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Radial integrals for electron pair production in a point-Coulomb potential

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A theoretical formalism is given for the calculation of electron pair production in a point-Coulomb potential using the method developed by Sud, Wright, and Onley for expressing the radial integrals over the product of Dirac-Coulomb functions in terms of a matrix generalization of the Γ function. A partial differential equation in lepton energy satisfied by the Γ function is obtained. This partial differential equation can be integrated numerically to obtain the positron spectrum for high-energy photons.

INTRODUCTION

Calculations of the pair-production cross section in the point-Coulomb field of the nucleus, in the plane-wave Born approximation (PWBA), have been performed by Heitler and Sauter[1] and Bethe and Heitler.[2] The PWBA calculations are valid for light elements but the results are about 10% high for heavy elements. Many calculations have been performed by using Sommerfeld-Maue wave functions[3] for leptons, which are not valid for all values of Z in the intermediate-energy range (e.g., Maximon and Bethe,[4] Bethe and Maximon,[5] and Davies *et al.*[6]). Even for low energies it is well known from experiments that large deviations from Bethe-Heitler results are expected owing to the distortion of the lepton wave functions in the Coulomb field. These calculations are valid for high-energy photons. These difficulties can be overcome by doing an exact calculation. For exact calculations [the distorted-wave Born approximation (DWBA)] the matrix elements are obtained by using electron (positron) wave functions which are obtained by solving the Dirac equation in the static nuclear Coulomb field. A number of DWBA calculations for pair production are available in the literature, e.g., Jaeger and Hulme[7]; Øverbø *et al.*,[8] Dugne and Proriol,[9] and Tseng and Pratt.[10] It has been possible so far to compute the pair-production cross section only for low-energy photons. The major difficulty in extending the DWBA calculations to intermediate- and high-energy photons lies in the computation of a large number of Appell's hypergeometric functions F_2 for the radial elements. Recently a new method has been developed to handle the radial integrals involving Dirac-Coulomb functions. (For details see Sud *et al.*,[11] Wright *et al.*,[12] and Sud and Sud.[13]) In this method the radial matrix elements for the pair-production process are obtained from the elements of the matrix Γ function. This matrix Γ function has many useful properties, including a recursion relation similar to that of the Γ function. This property will be used to reduce the number of Appell hypergeometric functions F_2 to a minimum, thus reducing the computational work. In the calculations hitherto performed one repeats the complete calculation at many values of the electron or positron energy in order to generate the positron spectrum. Thus one has to compute the Appell hypergeometric function F_2 afresh at every energy. The matrix Γ function satisfies a partial differential matrix equation in energy.[12] Given the matrix Γ function at some energy of the electron (or positron), this equation can be integrated to evaluate the radial matrix elements as a function of energy. This technique has been used by Soto Vargas[14] to compute the spectrum of the virtual photons emitted in an electron-nucleus scattering. In Sec. II we briefly describe the technique and obtain the partial differential equation used to propagate the radial integrals as a function of energy. An expression for the pair-production cross section is given in Sec. I.

I. PAIR-PRODUCTION CROSS SECTION

The differential cross section for pair production is given as

$$d\sigma=(2\pi)^{-4}(r_0^2/\alpha)k^{-1}p_+E_+p_-E_-dE_+d\Omega_+d\Omega_-|H_\rho|^2, \quad (1)$$

where $r_0=e^2/mc^2$, $\alpha=e^2/\hbar c$, k is the photon energy, and $E_\pm$ and $p_\pm$ are the energy and momentum of the positron (+) and electron (−). The matrix element H_ρ is given by

$$H_\rho=\int d^3x\,\psi_-(\vec{\gamma}\cdot\vec{e})\exp(i\vec{k}\cdot\vec{r})\psi_+. \quad (2)$$

The functions ψ_- and ψ_+ are the wave functions for the electron and positron. Their asymptotic forms contain a plane wave plus incoming spherical waves. These functions are solutions of the Dirac equation with a Coulomb potential.[15] The wave function for the electron is given as

$$\psi_- = 4\left(\frac{\pi}{2E_- p_-}\right)^{1/2} \sum_{\mu_-} \sum_{m_- \kappa_-} S_{\kappa_-} C_{m_-} i^{l_-} C(l_-, \tfrac{1}{2}, j_-;\ \mu_- - m_-, m_-) Y_{l_-}^{\mu_- - m_-^*}(\hat{p}) \psi_{\kappa_-}^{\mu_-}, \tag{3}$$

with

$$\psi_\kappa^\mu = \begin{pmatrix} g(r)\ \chi_\kappa^\mu \\ if(r)\ \chi_{-\kappa}^\mu \end{pmatrix}.$$

