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Inverse-bremsstrahlung absorption rate in an intense laser field

L. Schlessinger and J. Wright*
R & D Associates, Marina del Rey, California 90291
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Inverse bremsstrahlung is one of the important mechanisms for heating plasmas with lasers. In the intense fields required for laser fusion, processes involving multiphoton effects are very important. This paper reviews and compares the different approaches used to quantify the effect of intense fields on the inverse-bremsstrahlung absorption rate. The quantum-mechanical treatment of this problem requires summation of the contribution of large numbers of photons. Numerical problems associated with this summation have made direct evaluation difficult. In this paper the authors describe a method, applicable to many potentials of interest, for transforming the sum over a large number of complicated terms into a rapidly converging sum. Numerical results for both classical and quantum treatments are presented and compared, and the errors introduced by several approximations are quantified.

I. INTRODUCTION

Inverse bremsstrahlung is one of the important mechanisms for transferring energy from laser light to matter. In the very intense field used in the laser fusion program, processes involving multiphoton absorption and emission are very important.[1-3] There have been a number of different formalisms suggested for treating inverse bremsstrahlung in intense fields[4-17] and a few numerical calculations.[18] There are some errors in the numerical calculations in the literature and there has been no systematic comparisons of the numerical results of the different formulations. In this paper, we review the present status of the theory and present numerical comparisons which show clearly the magnitude of the errors due to some of the approximations.

The main difficulty with the more elaborate theories is that, at high intensities, large numbers of photons must be included. The numerical problem associated with summing the contributions from many photons make the direct evaluation difficult. For the pure Coulomb potential it is possible to do enough of the problem analytically to avoid the numerical cancellations that occur, but those same methods will not work if shielding effects are included or if the plasma is only partially ionized. In the work discussed here we have succeeded in transforming the sum over a large number of intractable terms into a relatively rapidly converging sum.

The previous analytical works have made a number of approximations, and it is important that these be identified and their reliability assessed since some approximations will have to be made when more exact calculations, which go beyond the Born approximation and which include the effects of the atomic electrons on bremsstrahlung absorption, are done.

The intense-field bremsstrahlung problem has been addressed using two somewhat different approaches. The approach used here has two steps: (i) The cross section for absorption or emission of bremsstrahlung by an electron in an external field is calculated. (ii) The rate at which energy is absorbed by electrons with a specified energy distribution is calculated.

The assumptions or approximations we make for obtaining the cross sections are (a) the electrons interact with infinitely heavy ions via a static, shielded, Coulomb potential; (b) the laser field is treated as a classical plane electromagnetic wave; and (c) the Born approximation is used for the scattering of electrons from ions. The appropriate generalization of plane waves is the exact relativistic wave functions for an electron in a classical plane electromagnetic wave.

The cross section for these approximations is calculated exactly. Several other approximate forms for the cross section such as nonrelativistic, classical nonrelativistic, and classical relativistic are also derived. The approximate forms are useful in numerical computation of the energy absorption rate.

In our approach, to obtain analytic or numerical values for the energy absorption rate, we must specify an electron distribution function. We derive the expression for the energy absorption rate using the various approximate forms of the cross section obtained above by assuming that the electrons have a Maxwellian distribution in their canonical momenta (the random part of their velocities).

The numerical results for the correction factor for the bremsstrahlung absorption rate, calculated according to these assumptions, are presented in Sec. V. The range of applicability of these assumptions will be discussed in Sec. VI. There we also describe methods which allow the assumptions to be relaxed or eliminated.

The other approach[1,9,10] to the calculation of the bremsstrahlung absorption rate involves solving the Boltzmann equation or some approximation to it for the plasma in the presence of the laser beam. Using the distribution function thus obtained, the current J and electric field E are calculated. The rate at which the plasma gains energy from the laser beam is given by the average over a laser period of $J\cdot E$. The assumptions involved in this treatment concern the specification of the collision term in the Boltzmann equation and the approximations used to obtain a solution. In Sec. VI we discuss the relationship between the energy absorption rate obtained using this method and various different approximations and the rate obtained according to our prescription. We show that qualitatively and quantitatively there is little difference in the results obtained via these different approaches for the range of physical parameters of interest.

For simplicity and brevity we will limit our calculations to wavelengths of 1 or 10 μ, intensities from 1.5×10^{14} to 5×10^{16} W/cm^2, and electron temperatures between 100 eV and 10 keV.

Section II contains the derivation of the bremsstrahlung cross section and various approximate forms useful in the numerical calculations. Section III contains the derivation of the expression for the energy absorption rate for a general distribution function and special forms for various approximations to the cross section. Section IV contains a discussion of the alternative methods of derivation of the energy absorption rate. Section V presents the numerical results and comparison among the different expressions. Section VI presents a discussion of the validity of the assumptions and methods to relax the approximations.

II. CROSS SECTION AND TRANSITION RATES

We use unrationalized CGS units throughout with $m_e=\hbar=c=1$. We use the conventions of Ref. 19. The inner product of two four-vectors A and B is represented by $A\cdot B=A_0B_0-\vec{A}\cdot\vec{B}$. We work in the gauge where $k\cdot A=\vec{k}\cdot\vec{A}=0$.

In this section we calculate the cross section and transition rate in the Born approximation for electron scattering in a static Coulomb field in the presence of an intense laser field. First, we solve the exact relativistic wave equation for a scalar (spinless) particle of mass equal to the electron mass in a classical electromagnetic field. Since we are interested in laser beams of high intensity, treating the laser field as a classical external field is a good approximation.[4,7] The purpose of using the relativistic wave equation is to obtain a form for the cross section which allows us to assess the validity of using the dipole approximation for the laser potential in the calculation of the cross section. Since the Dirac equation in second-order form differs only by a spin term from the Klein-Gordon equation, it is simpler and adequate for our purposes to consider a scalar wave equation. If it were necessary, the formalism developed below could be extended to the Dirac equation.[14] Second, the relativistic wave functions are used to obtain the Born approximation expressions for the transition rates and cross sections in an intense field. Using these expressions, we can obtain the nonrelativistic limit for the transition rates. Finally, the classical transition rates are derived for nonrelativistic and relativistic kinematics.

