Cluster decomposition of multiparticle scattering operators

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Cluster properties of multiparticle scattering operators defined in terms of a separation variable are shown to be equivalent to cluster properties specified in terms of an angular variable. It is shown that these formulations lead to a formulation of cluster properties in terms of an angular momentum limit. Finally, using a spectral representation of the scattering operators derived from unitarity, the author shows that scattering operators satisfy cluster requirements if the eigenvectors in the spectral representation split into products of eigenvectors in the large-angular-momentum limit.

I. INTRODUCTION

Any theory dealing with particles for which the interactions are of short range must have a separability or cluster property. Separability means that if a cluster of interacting particles is broken into two subclusters and the two subclusters are moved far apart from one another, there should be no interaction between the subclusters. Such a requirement is automatically satisfied for nonrelativistic theories with a local Hamiltonian, or in a relativistic quantum field theory where the interactions are mediated by local fields.[1] However, in relativistic particle theories the condition must be imposed separately. In fact, it is easy to construct models involving multiparticle processes that do not satisfy cluster properties.[2]

Attempts at constructing a relativistic particle theory have had a long history.[3] Recently, cluster properties have been formulated for N-particle systems with relativistic Hamiltonians.[4] In this paper, however, cluster properties will be discussed in the context of an S-matrix theory of the type first envisaged by Heisenberg,[5] in which there are no equations of motion; rather, scattering amplitudes are the basic theoretical objects and the goal is to impose enough physical conditions—such as relativity and unitarity—so as to fix to the greatest extent possible the form of the scattering amplitudes.

A variant of this point of view forms the background for this paper. The theory under consideration is an operator-S-matrix theory, in which physical conditions are expressed as operator conditions on the scattering operator. It is clear that unitarity can be expressed as an operator condition and in a previous paper[6] the notion of a crossed scattering operator was introduced to express crossing relations as operator relations. Further, Coester[7] has shown how cluster properties can be formulated as operator relations with the help of a "separation" operator. If $\vec{P}$ is the momentum operator, then $e^{-i\vec{P}\cdot\vec{a}}\varphi$ translates the wave function φ by an amount $\vec{a}$. To separate a cluster of particles into two subclusters let $\vec{p}$ be the relative momentum operator of the two subclusters, so that $e^{-i\vec{p}\cdot\vec{a}}$ separates the two subclusters by an amount $\vec{a}$. Then a way of formulating a cluster property as an operator relation is to demand that as $|\vec{a}|$ gets large, $e^{i\vec{p}\cdot\vec{a}}S_{B,A}e^{-i\vec{p}\cdot\vec{a}}$ should tend towards $S_{B_1A_1}\otimes S_{B_2A_2}$, where $S_{B,A}$ is the channel scattering operator for the $A\to B$ reaction, and A_1 and A_2 are subclusters of the initial cluster A, while B_1 and B_2 are subclusters of the final cluster B. The sense of the limit is a strong operator limit and will be discussed more fully in Sec. II. By ranging over all possible subclusters of A and B, a set of disconnected operators $S_{B_1A_1}\otimes S_{B_2A_2}$ is formed, so that $S_{B,A}$ can be written as a sum over disconnected operators plus a remainder, which is the connected operator.

Because the cluster property is formulated as an operator property, no definite set of variables for the wave functions are singled out. But partial-wave variables are of interest because they are the simplest variables in which to express the content of unitarity. In the previous paragraph the relative momentum operator of the two subclusters was used to define the separation operator $U_{\vec{a}}=e^{-i\vec{p}\cdot\vec{a}}$. But the relative momentum operator does not commute with the relative orbital angular momentum operator. Hence, in using partial-wave variables, it is convenient to replace $|\vec{a}|$ by the relative orbital angular momentum; this is what is done in nonrelativistic scattering theory in going from an impact parameter to an orbital angular momentum. But the orbital angular momentum is in general not a diagonal variable of a multiparticle scattering amplitude, so what is used in its stead is the total angular momentum. The goal of Sec. II is to show that cluster properties of the channel scattering operators, expressed as a strong operator limit in a multi-

particle Hilbert space as $|\vec{a}|$ gets large, implies a cluster property for the channel scattering operators as a strong operator limit in so-called partial-wave Hilbert spaces, when the total angular momentum gets large. As an intermediate step it will be shown that the strong operator limit as $|\vec{a}|$ gets large is equivalent to a strong operator limit with respect to an angular variable $\mathcal{L}$. $\mathcal{L}$ occurs in the spherical Bessel function $j_{\mathcal{L}}(pa)$ and is thus related to the separation of two clusters of particles with relative momentum p as a gets large.