The radial functions g_κ and f_κ are given by

$$rf_\kappa = \frac{e^{\pi y/2}|\Gamma(\gamma+iy)|}{p(2pr)^{1/2}\Gamma(2\gamma+1)} \operatorname{Im}\{(\gamma+iy)e^{-i(\gamma+1/2)\pi/2} e^{i\eta} M_{-\frac{1}{2}-iy,\gamma}(2ipr)\},$$

$$rg_\kappa = -\left(\frac{E-1}{E+1}\right)^{1/2} \frac{e^{\pi y/2}|\Gamma(\gamma+iy)|}{p(2pr)^{1/2}\Gamma(2\gamma+1)} \operatorname{Re}\{(\gamma+iy)e^{-i(\gamma+1/2)\pi/2} e^{i\eta} M_{-\frac{1}{2}-iy,\gamma}(2ipr)\}, \tag{4}$$

where M is the Whittaker function;

$$\chi_\kappa^\mu = \sum_m C(l, \tfrac{1}{2}, j;\ \mu - m, m) Y_l^{\mu-m}(\hat{r}) \chi^m$$

are the spin-angle functions with

$$\kappa = \begin{cases} l & \text{for } j = l - \frac{1}{2} \\ -l-1 & \text{for } j = l + \frac{1}{2} \end{cases},$$

where l and j are the orbital angular momentum and the total angular momentum, and μ is the eigenvalue of j_z. Some other constants in Eq. (3) are

$$S_\kappa = \exp|i\delta_\kappa|,$$

with

$$\delta_\kappa = \eta - \tfrac{1}{2}\pi\gamma - \arg\Gamma(\gamma+iy) + \tfrac{1}{2}(l+1)\pi,$$

in which

$$\gamma = [\kappa^2 - (Z\alpha)^2]^{1/2};\quad y = \alpha ZE/p,$$

and where η is defined by

$$\exp(2i\eta) = -(\kappa - iy/E)/(\gamma + iy).$$

The function ψ_+ is chosen according to the rules given by Rose.[15] By using the wave function as given in (3), the positron spectrum is given as[8]

$$\frac{d\sigma}{dE_+} = r_0^2 \frac{2}{\alpha k^3} \sum_{\kappa_+ \kappa_- M} |A_{\kappa_+ \kappa_- M}|^2, \tag{5}$$

in which

$$A_{\kappa_+\kappa_-} = f_{\kappa_+\kappa_-} \sum_{L=L_{\min}}^{L_{\max}} (-1)^{(L-L_{\min})/2} \sum_{n=0}^{L} \frac{(L+n)!}{n!(L-n)!} \left(\frac{1}{2k}\right)^n \frac{\Gamma(a)}{(k+p_+ + p_-)^a}$$
$$\times \left[[(E_+ + 1)(E_- + 1)]^{1/2} \left(\frac{(\kappa_+ - M)(\kappa_- - M)}{(2\kappa_+ - 1)(2\kappa_- + 1)}\right)^{1/2} V(l_- L l_+' M) R_{\kappa_+\kappa_-}^{+} \right.$$
$$\left. - [(E_+ - 1)(E_- - 1)]^{1/2} \left(\frac{(\kappa_+ + M)(\kappa_- + M)}{(2\kappa_+ + 1)(2\kappa_- - 1)}\right)^{1/2} V(l_-' L l_+ M) R_{\kappa_+\kappa_-}^{-} \right],$$

where

$$f_{\kappa_+\kappa_-} = \frac{(2p_+)^{\gamma_+ - 1/2}(2p_-)^{\gamma_- - 1/2} e^{\pi(y_+ + y_-)/2} |\Gamma(\gamma_+ + iy_+)|\,|\Gamma(\gamma_- + iy_-)|}{\Gamma(2\gamma_+ + 1)\Gamma(2\gamma_- + 1)},$$

$$V(l_- L l_+' M) = (2L+1)[(2l_- + 1)/(2l_+' + 1)] C(l_-, L, l_+'; 0, 0, 0) C(l_-, L, l_+'; M, M, 0),$$