A. Electron wave function

The Klein-Gordon equation for a spinless particle having the mass of the electron in a classical electromagnetic field A (describing the laser field) and a static potential describing the electron Coulomb interactions is

$$\left(\frac{i\partial}{\partial x_\mu}-e(A_\mu+V_\mu)\right)^2\psi=\psi\,. \tag{2.1}$$

For a static Coulomb field, $V_\mu=(V,0,0,0)$, with V given in Eq. (2.9). For this work, we treat V in the Born approximation. The exact solution to Eq. (2.1) with $V=0$ and

$$A_\mu=(0,\vec{A})\,,\quad \vec{A}(\vec{r},t)=\vec{A}(\omega t-\vec{k}\cdot\vec{r}) \tag{2.2}$$

is given by

$$\psi_p=\left(\frac{1}{2\pi}\right)^{3/2}\frac{1}{\sqrt{(2E_p)}}\,e^{-ip\cdot x}\,e^{i\phi_p}\,, \tag{2.3}$$

where

$$\phi_p=\frac{\omega}{2p\cdot k}\int_0^\tau(2e\vec{A}\cdot\vec{p}-e^2\vec{A}\cdot\vec{A})\,d\tau\,, \tag{2.4}$$

with

$$\tau=t-\hat{k}\cdot\vec{r}\,.$$

For a linearly polarized plane wave with $\vec{A}=\vec{\epsilon}\,a\sin(k\cdot x)$, ϕ is given by

$$\phi_p=(1/2p\cdot k)\{-2ea\vec{\epsilon}\cdot\vec{p}\cos(k\cdot x)-\tfrac{1}{2}e^2a^2\times[k\cdot x-\tfrac{1}{2}\sin(2k\cdot x)]\}\,. \tag{2.5}$$

B. Scattering amplitude and cross section

The scattering amplitude for scattering from a state labeled by p_1 to one labeled by p_2 is given by[19]

$$M(p_2,p_1)\,\delta(E_2-E_1-\Delta E) = (2\pi)^3\int d^4y\,\psi^*_{p_1}(y)\left(-2i\,V(y)\,\frac{\partial}{\partial y}\right)\psi_{p_1}(y)\,. \quad (2.6)$$

The transition rate from state p_1 to state p_2 in the interval d^3p_2 is then

$$dR(p_1,p_2)=\frac{|M(p_1,p_2)|^2}{(2\pi)^4}\,d^3p_2\,\delta(E_2-E_1-\Delta E)\,. \quad (2.7)$$

Equations (2.1)–(2.6) lead to the formula

$$M(p_2,p_1)\,\delta(E_2-E_1-\Delta E) = \frac{Ze^2}{4\pi\sqrt{E_1E_2}}\int d^4y\,e^{i(\bar p_2-\bar p_1)\cdot y}\,\frac{e^{-\mu r}}{r}\exp\left\{i\left[\sqrt{\nu}\left(\frac{\vec\epsilon\cdot\vec p_2}{p_2\cdot k}-\frac{\vec\epsilon\cdot\vec p_1}{p_1\cdot k}\right)\cos k\cdot y-\nu\left(\frac{1}{8p_2\cdot k}-\frac{1}{8p_1\cdot k}\right)\sin 2k\cdot y\right]\right\}$$
$$\times\left[\frac{\bar E_1+\bar E_2}{2}-\frac{\sqrt{\nu}\,\omega}{2}\left(\frac{\vec\epsilon\cdot\vec p_2}{p_1\cdot k}+\frac{\vec\epsilon\cdot\vec p_1}{p_2\cdot k}\right)\sin k\cdot y-\frac{\nu\omega}{8}\left(\frac{1}{p_1\cdot k}+\frac{1}{p_2\cdot k}\right)\cos 2k\cdot y\right], \quad (2.8)$$

where we have taken the Coulomb (shielded) potential

$$V(r)=eZe^{-\mu r}/4\pi r \quad (2.9)$$

and

$$\nu=e^2a^2\,. \quad (2.10)$$

Restoring units for the moment we have

$$\nu=r_eE^2\lambda^2/4\pi^2m_ec^2\,,$$

where E is the (peak) local electric field and λ is the laser wavelength. For a traveling wave, E is related to the intensity I by $E^2=8\pi I/v_g$, where v_g is the group velocity. In that case

$$\nu=(2r_e\lambda^2/\pi m_ec^2)(I/v_g)\,.$$

The variable ν is four times the electron jitter energy divided by the rest energy. The jitter energy is the average value of the energy of the electron caused only by the laser field. The momentum $\bar p$ in Eq. (2.8) is given by

$$\bar p_\mu=p_\mu+[\nu/4(p\cdot k)]k_\mu\,. \quad (2.11)$$

The electron within the field $\vec A$ gains momentum from the field. $\bar p$ is the average value of the electron momentum in the field. Discussion of the origin of the momentum shift and its consequences can be found in Refs. 4 and 6.

To proceed further, we use the generating function for the Bessel functions,[20]

$$\exp\left[\frac{Z}{2}\left(t-\frac{1}{t}\right)\right]=\sum_{k=-\infty}^{\infty}t^kJ_k(Z) \quad (2.12)$$

to expand the sin and cos in the exponential of Eq. (2.8). Integrating over y_0 [in Eq. (2.8)] produces an energy-conserving δ function while integrating over y produces the Fourier transform of the shielded Coulomb potential.

$$M(p_2,p_1)\,\delta(E_2-E_1-\Delta E) = \sum_{l=-\infty}^{\infty}M^l(p_1,p_2)\,\delta(\bar E_2-\bar E_1-l\omega)\,, \quad (2.13)$$

where

$$M^l(p_1,p_2) = \frac{2\pi Ze^2}{\sqrt{E_1E_2}(\bar Q_l^2+\mu^2)}\sum_{m=-\infty}^{\infty}(-1)^mJ_m(V_2)\left[J_{l-2m}(V_1)\frac{\bar E_1+\bar E_2}{2}-\frac{\sqrt{\nu}\,\omega}{4}\left(\frac{\vec\epsilon\cdot\vec p_1}{p_1\cdot k}+\frac{\vec\epsilon\cdot\vec p_2}{p_2\cdot k}\right)[J_{l-2m-1}(V_1)+J_{l-2m+1}(V_1)]\right.$$
$$\left.+\frac{\nu\omega}{16}\left(\frac{1}{p_1\cdot k}+\frac{1}{p_2\cdot k}\right)[J_{l-2m-2}(V_1)+J_{l-2m+2}(V_1)]\right], \quad (2.14)$$

with

$$V_1=\sqrt{\nu}\left(\frac{\vec p_1\cdot\vec\epsilon}{p_1\cdot k}-\frac{\vec p_2\cdot\vec\epsilon}{p_2\cdot k}\right), \quad (2.15)$$

$$V_2=\frac{\nu}{8}\left(\frac{1}{p_1\cdot k}-\frac{1}{p_2\cdot k}\right), \quad (2.16)$$

$$\bar Q_l=\vec p_1-\vec p_2-l\vec k\,. \quad (2.17)$$

The rate of transition from state p_1 to state p_2 in the interval d^3p_2 is given by

$$dR(p_1,p_2)=\sum_{l=-\infty}^{\infty} dR^l(p_1,p_2) \tag{2.18}$$

and

$$dR^l(p_1,p_2)=\frac{4Z^2r_e^2}{|\sqrt{\overline{E}_1\overline{E}_2}(Q_l^2+\mu^2)|^2}\,d^3p_2\,\delta(\overline{E}_2-\overline{E}_1-l\omega)$$
$$\times\Bigg|\sum_m(-1)^m J_m(V_2)\bigg[J_{l-2m}(V_1)\frac{\overline{E}_1+\overline{E}_2}{2}-\frac{\sqrt{\nu}\,\omega}{4}\left(\frac{\vec{\epsilon}\cdot\vec{p}_1}{p_1\cdot k}+\frac{\vec{\epsilon}\cdot\vec{p}_2}{p_2\cdot k}\right)[J_{l-2m-1}(V_1)+J_{l-2m+1}(V_1)]$$
$$+\frac{\nu\omega}{16}\left(\frac{1}{p_1\cdot k}+\frac{1}{p_2\cdot k}\right)[J_{l-2m-2}(V_1)+J_{l-2m+2}(V_1)]\bigg]\Bigg|^2, \tag{2.19}$$

where r_e is the classical radius of the electron.