Using the parameter $\mathcal{L}$, it is possible to formulate the main result in a more group-theoretical language. Let a set A of noninteracting particles $1,\ldots,N$ be broken into subclusters, with subcluster A_1 containing particles $1,\ldots,M$ and subcluster A_2 particles $M+1,\ldots,N$. Couple all of the particles in cluster A_1 together (i.e., take the M-fold tensor product of the wave functions of particles $1,\ldots,M$) and all of the particles in cluster A_2 together. The tensor-product reduction is naturally expressed in terms of partial-wave Hilbert spaces because in the reduction there will be a set of labels that can be thought of as particlelike labels of clusters A_1 and A_2; that is, A_1 and A_2 can be thought of as "particles" with a given mass, spin, momentum, and spin projection, along with additional labels that specify the internal configurations of the subclusters.[8] If these two "particles" are coupled together, then, as their relative orbital angular momentum gets large, cluster A breaks up into two subclusters A_1 and A_2. It will be shown that in this limit the parameter $\mathcal{L}$ becomes identical with the total angular momentum so that as $\mathcal{L}$ gets large, the total angular momentum gets large. In fact, it is not even necessary to refer to the relative orbital angular momentum for the parameter $\mathcal{L}$ is identified not only with the total angular momentum, but also indicates which set of particles are being separated—that is, which particles go into cluster A_1 and which particles go into cluster A_2.

Section III will discuss a coupling scheme that does not refer to the relative orbital angular momentum. The variables that replace the relative orbital angular momentum L and the total intrinsic spin j of the two subclusters are the spin projections of the two subclusters. Such variables are appropriate for investigating the relationship of a cluster property with the unitarity equations. More specifically, unitarity can be used to spectrally represent channel S operators.[9] Combining the spectral representation of the channel S operators with the formulation of a cluster property in terms of large angular momentum suggests conditions that the eigenvectors of the projected S operators should satisfy. The main result of Sec. III is to show that if the eigenvectors of the projected S operators factor into two parts as the angular momentum gets large, the cluster property in terms of large angular momentum is guaranteed to hold.

II. CLUSTER LIMITS

Let $\mathcal{K}_A$ denote the Hilbert space of a cluster A of free noninteracting particles so that $\mathcal{K}_A=\Lambda_A\mathcal{K}$, where Λ_A is a projection operator from the full Fock space $\mathcal{K}$ to $\mathcal{K}_A$. $\mathcal{K}_A$ is formed out of a tensor product of single-particle Hilbert spaces. If $\mathcal{K}_B=\Lambda_B\mathcal{K}$ is the Hilbert space for a cluster B of free particles, then the projected or channel scattering operator for the reaction $A\to B$ carrying elements from $\mathcal{K}_A$ to $\mathcal{K}_B$ is defined to be

$$S_{B,A}=\Lambda_B S\Lambda_A\,, \tag{2.1}$$

where S is the total scattering operator acting on the full Fock space $\mathcal{K}$.

The goal of this section is to analyze the reaction $A\to B$ when A is broken into two subclusters A_1 and A_2, which are spatially separated so that only the subreactions $A_1\to B_1$ and $A_2\to B_2$ can proceed; here B_1 and B_2 are two spatially separated subclusters of B, whose union is B.

To define a separation operator, it is most convenient to replace the momenta of subclusters A_1 and A_2, $\vec{P}_1$ and $\vec{P}_2$ with the total momentum $\vec{P}=\vec{P}_1+\vec{P}_2$ and a relative momentum $\vec{p}$, defined by $p=B^{-1}(P)P_1$; $B(P)$ is a boost Lorentz transformation defined in Appendix A. (p will stand for both a four-vector and the magnitude of a three-vector; the context should make it clear which meaning is intended.) For the four-momenta P, P_1, and P_2, invariant masses are given by $s=(\Sigma_{i\varepsilon A}P_i)^2$, $s_1=(\Sigma_{i\varepsilon A_1}P_i)^2$, and $s_2=(\Sigma_{i\varepsilon A_2}P_i)^2$, respectively.

