the $2N$-dimensional hypersurface should be equivalent. We prefer to delay more serious consideration of the overall quantization problem until later, after the formal classical covariance theory has been constructed.

We make one further brief remark concerning quantization. If we follow the usual procedure for handling constraints, first used by Dirac,[12] $\xi-\xi'\approx 0$ and $p+p'\approx 0$ would be realized as "wave" equations restricting Ψ. It is easy to show that this would so restrict the physical states that the right-hand side of Eq. (4.4) would vanish identically. All the dynamical information would be lost. The method of Bergmann and Goldberg[23] fails in the same way. There the dynamical variables are identified first in the classical theory; but the test is so strict when applied to the present case that there are *no* dynamical variables in the theory (the entire phase space is annihilated at the start).[24,25]

In the present paper we have treated the constraint problem by "peeling apart" the faces of the hypersurface, "refilling" the big phase space, and representing the system motion with duplicate principal-coordinate equations (Fig. 1).

We close by noting a remarkable feature of the present formalism, when we quantize formally from central coordinates. Starting from Eq. (3.2), we obtain

$$i\hbar\frac{\partial\Phi}{\partial t} = \left[F_1(x,p_z,t)i\hbar\frac{\partial}{\partial p_z} + F_2(x,p_z,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\right]\Phi\ , \tag{4.4'}$$

where Φ is defined over the manifold of the constraint hypersurface, and x and p_z commute. The $i\hbar$ factors *cancel out* of Eq. (4.4'), and the equation defines a family of integral surfaces in the state space of the motion, which are composed of the classical histories.[26] Thus quantization from principal coordinates gives a generalized Schrödinger equation, Eq. (4.4), while quantization from central coordinates gives the *classical limit*, from Eq. (4.4'), on a phase space having no Poisson-bracket algebra.

V. CONCLUSION

We have presented the beginnings of a formalism extending the classical Hamilton-Lagrange mechanics to a large class of open systems, for which the Helmholtz conditions for the existence of a Lagrange function for the equations of motion are not satisfied. The new scheme is a generalization of the traditional subject of mechanics and subsumes the latter in a natural way. The equations of motion display a new, generalized, gauge-invariance feature. We have derived classical generalized Hamiltonian equations, and we have explored certain preliminary aspects of formal quantization.

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od, which employs complex coordinates, Lagrangians, and Hamiltonians *in a certain way*, is based on an ingenious device of decomposing second-order equations of real variables into first-order equations of complex variables. Despite certain innocent appearances, Dekker's mechanics is not just the usual mechanics with complex coordinates, but rather something quite new. It is probably related to the mechanics presented herein, but exactly how is not clear to us yet.

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[21]This procedure of shrinking the phase space is not new. It has been used previously by Dirac [Proc. R. Soc. A 246, 333 (1958)] in his treatment of the gravitational field. There the arbitrary coordinates are the $g_{\mu o}$, and their conjugate momenta $p^{\mu o}$ are constraints. The "contracted" phase-space coordinates are the g_{rs} and p^{rs}; $r, s = 1, 2, 3$.

[22]We have used several times the property that if $X \approx 0$ and $Y \approx 0$, then $X \cdot Y \simeq 0$.

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[24]The test identifies as dynamical variables only those quantities satisfying certain Poisson-bracket relations with the constraints. These involve the null vectors of the matrix whose entries are the Poisson brackets of the constraints with one another. Since the universal constraints all are first class, this matrix is the zero matrix and every vector a null vector. One finds then that a function G must have vanishing Poisson bracket with all the constraints if it is to be a dynamical variable. So it must be independent of the manifold coordinates. All the dynamics is lost and we are left only with the uninspiring assurance that the constraints will stay equal to zero.

[25]If the Dirac approach is in the spirit of the Schrödinger picture for restricting ψ, the Bergman-Goldberg approach is in the spirit of the Heisenberg picture for its focus on the dynamical variables. The two methods seem to be equivalent [E. Marx, (private communication)].

[26]In fact, Eq. (4.4′) without the $i\hbar$ is identical to the Hamilton-Jacobi equation which results after the canonical transformation $zp_z \rightarrow -p_z z$.