The cross section for absorption ($l<0$) or emission ($l>0$) of l photons is then

$$d\sigma^l=(E_1/p_1)\,dR^l(p_1,p_2). \tag{2.20}$$

C. Nonrelativistic limit

The formula that is usually quoted in the literature[7,9] comes from solving the Schrödinger equation in an oscillating, spatially uniform electric field. Other derivations of the scattering cross section can be found in Ref. 7. This result may be obtained from Eq. (2.19) by taking $V_2\to 0$ and neglecting terms of $O(v/c)$. To this order the cross section with absorption or emission of l photons is given by

$$\frac{d\sigma^l}{d\Omega}=\frac{4Z^2r_e^2}{(Q^2+\mu^2)^2}J_l^2(x)\frac{p_2}{p_1}, \tag{2.21}$$

where

$$\vec{Q}=\vec{p}_1-\vec{p}_2, \tag{2.22}$$

$$\tfrac{1}{2}p_2^2=\tfrac{1}{2}p_1^2\pm l\omega, \tag{2.23}$$

$$x=\sqrt{\nu}(\vec{Q}\cdot\vec{\epsilon})/\omega. \tag{2.24}$$

The total differential cross section is

$$\frac{d\sigma}{d\Omega}=\sum_{l=-\infty}^{\infty}\frac{d\sigma^l}{d\Omega}. \tag{2.25}$$

The Bessel functions satisfy a sum rule

$$1=\sum_{-\infty}^{\infty}J_l^2, \tag{2.26}$$

which is applicable here if we realize that for moderate values of l and velocities of interest, the quantities of interest depend only weakly on l. This leads to a sum rule

$$\frac{d\sigma}{d\Omega}=\sum_{-\infty}^{\infty}\frac{d\sigma^l}{d\Omega}\sim\frac{d\sigma}{d\Omega}\ \text{(elastic, zero field)}. \tag{2.27}$$

Physically if the absorbed and emitted photons are soft, they have little effect on the orbit or scattering of a particle. Experimental confirmation of the results (2.21) and (2.27) can be found in Ref. 2.

The nonrelativistic limit for the transition amplitude can be obtained directly from (2.8) by dropping the term proportional to ν in the exponential, replacing $\overline{p}$ by p, $p\cdot k$ by $m_e\omega$, and neglecting terms of the order v/c. The V_2 terms in Eq. (2.8) represent the leading relativistic corrections. Note that in taking the limit of a spatially uniform field, one must be careful to take the limit in the transition amplitude [i.e., in Eq. (2.8) replace $\sin 2k\cdot y$ by $\sin 2\omega t$] and not the wave function. Taking the limit in the wave function results in the loss of the leading relativistic correction. This correction only matters for longer wavelength lasers such as CO_2 lasers at forseeable intensities. The nonrelativistic expression for the wave function, obtained from Eqs. (2.2) and (2.4), is

$$\psi_{\rm NR}=\frac{1}{\sqrt{2}(2\pi)^{3/2}}\exp\left(i\vec{p}\cdot\vec{x}-\frac{ip^2}{2}t+ie\int^t\vec{A}\cdot\vec{p}\,dt'\right)$$

and for the linearly polarized wave (2.5) the wave function is

$$\psi_{\rm NR}=\frac{1}{\sqrt{2}(2\pi)^{3/2}}\exp\left(i\vec{p}\cdot\vec{x}-\frac{ip^2t}{2}-\frac{iea\vec{\epsilon}\cdot\vec{p}}{\omega}\cos\omega t\right).$$

D. Classical calculations of cross section

A simple classical treatment may be used to obtain an approximate form for the cross section. The basis of the classical approximation, as described in Ref. 7, is that the scattering time is small compared to the period of the laser light. The real value of the classical approximation is that it can easily be extended to the relativistic case. For the relativistic case, calculations using the full formula (2.19) are slowly convergent and quite time consuming. The magnitude of the correction for the relativistic case can easily be estimated using the classical approximation. Furthermore, as we will discuss in Sec. VI, the classical approximation is expected to be excellent in the region where relativistic effects would be important.

The idea of the classical or instantaneous approximation is that electron-ion scattering takes place instantaneously and elastically according to the ordinary Coulomb cross section. Furthermore, the instantaneous energy is conserved in the collision. When this process is rewritten in terms of the canonical momenta, there is an effective energy transfer caused by the laser field. If $v(t)$ is the instantaneous electron velocity, we can write

$$\begin{aligned} v_1(t) &= \vec{p}_1 - e\vec{a}\cos\omega t\,, \\ v_2(t) &= \vec{p}_2 - e\vec{a}\cos\omega t\,. \end{aligned} \tag{2.28}$$

The momenta p_i are constants in the absence of collision. If the time of a collision is short such that ωt does not change appreciably, then the conservation of energy

$$\tfrac{1}{2}v_1^2 = \tfrac{1}{2}v_2^2 \tag{2.29}$$

leads to an effective energy transfer $\frac{1}{2}(p_2^2 - p_1^2)$ given by

$$\Delta E \equiv \xi_{\rm NR} = \sqrt{\nu}\,\vec{Q}\cdot\vec{\epsilon}\cos\alpha\,, \tag{2.30}$$

where $\alpha = \omega t$, the phase of the electromagnetic wave at time of scattering. Since the energy transfer depends on the phase α, it is necessary to average over α to obtain the energy absorption rate.

This approximation is easily extended to the relativistic case with the result

$$\xi_R = (\sqrt{\nu}\,\vec{Q}\cdot\vec{\epsilon}\cos\alpha - \tfrac{1}{2}\nu\vec{Q}\cdot\vec{k}\cos^2\alpha)/(1 + \tfrac{1}{2}\nu\cos^2\alpha)\,. \tag{2.31}$$

[This calculation assumes that the electron is nonrelativistic in its canonical momentum (i.e., $p \ll 1$), but that the jitter velocity ($ea = \sqrt{\nu}$) can be near unity.]