As discussed in the Introduction, the action of the separation operator is given by

$$(U_{\vec{a}}^{\mp}F)(\vec{P},\vec{p},\ldots)=e^{-i\vec{p}\cdot\vec{a}}F(\vec{P},\vec{p},\ldots)\,, \tag{2.2}$$

where F is an element of $\mathcal{K}_A$ depending on $\vec{p}$. The remaining variables in F will be specified after a transition to partial-wave variables is made. In any event, $U_{\vec{a}}^{\mp}$ acts only on $\vec{p}$ and leaves all the other variables untouched. It would be possible to define a separation operator that moves only one cluster, leaving the other cluster fixed; however, moving both clusters involves the relative momentum $\vec{p}$, which can be transformed to a relative orbital angular momentum that is symmetric with respect to both clusters.

The scattering operator of Eq. (2.1) is cluster decomposed by considering $U_{\vec{a}}^{\dagger}S_{B,A}U_{\vec{a}}^{\mp}$ in the limit as $|\vec{a}|\to\infty$. As in Ref. 7, this limit is given a

precise mathematical sense by taking the strong operator limit:

$$\underset{|\vec{a}|\to\infty}{\text{s-lim}}\ U_{\vec{a}}^{\dagger} S_{B,A} U_{\vec{a}} = S_{B_1A_1}\otimes S_{B_2A_2}$$

means

$$\lim_{|\vec{a}|\to\infty} \|(U_{\vec{a}}^{\dagger} S_{B,A} U_{\vec{a}} - S^d)F\| = 0\,,$$

for all F in $\mathcal{H}_A$ and for all directions $\hat{a}=\vec{a}/|\vec{a}|$. This strong operator limit can also be written

$$\underset{|\vec{a}|\to\infty}{\text{s-lim}} \|(S_{B,A} - S^d)U_{\vec{a}}F\| = 0\,, \tag{2.3}$$

since $U_{\vec{a}}$ commutes with the disconnected scattering operator $S^d \equiv S_{B_1A_1}\otimes S_{B_2A_2}$.

The main point of this paper is to show that the limit of spatial separation $|\vec{a}|\to\infty$ of Eq. (2.3) implies a cluster property expressed in terms of a limit of large angular momentum. Because the limit involves angular momentum, it is convenient to introduce partial-wave variables. Now partial-wave variables can be thought of as variables that treat a cluster as though it were a particle of "mass" $\sqrt{s}$, momentum $\vec{P}$, spin J, and spin projection σ, along with any other variables that are needed to describe the internal configurations of the particles in the cluster. Thus one can write an element of $\mathcal{H}_A$ as $f(\vec{P}sJ\sigma,A)$, where f will denote angular-momentum-type wave functions and A any set of variables that specify the internal configuration of the A cluster of particles. For example, if A is a two-particle cluster, the momentum wave function $F(\vec{p}_1,\vec{p}_2)$ becomes $f(\vec{P}sJ\sigma)$, where $\vec{P}=\vec{p}_1+\vec{p}_2$, $s=(p_1+p_2)^2$, and J,σ come from a spherical harmonic transform of $\hat{p}_1$, the direction of particle 1 in the two-particle center-of-mass (CM) frame. Then the action of $S_{B,A}$ on f can be written

$$(S_{B,A}f)(\vec{P}sJ\sigma,B) = \sum_{\sigma_A} S(sJ;B,A) f(\vec{P}sJ\sigma,A)\,, \tag{2.4}$$

where $S(\ )$ is the kernel for the $S_{B,A}$ operator. That S does not depend on $\vec{P}$ and σ, and is diagonal in s and J, is a reflection of the relativistic invariance of $S_{B,A}$. In fact, since both $S_{B,A}$ and $U_{\vec{a}}$ do not change the total momentum, the label $\vec{P}$ will be suppressed in the subsequent discussion; further, since both $S_{B,A}$ and $U_{\vec{a}}$ are diagonal in the total invariant mass $\sqrt{s}$, we will as a matter of convenience assume that all wave functions f are sharply peaked about some value of $\sqrt{s}$.

To examine the consequences of an angular momentum limit it is necessary to examine the variables in Eq. (2.2) somewhat more carefully. Separating subcluster A_1 from A_2 suggests first coupling all the A_1 particles together, all the A_2 particles together, and then coupling the two subclusters together to give the state of the entire A cluster. That is, the remaining variables that go into the function F of Eq. (2.2) can be chosen as s_1, J_1, σ_1, the "mass", spin, and spin projection of subcluster A_1, and other variables (denoted by A_1) necessary to specify internal configurations of subcluster A_1, along with a similar set of variables for subcluster A_2. Thus Eq. (2.2) can be written more precisely as

$$(U_{\vec{a}}F)(\vec{p}, s_1J_1\sigma_1A_1, s_2J_2\sigma_2A_2) = e^{-i\vec{p}\cdot\vec{a}}F(\vec{p}, s_1J_1\sigma_1A_1, s_2J_2\sigma_2A_2)\,. \tag{2.5}$$