$$R_{\kappa_+\kappa_-}^{\pm} = \operatorname{Im}\{\exp[-i(\pi/2)(\gamma_+ + \gamma_- - L - 1)]\ [K_+ K_- F_2(a, b_+, b_-; c_+, c_-; x_+, x_-) \pm K_+ K_-^* F_2(a, b_+, b_- - 1; c_+, c_-; x_+, x_-)$$
$$\mp K_+^* K_- F_2(a, b_+ - 1, b_-; c_+, c_-; x_+, x_-) - K_+^* K_-^* F_2(a, b_+ - 1, b_- - 1; c_+, c_-; x_+, x_-)]\}, \tag{6}$$

$$y_\pm = Z\alpha E_\pm / p_\pm,$$

$$\gamma_\pm = [\kappa_\pm^2 - (Z\alpha)^2]^{1/2},$$

and

$$l_\pm = \begin{cases} \kappa_\pm, & \kappa_\pm > 0 \\ -\kappa_\pm - 1, & \kappa_\pm < 0, \end{cases} \qquad l_\pm' = \begin{cases} \kappa_\pm - 1, & \kappa_\pm > 0 \\ -\kappa_\pm, & \kappa_\pm < 0. \end{cases}$$

The values of some parameters are

$$a = \gamma_+ + \gamma_- - n,\quad b_\pm = \gamma_\pm + iy_\pm + 1,\quad c_\pm = 2\gamma_\pm + 1,$$

$$K_\pm = (\gamma_\pm + iy_\pm) e^{i\eta_\pm},$$

where $x_\pm$ and η are defined by

$$x_\pm=\frac{2p_\pm}{p_+ + p_- + k}, \quad e^{2i\eta_\pm}=-\frac{\kappa_\pm - i(y_\pm/E_\pm)}{\gamma_\pm + iy_\pm}. \tag{7}$$

$C(l_-, L, l'_+; M, M, 0)$ is the Clebsch-Gordan coefficient and F_2 is the Appell hypergeometric function.[16] The positron spectrum is obtained by numerical evaluation of Eq. (5). One computes the Appell hypergeometric functions F_2 in the radial integrals $R^{\pm}_{\kappa_+\kappa_-}$ for a particular value of the positron energy; it has then to be reevaluated for each value of the parameter $a (a=\gamma_+ + \gamma_- - n$ for $n=0$ to $L)$. In Sec. II we describe a method which will facilitate the computation of the $R^{\pm}_{\kappa_+\kappa_-}$.

II. ENERGY DEPENDENCE OF THE RADIAL INTEGRALS

If we take the nucleus to be a point charge, the radial Dirac-Coulomb functions for a lepton of energy E and mass m satisfy the first-order matrix differential equation

$$\frac{dU(r)}{dr}=\left(\frac{A}{r}-B\right)U(r), \tag{8}$$

where the constant 2×2 matrices A and B corresponding to the standard form of this equation are given by

$$A=\begin{pmatrix} -\kappa & \alpha Z \\ -\alpha Z & \kappa \end{pmatrix} \text{ and } B=\begin{pmatrix} 0 & -(E+1) \\ E-1 & 0 \end{pmatrix},$$

where κ is the eigenvalue of the Dirac operator $K=\beta(\vec{\sigma}\cdot\vec{L}+1)$. The normalized solution for this particular choice of A and B can be written as

$$U(r)=\begin{pmatrix} V^R(r) & V^I(r) \\ U^R(r) & U^I(r) \end{pmatrix}, \tag{9}$$

where the labels R (I) designate the regular (irregular) solutions of Eq. (8), and are given explicitly in terms of the Whittaker function (compare Eq. 4):

$$V^R(r)=\frac{\exp(\pi y/2)}{p\sqrt{(2pr)}}\,\frac{|\Gamma(\gamma+iy)|}{(2\gamma+1)}\,\mathrm{Re}\{(\gamma+iy) \times\exp[-\tfrac{1}{2}i(\gamma+\tfrac{1}{2})\pi]\exp(i\eta)M_{-1/2-iy,\gamma}(2ipr)\}. \tag{10}$$

$$U^R(r)=-\left(\frac{E-1}{E+1}\right)^{1/2}\frac{\exp(\pi y/2)}{p\sqrt{(2pr)}}\,\frac{|\Gamma(\gamma+iy)|}{(2\gamma+1)}\,\mathrm{Im}\{(\gamma+iy) \times\exp[-\tfrac{1}{2}i(\gamma+\tfrac{1}{2})\pi]\exp(i\eta)M_{-1/2-iy,\gamma}(2ipr)\}.$$