III. ENERGY ABSORPTION RATE

A. General results

The quantity of primary interest is $\dot{W}$, the rate per unit volume at which the electrons absorb energy from the laser. If we define $f(p,E)$ as the electron distribution function normalized such that $\int f(p,E)\,d^3p = 1$ and $dR(p_1,p_2)$ as the transition rate per electron, per ion from state p_1 to state p_2 in the interval d^3p_2 with energy change ΔE, then $\dot{W}$ is given by

$$\dot{W} = N_i N_e \int d^3p_1 \int d^3p_2 [f(\hat{p}_1,E_1) - f(\hat{p}_1,E_1+\Delta E)] \times \Delta E\, dR(p_1,p_2)\,, \tag{3.1}$$

where N_i and N_e are the number densities of ions and electrons.

For convenience we collect the various forms of $\Delta E\, dR(p_1,p_2)$ here. For the general relativistic case, $\Delta E\, dR(p_1,p_2)$ is given by

$$\Delta E (dR)_{\rm rel} = (E_2 - E_1)\sum_l dR^l_{\rm rel} \tag{3.2}$$

and $dR^l_{\rm rel}$ is given in Eq. (2.19). For the nonrelativistic case, $\Delta E\, dR$ is given by

$$\Delta E (dR)_{\rm NR} = (E_2 - E_1)\sum_l dR^l_{\rm NR}\,, \tag{3.3}$$

$$dR^l_{\rm NR} = \delta(E_2 - E_1 - l\omega)\,\frac{4Z^2 r_e^2}{(Q^2+\mu^2)^2}\,J_l^2(x)\,, \tag{3.4}$$

$$x = \sqrt{\nu}\,(\vec{Q}\cdot\vec{\epsilon})/\omega\,. \tag{3.5}$$

For the nonrelativistic classical case, $\Delta E\, dR$ is given by

$$(\Delta E\, dR)^{\rm cl}_{\rm NR} = \int_0^\pi \frac{d\alpha}{\pi}(E_2 - E_1)\,\delta(E_2 - E_1 - \xi_{\rm NR})\,\frac{4Z^2 r_e^2}{(Q^2+\mu^2)^2}\,, \tag{3.6}$$

where

$$\xi_{\rm NR} = \sqrt{\nu}\,\vec{Q}\cdot\vec{\epsilon}\cos\alpha \tag{3.7}$$

and for the relativistic classical case, $\xi_{\rm NR}$ is replaced in Eqs. (3.6) and (3.7) by ξ_R, where

$$\xi_R = \frac{\sqrt{\nu}\,\vec{Q}\cdot\vec{\epsilon}\cos\alpha - \frac{1}{2}\nu\vec{Q}\cdot\vec{\epsilon}\cos^2\alpha}{1 + \frac{1}{2}\nu\cos^2\alpha}\,. \tag{3.8}$$

Equation (3.8) assumes that the electron canonical momentum p is nonrelativistic but that the jitter velocity ($ea = \sqrt{\nu}$) is near unity and thus relativistic.

B. Absorption rate for Maxwellian distribution, shielded Coulomb potential, and nonrelativistic cross-section approximation

We have found that it is extremely time consuming (and probably not necessary) to evaluate $\dot{W}$ using the fully relativistic formula (2.19). Thus we will evaluate $\dot{W}$ nonrelativistically and in Sec. VI discuss the relativistic corrections.

Here we evaluate $\dot{W}$ assuming that the electron distribution function $f(p,E)$ is Maxwellian in the electron canonical momentum (the random part of the electron momentum). Thus

$$f(E) = \exp(-E/T)/(2\pi T)^{3/2}\,, \tag{3.9}$$

$$E = \tfrac{1}{2}p^2\,. \tag{3.10}$$

We define new variables

$$p = \tfrac{1}{2}(p_1 + p_2)\,, \tag{3.11}$$

$$Q = p_1 - p_2\,. \tag{3.12}$$

With these definitions and using Eqs. (3.1), (3.2), and (3.4) several of the integrals can be done, which yields

$$\dot{W} = \frac{16\pi N_i N_e \omega Z^2 r_e^2}{(2\pi T)^{1/2}} \times \sum_{l=1}^{\infty} l \sinh\frac{\omega l}{2T} \int_0^\infty Q\,dQ \times \int_{-1}^{1} dx \exp\left[-\left(\frac{\omega l}{Q}\right)^2 \Big/ 2T - \frac{Q^2}{8T}\right] \times J_l^2\left(\sqrt{\nu}\,\frac{Qx}{\omega}\right)\Big/(Q^2+\mu^2)^2 . \tag{3.13}$$

The weak-field or "standard" result for the energy absorption rate can be obtained from Eq. (3.13) as the leading term in ν. This term is obtained by keeping only the $l=1$ term in Eq. (3.13) and using the small argument expansion of $J_1(x)$; i.e., $J_1(x) \to x/2$. Thus

$$\dot{W}_{\rm std} = \tfrac{8}{3}\pi Z^2 r_e^2 N_i N_e \frac{\pi}{(2\pi T)^{3/2}} \frac{\sinh(\omega/2T)}{\omega/2T} \cdot \times \int_0^\infty Q^3\,dQ \exp\left(-\frac{(\omega/Q)^2}{2T} - \frac{Q^2}{8T}\right)\Big/(Q^2+\mu^2)^2 . \tag{3.14}$$

The standard result usually found in the literature[1] is obtained from Eq. (3.14) by neglecting shielding (letting $\mu \to 0$). We shall briefly discuss the validity of that result in Sec. VI. Letting $\mu \to 0$, we have

$$\dot{W}_{\rm std}^{\rm No\ shield} = \tfrac{8}{3}\pi Z^2 r_e^2 N_i N_e \frac{\pi}{(2\pi T)^{3/2}} \nu \frac{\sinh(\omega/2T)}{\omega/2T} K_0 \frac{\omega}{2T} . \tag{3.15}$$

Although Eq. (3.13) is a relatively simple expression, for the parameters of interest, it is necessary to consider terms in the sum for $l \geq 1000$. To avoid this problem, we proceed to evaluate the l sum. To accomplish this, we show that

$$\sum_{l=1}^{\infty} l \sinh\frac{\omega l}{2T} J_l^2\left(\sqrt{\nu}\,\frac{Qx}{\omega}\right) e^{-(\omega l/Q)^2/2T} = \frac{1}{\pi}\int_0^\infty d\psi\, J_0\left(2\sqrt{\nu}\,\frac{Qx}{\omega}\sin\psi\right) \times \frac{1}{2\pi}\int_{-\infty}^{\infty} dl\, e^{2il\psi}\, l \sinh\frac{\omega l}{2T} e^{-(\omega l/Q)^2/2T} . \tag{3.16}$$