To find the action of $U_{\vec{a}}$ on the total angular momentum wave functions, we transform $\vec{p}$ to a relative orbital angular momentum $L\sigma_L$ and then couple L to the total intrinsic spin j to get the total angular momentum J. Equation (A11) gives the connection between the momentum wave function of Eq. (2.5) and the various angular momentum wave functions defined in the Appendix:

$$\begin{aligned}
&(U_{\vec{a}}f)(sL\sigma_L, s_1J_1\sigma_1A_1, s_2J_2\sigma_2A_2)\\
&\quad=\left(\frac{2L+1}{4\pi}\right)^{1/2}\int d\hat{p}\, D^{L}_{\sigma_L0}(\hat{p})e^{-i\vec{p}\cdot\vec{a}}F(\vec{p}, s_1J_1\sigma_1A_1, s_2J_2\sigma_2A_2)\\
&\quad=\int d\hat{p}\, Y_{L\sigma_L}(\hat{p})\sum_{\mathfrak{L}M} i^{\mathfrak{L}}j_{\mathfrak{L}}(pa)Y_{\mathfrak{L}M}(\hat{p})Y^*_{\mathfrak{L}M}(\hat{a})\sum_{L'\sigma'_L} Y^*_{L'\sigma'_L}(\hat{p})f(sL'\sigma'_L, s_1J_1\sigma_1A_1, s_2J_2\sigma_2A_2)\\
&\quad=\sum_{\substack{\mathfrak{L}\mathfrak{M}\\ L'\sigma'_L}} i^{\mathfrak{L}}Y^*_{\mathfrak{L}\mathfrak{M}}(\hat{a})j_{\mathfrak{L}}(pa)\langle L'\sigma'_L|\mathfrak{L}\mathfrak{M}L\sigma_L\rangle\langle L'0|\mathfrak{L}0L0\rangle f(sL'\sigma'_L, s_1J_1\sigma_1A_1, s_2J_2\sigma_2A_2)\,,\\
&(U_{\vec{a}}f)(sJ\sigma Lj, s_1J_1A_1, s_2J_2A_2)\\
&\quad=\sum_{\substack{\sigma_Lm\\ J'\sigma'\\ \mathfrak{L}\mathfrak{M}\\ L'\sigma'_L}}\langle J\sigma|L\sigma_L jm\rangle\langle L'\sigma'_L|\mathfrak{L}\mathfrak{M}L\sigma_L\rangle\langle L'0|\mathfrak{L}0L0\rangle i^{\mathfrak{L}}Y^*_{\mathfrak{L}\mathfrak{M}}(\hat{a})j_{\mathfrak{L}}(pa)\langle J'\sigma'|L'\sigma'_L jm\rangle f(sJ'\sigma'L'j, s_1J_1A_1, s_2J_2A_2)\\
&\quad=\sum_{\substack{\mathfrak{L}\mathfrak{M}\\ J'\sigma'\\ L'}} i^{\mathfrak{L}}j_{\mathfrak{L}}(pa)Y^*_{\mathfrak{L}\mathfrak{M}}(\hat{a})[(2L'+1)(2J+1)]^{1/2}W(jLJ'\mathfrak{L};JL')\langle J'\sigma'|\mathfrak{L}\mathfrak{M}J\sigma\rangle\langle L'0|\mathfrak{L}0L0\rangle f(sJ'\sigma'L'j, s_1J_1A_1, s_2J_2A_2)\,.
\end{aligned} \tag{2.6}$$

$j_{\mathcal{L}}(pa)$ is a spherical Bessel function and p is connected with the total invariant "mass" of cluster A by $\sqrt{s}=(s_1+p^2)^{1/2}+(s_2+p^2)^{1/2}$; $W(\ )$ is a Racah coefficient.[10]

Now one expects the limits as $|\vec{a}|\to\infty$ to be connected with the limit as the relative orbital angular momentum of the clusters get large. But in order to express this limit in terms of a diagonal quantity like the total angular momentum J, it is necessary to couple j, the total intrinsic spin of the two clusters, to the orbital angular momentum L. Then, when J gets large, the Clebsch-Gordan (or Racah) coefficient involving J, L, and j will force L to get large, which means that the large J limit controls the separation of the A_1 and A_2 clusters by forcing their relative orbital angular momentum to get large.