The solutions of Eq. (8) can also be written in a representation in which B is diagonal. The matrices A and B corresponding to these solutions with B in diagonal form are

$$A_\pm=\begin{pmatrix} iy_\pm & \gamma_\pm - iy_\pm \\ \gamma_\pm + iy_\pm & -iy_\pm \end{pmatrix}, \quad B'=\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{11}$$

where the $+(-)$ corresponds to the positron (electron) and $B_\pm = p_\pm B'$. To evaluate $R^{\pm}_{\kappa_+\kappa_-}$ we form the integrand of the direct product of the radial functions $U_\pm$, which is given as

$$W(A,B;r)=U_-(k_1r)\otimes U_+(k_2r)(e^{ikr}/r). \tag{12}$$

The integrand W satisfies a 4×4 matrix differential equation of the type (8). The A and B matrices for this particular case are given as

$$A=A_-\otimes I_2 + I_2\otimes A_+ - I_4,$$
$$B=B_-\otimes I_2 + I_2\otimes B_+ - ikI_4. \tag{13}$$

In these equations I is a unit matrix of the dimension of the subscript level. The integral of such an integrand (Eq. 12) is defined as a matrix Γ function (for details see Ref. 11),

$$\Gamma(A+1,B)=\int_{(0)}^{\infty} W(A,B;r)\,dr, \tag{14}$$

where it is assumed that the integral is convergent at the upper limit and the (0) indicates that any simple poles present at the origin have been removed. The four F_2 functions required for $R^{\pm}_{\kappa_+\kappa_-}$ are obtained from the first column of the 4×4 matrix Γ function. The radial integrals $R^{\pm}_{\kappa_+\kappa_-}$ have to be computed for every value of a $(a=\gamma_+ + \gamma_- - n,\ n=0, L)$ in Eq. (6). We will calculate $R^{\pm}_{\kappa_+\kappa_-}$ by computing the Γ matrix for only one value of n. This is made possible by using the following properties of the integrand:

$$r^a e^{-br} W(A,B;r)=W(A+aI, B+bI; r),$$

and the recursion relation satisfied by the matrix Γ function

$$A\Gamma(A,B)=B\Gamma(A+1,B). \tag{15}$$

Thus we compute the Γ matrix for $n=0$ and solve for other values of n by using Eq. (15). It has been shown by Wright *et al.*[12] that the matrix Γ function satisfies two first-order partial differential equations in the momentum:

$$\frac{\partial\Gamma}{\partial k_i}=T_i\Gamma \quad (i=1,2;\ k_1=p_-,\ k_2=p_+), \tag{16}$$

where the matrices T_i for our case are given in terms of $A_\pm$ and B':

$$T_1=(A_-\otimes I_2)/p_- - (B'\otimes I_2)B^{-1}(A+I_4),$$
$$T_2=(I_2\otimes A_+)/p_+ - (I_2\otimes B')B^{-1}(A+I_4), \tag{17}$$

where A and B are as defined in Eq. (13). To generate the positron spectrum we will require $R^{\pm}_{\kappa_+\kappa_-}$ at various positron energies. This is done by

computing the matrix Γ function as a function of E_+. Thus we have

$$\frac{\partial\Gamma}{\partial E_+}=\frac{\partial\Gamma}{\partial p_+}\frac{\partial p_+}{\partial E_+}+\frac{\partial\Gamma}{\partial p_-}\frac{\partial p_-}{\partial E_+},$$
$$\frac{\partial\Gamma}{\partial E_+}=\left(\frac{E_+}{p_+}T_2-\frac{E_-}{p_-}T_1\right)\Gamma. \tag{18}$$

Equation (18) is straightforwardly solved by numerical integration given the initial value of Γ, thereby evaluating the matrix elements over a complete range, including regions where direct evaluation of Γ may not be feasible.

III. CONCLUSIONS

In the derivation of Eq. (18) we have made the implicit assumption that the matrices $A_\pm$ and B' are independent of the lepton momenta. However, the $A_\pm$ matrices are dependent on the momentum variables through the parameters $y_\pm = \alpha Z E_\pm / p_\pm$. In the calculation of the positron spectrum from a high-energy photon in the field of the nucleus, for positron energy $E_+ \gg m$ (mass of the positron), we can write $y_+ \simeq \alpha Z(1+m^2/2p_+^2)$. We intend to use the technique in the energy range $E_+ \gg m$ where this approximation is valid. For the low-energy end of the spectrum, where the approximation is not valid, the matrix Γ function will be calculated directly. The method results in a considerable saving of computational time and therefore it is possible to perform the calculation for high-energy photons. The calculation of the positron spectrum is in progress and will be published later.

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