There are two steps involved; first we use

$$J_l^2(y) = \frac{1}{\pi}\int_0^\pi J_0(2y\sin\psi)\cos 2l\psi\, d\psi \tag{3.17}$$

and then use

$$\sum_{-\infty}^{\infty} g(n) = \sum_{-\infty}^{\infty} G(n) , \tag{3.18}$$

where g and G are Fourier transforms of each other. In this case,

$$g(l) = l \sinh\frac{\omega l}{2T}\cos 2l\psi \exp\left[-\left(\frac{\omega l}{Q}\right)^2\Big/2T\right], \tag{3.19}$$

$$G(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{2i\pi n l} g(l)\, dl . \tag{3.20}$$

Define

$$S(\psi) = \frac{1}{8\sqrt{\pi}}\left(\frac{Q}{\omega}\right)^3 (2T)^{3/2} \exp\left(-\frac{2\psi^2 Q^2 T}{\omega^2} + \frac{Q^2}{8T}\right) \times \left(\frac{\omega}{T}\cos\frac{Q^2\psi}{\omega} - 4\psi \sin\frac{Q^2\psi}{\omega}\right). \tag{3.21}$$

Then $G(n) + G(-n) = S(\psi+\pi n) + S(\psi - \pi n)$ and

$$\int_0^\pi d\psi\left(\sum_{n=1}^{\infty}[S(\psi+n\pi) + S(\psi-n\pi)] + S(\psi)\right) = \int_{-\infty}^{\infty} S(\psi)\,d\psi = 2\int_0^\infty S(\psi)\,d\psi . \tag{3.22}$$

Thus Eq. (3.13) becomes

$$\dot{W} = \frac{16\pi Z^2 r_e^2 N_e N_i \omega}{(2\pi T)^{1/2}} \int_0^\infty Q\,dQ \exp\left(-\frac{Q^2}{8T}\right)\Big/(Q^2+\mu^2)^2 \times \int_{-1}^{1}\frac{dx}{\pi}\int_0^\infty d\psi\, J_0\left(2\sqrt{\nu}\,\frac{Qx}{\omega}\sin\psi\right) S(\psi) . \tag{3.23}$$

We now use the expansion for $J_0(2\rho x)$ from Ref. 20 to obtain

$$\int_{-1}^{1} dx\, J_0(2\rho x) = \sum_0^\infty \frac{2}{2n+1}(-1)^n \frac{\rho^{2n}}{(n!)^2} , \quad \rho = (\sqrt{\nu}/\omega)\,Q\sin\psi . \tag{3.24}$$

To evaluate $\dot{W}$ we still have to do two integrals and one sum, but there are no Bessel functions and the sum converges much faster than the sum over Bessel functions (3.13). This expression is

$$\dot{W} = \frac{4TZ^2 r_e^2 N_e N_i}{\omega^2 \pi}\int_0^\infty \frac{dQ\,Q^4}{(Q^2+\mu^2)^2}\int_0^\infty d\psi \exp\left(-2\frac{\psi^2 Q^2 T}{\omega^2}\right)\left[\frac{\omega}{T}\cos\left(\frac{Q^2\psi}{\omega}\right) - 4\psi\sin\left(\frac{Q^2\psi}{\omega}\right)\right] \times \sum \frac{2}{2n+1}\frac{(-1)^n}{(n!)^2}\left(\frac{\nu Q^2 \sin^2\psi}{\omega^2}\right)^n . \tag{3.25}$$

For a general potential, the integrals would have to be done numerically; however, for a Coulomb potential, we can do one more (the Q) integral analytically.

If we define

$$Re^{i\theta}=\psi+i\omega/2T\,,\quad \tan\theta=\omega/2T\psi\,, \tag{3.26}$$

$$I_n=\psi[(\omega/R)\tan\theta]^{n+1/2}\,\Gamma(n+\tfrac{1}{2})\times[\tan\theta\cos(n+\tfrac{1}{2})\,\theta-2\sin(n+\tfrac{1}{2})\,\theta]\,. \tag{3.27}$$

Defining

$$A(\psi)=2\sum_{1}^{\infty}\frac{(-1)^n}{(n!)^2(2n+1)}\left(\frac{\nu}{\omega^2}\sin^2\psi\right)^n I_n\,, \tag{3.28}$$

then

$$\dot{W}=\frac{4TZ^2r_e^2N_eN_i}{\omega^2\pi}\int_0^\infty d\psi\,A(\psi)\,. \tag{3.29}$$

In this form it is easy to evaluate the expression numerically. The "standard" expression is the term in Eq. (3.28) which is proportional to ν. Thus the correction F_{nr} to the standard expression is the ratio of the sum to the first term in the sum. In the form (3.28) the expression appears to be a function of both ν and ν/T. For practical purposes, the correction factor is essentially a function of only ν/T. This fact is demonstrated in Figs. 1 and 2. The parameter ν/T is proportional to the ratio of the electron jitter energy to its thermal energy.

One can obtain a slightly different form by expanding the Bessel function in Eq. (3.13) [and using (3.17)] in powers of its argument. This is the procedure adapted in Ref. 15. For the Coulomb potential this leads to

$$W=\frac{16N_eN_iZ^2r_e^2}{(2\pi T)^{1/2}}\times\sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{(-1)^{n+l}}{2n+1}\frac{(2n)!}{n!\,n!}\frac{1}{(n+l)!}\frac{1}{(n-l)!}\times\left(\frac{\nu l}{2\omega}\right)^n K_{n-1}\frac{\omega l}{2T}\sinh\left(\frac{\omega l}{2T}\right)\,.$$

Unfortunately, the modified Bessel functions are all singular for small values of their agrument and large cancellations take place. It is possible to exhibit this cancellation for the Coulomb potential, but it will probably prove quite difficult for other potentials. The loss of accuracy is such that it is not possible to directly evaluate the expression.

C. Energy absorption rate for Maxwellian distribution and classical nonrelativistic cross-section approximation

The energy absorption rate for the classical nonrelativistic approximation, assuming that the electrons have a Maxwellian distribution in the canonical momentum, is obtained using Eqs. (3.6)–(3.8). Thus

$$\dot{W}^c_{\rm NR}=\int\frac{d\alpha}{\pi}\int d^3p_1\int d^3p_2\left(\frac{(E_2-E_1)\,\delta(E_2-E_1-\xi_{\rm NR})}{(2\pi T)^{3/2}}\right)\times\left[\exp\left(\frac{-p_1^2}{2T}\right)-\exp\left(\frac{-p_2^2}{2T}\right)\right]\frac{4Z^2r_e^2}{(Q^2+\mu^2)^2}\,, \tag{3.30}$$

where

$$\xi_{\rm NR}=\sqrt{\nu}\,\vec{Q}\cdot\vec{\epsilon}\cos\alpha\,.$$

Defining the same variables as before [Eqs. (3.11) and (3.12)], we find that several integrals can be done, which yields

$$\dot{W}^c_{\rm NR}=\frac{16Z^2r_e^2N_eN_i}{(2\pi T)^{1/2}}\int_0^\infty Q\,dQ\,\frac{e^{-Q^2/8T}}{(Q^2+\mu^2)^2}\int_0^\infty d\alpha\times\int_{-\pi}^{\pi}\sin\theta d\theta\,e^{-\xi^2/2Q^2T}\,\xi\sinh\frac{\xi}{2T}\,. \tag{3.31}$$

If we approximate

$$\sinh\frac{\xi}{2T}\approx\frac{\xi}{2T}\,, \tag{3.32}$$

then the integral over Q separates from the others. Note that for the classical approximation the integral over momentum transfer requires a shielding factor for convergence. For the analogous quantum-mechanical case [see Eq. (3.14)], the integral is finite without the shielding factor.