In working out these limits it is convenient to express the action of $U^{\vec{a}}_{a}$ on elements in $\mathcal{H}_A$ in such a way as to distinguish between parameters and internal Hilbert-space variables. To that end, define the operator $R^{\mathcal{L}J}$ by

$$(R^{\mathcal{L}J}f)(sJ'\sigma'LjA)\equiv\sum_{L'}[(2L'+1)(2J+1)]^{1/2}\times W(jLJ'\mathcal{L};JL')\times\langle L'0|\mathcal{L}0L0\rangle f(sJ'\sigma'L'jA) \tag{2.7}$$

so that

$$(U^{\vec{a}}_{a}f)(sJ\sigma LjA)=\sum_{\substack{\mathcal{L}\mathfrak{M}\\J'\sigma'}} i^{\mathcal{L}}j_{\mathcal{L}}(pa)Y^{*}_{\mathcal{L}\mathfrak{M}}(\hat{a})\langle J'\sigma'|\mathcal{L}\mathfrak{M}J\sigma\rangle\times(R^{\mathcal{L}J}f)(sJ'\sigma'LjA)\,. \tag{2.8}$$

By using the operator $R^{\mathcal{L}J}$ it is possible to show that the strong operator limit as $|\vec{a}|\to\infty$, Eq. (2.3), is equivalent to a strong operator limit as the angular variable $\mathcal{L}$ gets large. Now the sphere is compact so that the strong operator limit of Eq. (2.3) can also be obtained after integrating over all directions $\hat{a}$. Then

$$\int d\hat{a}\,\|(S_{B,A}-S^{d})U^{\vec{a}}_{a}F\|^{2}=\sum_{\mathcal{L}}j^{2}_{\mathcal{L}}(pa)\sum_{J}\|(S_{B,A}-S^{d})R^{\mathcal{L}J}f\|^{2}\,; \tag{2.9}$$

here use has been made of the orthogonality properties of the spherical harmonics $Y_{\mathcal{L}\mathfrak{M}}(\hat{a})$ and the Clebsch-Gordan coefficients $\langle J'\sigma'|\mathcal{L}\mathfrak{M}J\sigma\rangle$. In fact, the operator $R^{\mathcal{L}JL}$ was defined in Eq. (2.7) so as to make use of the orthogonality properties of these functions. Appendix B shows that the limit as $|\vec{a}|\to\infty$ is equivalent to the limit as $\mathcal{L}\to\infty$; that is,

$$\lim_{|\vec{a}|\to\infty}\|(S_{B,A}-S^{d})U^{\vec{a}}_{a}f\|=0 \tag{2.3}$$

is equivalent to

$$\lim_{\mathcal{L}\to\infty}\sum_{J}\|(S_{B,A}-S^{d})R^{\mathcal{L}J}f\|=0\,, \tag{2.10}$$

for all f in $\mathcal{H}_A$. To arrive at a limit involving the total angular momentum it is necessary to broaden the class of Hilbert spaces by defining new Hilbert spaces $\mathcal{H}_{\vec{P}sJ\sigma,A}$ called partial-wave Hilbert spaces. A partial-wave Hilbert space is labeled not only by the cluster A, but also by the momentum $\vec{P}$, mass $\sqrt{s}$, and angular momentum and projection $J\sigma$ of this cluster. The norm for elements of a partial-wave Hilbert space is given by

$$\|f_{\vec{P}sJ\sigma}\|^{2}=\sum\!\!\!\!\!\!\int |f_{\vec{P}sJ\sigma}(\cdot)|^{2}<\infty\,; \tag{2.11}$$

$\sum\!\!\!\!\!\!\int$ is a sum and/or integral over the internal variables of the A cluster. It is clear that $\mathcal{H}_A$ can be written as a direct integral of partial-wave spaces, $\mathcal{H}_A=\sum_{J\sigma}\int d^{4}p\oplus\mathcal{H}_{\vec{P}sJ\sigma,A}$. More importantly, there are elements of the partial-wave spaces that are not elements of $\mathcal{H}_A$. This can be seen most readily by noticing that if $f\in\mathcal{H}_A$ has as one of its variables the total angular momentum, then $|f(J)|$ must go to zero as J gets large, whereas elements of partial-wave Hilbert spaces for which J is large do not in general tend to zero. As stated previously, the subscripts $\vec{P}$ and s can be dropped because none of the operators under consideration involve these variables. Thus we are interested in analyzing the cluster properties of scattering operators as they act on sequences $\mathcal{H}_{J\sigma,A}$ of partial-wave Hilbert spaces for large J.