The high-field correction factor, defined as

$$F=\frac{\dot{W}/\nu}{\lim_{\nu\to 0}(\dot{W}/\nu)}\,, \tag{3.33}$$

is given by

$$F^{\rm cl}_{\rm NR}=\frac{3}{\pi}\int_0^\pi d\alpha\int_{-1}^{1}dy\,y^2\cos^2\alpha\exp\left(-\frac{\nu y^2\cos^2\alpha}{2T}\right)\,. \tag{3.34}$$

This correction factor is obviously a function of ν/T.

D. Energy absorption rate for Maxwellian distribution and classical relativistic cross-section approximation

The energy absorption rate for the classical relativistic approximation, assuming that the electrons have a Maxwellian distribution in the canonical momentum, is obtained using Eqs. (3.8), (3.9), and (3.13). Defining the same variables as before [Eqs. (3.11) and (3.12)], we find that several integrals can be done, yielding

$$\dot{W}^c_R=\frac{16Z^2r_e^2N_eN_i}{(2\pi T)^{1/2}}\int_0^\infty\frac{Q\,dQ}{(Q^2+\mu^2)^2}\int_0^\pi\frac{d\alpha}{\pi}\int_{-1}^{1}d\cos\theta\times\int_0^{2\pi}d\phi\exp\left(-\frac{\xi_R^2}{2Q^2T}-\frac{Q^2}{8T}\right)\xi_R\sinh\frac{\xi_R}{2T}\,. \tag{3.35}$$

If we approximate

$$\sinh\frac{\xi_R}{2T}\approx\frac{\xi_R}{2T}\,, \tag{3.36}$$

the relativistic correction factor can be written as

$$F^{\,\rm cl}_{\rm rel}=\frac{3}{2\pi^2}\int_0^{\pi}d\alpha\int_{-1}^{1}dy\int_0^{2\pi}d\phi\,\exp\left(-\frac{\nu}{2T}h^2\right)h^2\,, \tag{3.37}$$

where

$$h=\frac{\cos\alpha[y-\frac{1}{2}\sqrt{\nu}\,\cos\alpha(1-y^2)\sin\phi]}{1+\frac{1}{2}\nu\cos^2\alpha}\,. \tag{3.38}$$

IV. PLASMA CALCULATIONS

A. Introduction

The other method used to derive the correction factor is a plasma-physics approach.[1,9,10] There is also extensive literature regarding this approach. The plasma approach has two major steps. First, the Boltzmann equation (or some approximation to it) for the electron distribution function in the laser field is solved simultaneously with Maxwell's equations which determine the net electromagnetic fields. The rate at which plasma electrons gain energy from the laser is then given by the average of $J\cdot E$, over a laser period. Here J is the net electron current and E the net electric field. The approximations involved in this approach are related to the specification of the collision term in the Boltzmann equation and to the approximation used to solve it. Below we discuss two representative solutions and their relationship to our results.

B. Kidder approach

In Kidder's approach,[1] the electric field used in the Boltzmann equation and in the calculation of $J\cdot E$ is taken to be the laser field. Furthermore, the collision integral is approximated by an effective collision frequency. Thus the Boltzmann equation to be solved is

$$\frac{\partial f}{\partial t}-\frac{eE_{\rm laser}\cdot\nabla_v f}{m}=\gamma f\,, \tag{4.1}$$

where γ is the standard Coulomb momentum transfer collision frequency, and the energy absorption rate is

$$\dot W=N_e e\left\langle\int d^3v\,\frac{\partial f}{\partial t}(v)\vec v\cdot\vec A_{\rm laser}\right\rangle, \tag{4.2}$$

where the angular brackets indicate average over a laser period. To lowest order in γ, the distribution function is Maxwellian in the canonical momentum, i.e.,

$$f_0(v)=\exp[-(v-eA)^2/2T]/(2\pi T)^{3/2} \tag{4.3}$$

and

$$\frac{\partial f}{\partial t}=\gamma f_0\,. \tag{4.4}$$

Taking

$$\gamma\propto\frac{1}{v^3}\,, \tag{4.5}$$

the correction factor, for linear polarization, defined in Eq. (3.33) is identical to the result in Eq. (3.34). The published curve[1] is for circular polarization and is not the same as that obtained from Eq. (3.34), which is for linear polarization.

C. Dawson-Oberman approach

In the model of Dawson and Oberman, the electrons are scattered by infinitely heavy ions. The equations solved in this approach are

$$\frac{\partial f}{\partial t}+\vec v\,\frac{\partial f}{\partial r}-\frac{e}{m}(\vec E_{\rm laser}-\vec\nabla\phi)\cdot\frac{\partial f}{\partial v}=0 \tag{4.6}$$

and

$$\nabla^2\phi=eN_e\int d^3v f(v)-Ze\sum_i\delta(r-r_i)\,. \tag{4.7}$$

Equations (4.6) and (4.7) are solved by linearizing f about a Maxwellian distribution in the electron canonical momentum; i.e.,

$$f=f_0(v-eA)+f_1\,. \tag{4.8}$$

Solving for f in this approximation and calculating

$$\dot W=eN_e\left\langle\int d^3v f(v)\,v(E_{\rm laser}-\nabla\phi)\right\rangle, \tag{4.9}$$

one obtains a result for W that is essentially the same as Eq. (3.13). The results differ only in the representation of the shielding and the necessity of introducing a high-velocity cutoff. The Dawson-Oberman results include both electron and ion shielding while the results quoted in Sec. III include only electron shielding. The representation of the shielding will affect the value of the cross section, but the strong-field correction factor is relatively insensitive to it. Thus the Dawson-Oberman approach gives numerical results for the strong-field correction factor essentially the same as Eq. (3.13).

V. NUMERICAL RESULTS

We have evaluated the previous expressions for ranges of parameters of interest. Where there is any significant deviation between the numerical results from different formulas we have displayed the deviation in a graph. The numerical results were obtained for two wavelengths, $\lambda=1\ \mu$ and $\lambda=10\ \mu$; intensities from 1.5×10^{14} to 5×10^{16}

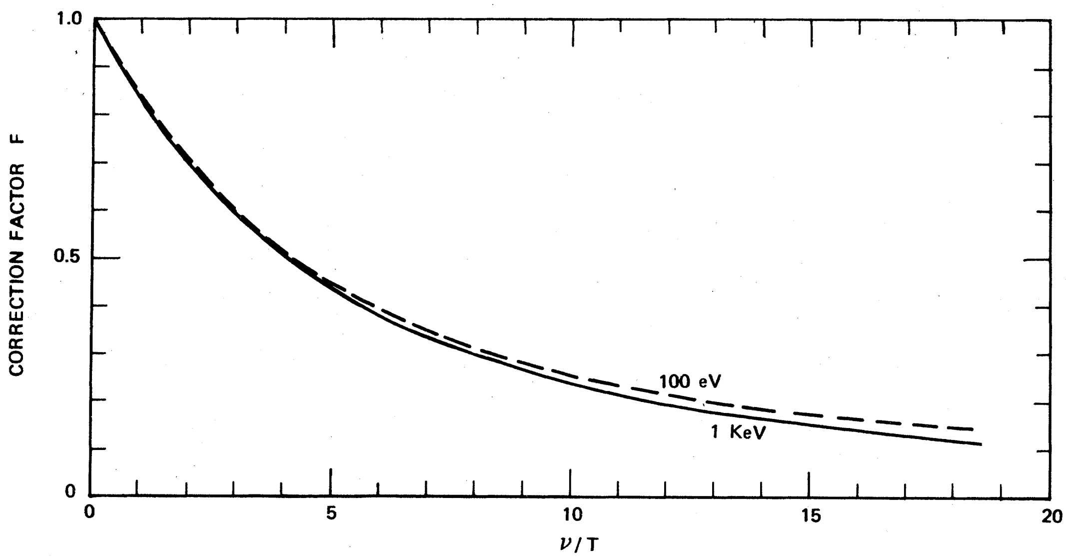

FIG. 1. Intense-field correction factor F for the inverse-bremsstrahlung absorption rate as a function of ν/T (ν/T is $2\langle v_{osc}^2\rangle|v_{th}^2$) for two values of the electron temperature. Wavelength equals 1.

W/cm^2, and temperatures from 100 eV to 2 keV. Our conclusions based on the numerical results are summarized below.

A. Summary

(a) For practical purposes the standard[1] or low-field bremsstrahlung absorption rate should be multiplied by a numerical correction factor. This factor is plotted in Fig. 1.

Note: The calculations performed by Brysk result in a correction factor F presented as a function of the variable x (see Fig. 1 of Ref. 15). These results agree with ours if x is identified with $\nu/2T$. According to Brysk's definition (see discussion on pp. 1260 and 1262),

$$x = 0.14\,\frac{I}{3\times 10^{15}\ \mathrm{W/cm^2}}\left(\frac{\lambda}{1\,\mu}\right)^2 \Big/ \frac{T}{1\ \mathrm{keV}}$$

or

$$x = \nu/8T\,.$$

This disagrees with our relation by a factor of 4. This error can be traced to the work of Osborne.[13] Brysk makes use of Osborne's results. Osborne's Eqs. (15) and (16) are in error (see Refs. 7 and 2), and the argument of the Bessel function in Osborne's Eqs. (15) and (16) should be $2\sqrt{\eta}x$ instead of $\sqrt{\eta}x$. Thus Osborne's results and hence Brysk's numerical results apply to an intensity of $\frac{1}{4}$ of that quoted by Brysk. Use of Brysk's results will underestimate the effect of intese fields on bremsstrahlung absorption.

(b) The correction factor is quite significant in the lower-temperature range at moderate intensities (1.5×10^{15} W/cm^2). For the higher intensities of some recent experiments (1.5×10^{16}), absorption is reduced by a factor of 2 even at a temperature of 2 keV, (see Fig. 2).

(c) The correction factor is, in principle, a function of two variables ν/T and T. For all practical purposes, at $\lambda = 1-10\ \mu$ the correction factor depends only on the variable ν/T. (This approximation has not yet been verified for shorter wavelengths and should be checked before it is used for $\lambda < 1\mu$.) The variable ν/T is proportional to the ratio of the electron jitter energy to its thermal energy. That is, ν/T is given by

$$\frac{\nu}{T} = \frac{2\langle v_{osc}^2\rangle}{v_{th}^2}\,,$$

where $\langle v_{osc}^2\rangle$ is the time average of the electron oscillation velocity in the laser field and v_{th}^2 equals kT.

The correction factor obtained using formula (3.29) is plotted as a function of ν/T for two extreme temperatures in Fig. 1. The variable ν is given by

$$\nu = \frac{r_e}{m_e c^2}\cdot\frac{E^2\lambda^2}{4\pi^2}\,, \qquad (5.1)$$

where E is the maximum value of the local electric field, r_e the classical electron radius, m_e the electron mass, λ the laser wavelength, and c the velocity of light. To obtain the correct deposition rate, the local electric field should be used in Eq. (5.1) to obtain ν. For a traveling wave the electric field is related to the intensity I by $E^2 = 8\pi I/v_g$, where v_g is the group velocity. For a plasma with

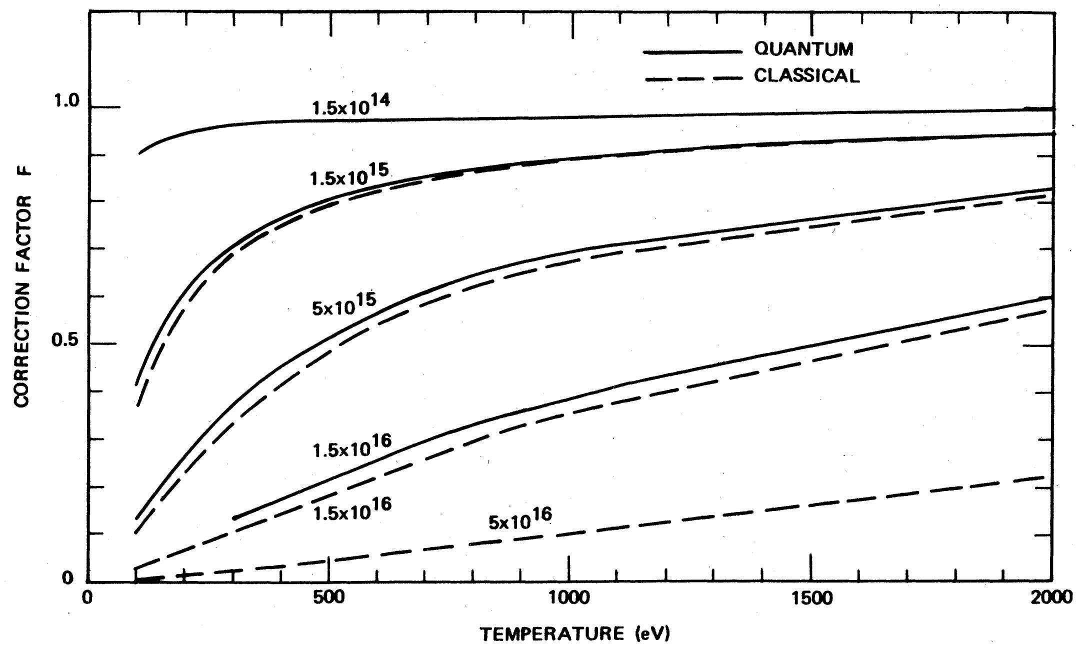

FIG. 2. Comparison of the classical and quantum-mechanical intense-field correction factor F for inverse-bremsstrahlung absorption rate as a function of electron temperature for several laser intensities. Here we have assumed a traveling wave and $\omega_p \ll \omega$. Wavelength equals 1μ.

$v_g = c(1-\omega_p^2/\omega^2)^{1/2}$ we have

$$\nu = \frac{2.194\,22}{\sqrt{(1-\omega_p^2/\omega^2)}} \times 10^{-3} \frac{I}{3\times 10^{15}\ \mathrm{W/cm^2}} \left(\frac{\lambda}{1}\ \mu\right)^2 \tag{5.2}$$

and

$$\frac{\nu}{T} = \frac{1.1212}{\sqrt{(1-\omega_p^2/\omega^2)}} \frac{I}{3\times 10^{15}\ \mathrm{W/cm^2}} \left(\frac{\lambda}{I}\ \mu\right)^2 \Big/ \frac{T}{1}\ \mathrm{keV}, \tag{5.3}$$

where ω_p is the plasma frequency. In relating ν to intensity in our calculations, we have taken $v_g = c$.

(a) The assumption that the scattering is instantaneous compared to the period of the wave (the classical approximation or the Kidder approach) which leads to the expression (3.34) gives results for the correction factor which are accurate to better than 10%. In Fig. 2 we compare the classical result (3.34) to the quantum result [(3.29) or (3.13)] for several values of the intensity. For the lower intensities the results are indistinguishable. The fact that the instantaneous approximation is so good may be significant in trying to incorporate strong-field effects into the usual corrections to the bremsstrahlung absorption coefficient (such as non-Born approximation effects, multiple scattering, etc.).

(b) For longer wavelengths (10 μ) the correction factor reduces the bremsstrahlung absorption coefficient by a large amount. Since the weak-field result is also substantially smaller for longer wavelengths, it is likely that inverse-bremsstrahlung absorption plays a very minor role in electron heating at 10-μ wavelengths.

(c) Relativistic effects. We evaluated the expression (3.37) for the correction factor including first-order relativistic effects. These effects were insignificant for $\lambda = 1\,\mu$. For $\lambda = 10\,\mu$, there was a small effect at the highest intensity of 5×10^{16} W/cm^2. Roughly, there is no correction until ν exceeds unity. In this range, the inverse bremsstrahlung absorption coefficient is so small that bremsstrahlung processes are probably no longer of interest for this application.

(d) The dipole approximation $[A(x,t) \approx A(t)]$, which is typically used to obtain the nonrelativistic formula, is a good approximation. Evaluation of the fully relativistic formula (2.19) indicates that the validity of the dipole approximation [neglect of V_2 terms and neglect of $l\vec{k}$ in Eq. (2.17)] is doubtful for the absorption of many photons. However, in calculating the energy absorption rate, the integration region where this approximation could be in error was strongly suppressed. Thus the dipole approximation is quite good.

(e) The approximate form for the correction factor

$$F \simeq (1 + \tfrac{1}{3}\langle v_{\mathrm{osc}}^2\rangle / v_{\mathrm{th}}^2)^{-3/2}$$

or

$$F \simeq (1 + \nu/6T)^{-3/2}$$

which is used by Faehl and Roderick[3] to address the absorption rate in a standing wave is in agreement with our correction factor (Fig. 1) to within about 10%.

VI. DISCUSSION OF RESULTS

Our approach to this problem has two major steps: (i) calculation of the rate (or cross section) at which an electron of energy E_1 makes transitions to energy E_2 in the presence of a laser field and (ii) calculation, using the rate obtained above, of the rate at which a distribution of electrons absorb energy from the laser beam.

The assumptions we have made for the first step are as follows:

(a) The laser field can be described as a classical plane electromagnetic wave.

(b) An electron interacts with a single, infinitely heavy ion via a static, shielded Coulomb potential. Ion-ion correlations have been neglected.

(c) The Born approximation is used for the scattering of electrons from the ions.

Assumption (a) is well satisfied in the intense coherent electromagnetic wave of the laser.[4] Assumption (b) concerns specification of the appropriate potential to include in the calculation of the cross section. A better model than used here would include the effects of partial ionization and ion-ion correlations. These effects, for cross sections at low intensity, are presently being addressed, and it is expected that they will contribute significantly to the cross sections. However, our results suggest that the high-intensity *correction factor* may not be sensitive to the form of the potential whenever the weak-field Coulomb cross section depends only on the momentum transfer. In general, the cross section depends on both energy and momentum transfer. However, the Born approximation cross section depends only on momentum transfer. The correction factor obtained using the instantaneous approximation is a good approximation to the exact result. Furthermore, the correction factor obtained using the instantaneous approximation is independent of the form of the potential as long as the Born approximation is valid or the cross section depends only on momentum transfer. These results suggest that the correction factor may be relatively insensitive to the form of the potential.

Assumption (c) concerns the validity of the Born approximation. There are important regions of electron temperature for which the Born approximation is in error by more than 50% even for the weak-field case.[21] Within the framework of the instantaneous approximation, Kroll and Watson[7] have derived a formalism for the intense-field correction which can be used for short-range potentials when the Born approximation is not valid. However, for pure Coulomb potentials or long-range potentials, the low-frequency theorem used by Kroll and Watson is not valid and there are important corrections even in the weak-field limit.[21] Thus to calculate the intense-field effects in cases dealing with low electron temperature or partially ionized atom scattering, where the Born approximation is not valid, new approximation techniques which specifically include the effects of the long-range Coulomb potential must be used.

For the second step of the calculation we have assumed only that the electron distribution function is Maxwellian in the canonical momenta. If the electron distribution is a sum of Maxwellians, a correction factor can be applied for each effective temperature. If the distribution is noniostropic, the integral in Eq. (3.10) must be evaluated with the appropriate distribution functions.

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*Present address: Physics Dept., Univ. of Illinois, Urbana, Ill. 61801.

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