To be more precise about these limits, note that Eq. (2.10) can be written in terms of partial-wave spaces as

$$\lim_{\mathcal{L}\to\infty}\sum_{JJ'\sigma'}\|(S_{B,A}-S^{d})R^{\mathcal{L}J}f_{J'\sigma'}\|=0\,, \tag{2.10}$$

$f_{J'\sigma'}\in\mathcal{H}_{J'\sigma',A}$, from which it follows that

$$\lim_{\mathcal{L}\to\infty}\|(S_{B,A}-S^{d})R^{\mathcal{L}J}f_{J'\sigma'}\|=0\,, \tag{2.12}$$

for J,J',σ' arbitrary but fixed. Now the coupled angular momenta $\mathcal{L}$, J, and J' are related to each other by $J=\mathcal{L}-J'+k$, $k=0,\ldots,2J'$ for $\mathcal{L}$ large, or $\mathcal{L}=J+J'-k$. Therefore, for fixed J', the limit in $\mathcal{L}$ is equivalent to a limit in J. And for fixed $J'\sigma'$ it is clear from the definition of $R^{\mathcal{L}J}f_{J'\sigma'}$ [Eq. (2.7)] that $R^{\mathcal{L}J}f_{J'\sigma'}$ is an element in the partial-wave Hilbert space $\mathcal{H}_{J_0,A}$. More importantly, $R^{\mathcal{L}J}f_{J'\sigma'}$ spans $\mathcal{H}_{J_0,A}$. It therefore follows that

$$\lim_{J\to\infty}\|(S_{B,A}-S^{d})F_{J_0}\|=0\,, \tag{2.13}$$

for all F_{J_0} in $\mathcal{K}_{J_0,A}$. However, it is important to stress that Eq. (2.13) is not a strong operator limit. Equation (2.13) is valid only when $R^{\mathcal{L}J}$ acts on elements of $\mathcal{K}_A$ whose variables are obtained by coupling together particles in A_1 and in A_2, and then coupling these two subclusters together. If other subclusters formed out of cluster A were coupled together, or if variables other than those of Eq. (2.6) were used, $R^{\mathcal{L}J}$ would have a different (albeit unitarily equivalent) form so that the limit in J of Eq. (2.13) would not hold. Thus the angular momentum limit in Eq. (2.13) is valid only when variables are used that refer to the subclusters being separated. In contrast, Eqs. (2.3) and (2.10) are genuine strong operator limits, valid for any choice of variables for elements of f in $\mathcal{K}_A$.

Also, the strong operator limit in the separation variable $|\vec{a}|$, Eq. (2.3), is equivalent to the strong operator limit in the angular variable $\mathcal{L}$, Eq. (2.10). But while these strong operator limits imply the angular momentum limit, Eq. (2.13), it does not seem possible to go in the opposite direction without some further control on the variable J. That is, the limit $\|(S_{B,A}-S^d)R^{\mathcal{L}J}f\|=0$ as $\mathcal{L}\to\infty$ for arbitrary but fixed J does *not* imply a limit $\sum_J\|(S_{B,A}-S^d)R^{\mathcal{L}J}f\|=0$ as $\mathcal{L}\to\infty$, now summed over all J, so that the angular momentum limit, Eq. (2.13), does not imply a strong operator limit in the separation variable $|\vec{a}|$, Eq. (2.3).

As examples of a cluster decomposition we consider one-line disconnected and $2\to N$ disconnected terms, as they are of interest in Sec. III, where cluster decomposition properties are combined with unitarity. A one-line disconnected operator means that $A_1\to B_1$ consists of just one particle not interacting with any other particles; hence, in writing $S_{B,A}\to S_{B_1A_1}\otimes S_{B_2A_2}$, the action of $S_{B_1A_1}\otimes S_{B_2A_2}$ on elements in $\mathcal{K}_A$ can be given as

$$(S_{B_1A_1}\otimes S_{B_2A_2})f(sJ\sigma Lj,s_1J_1,s_2J_2B_2)=\sum_{A_2}\!\!\!\!\!\!\int S(s_2J_2;B_2A_2)f(sJ\sigma Lj,s_1J_1,s_2J_2A_2)\,, \quad (2.14)$$

where $\sqrt{s_1}$ is the mass and $J_1(=0)$ is the spin of the single particle. The orbital angular momentum L goes to infinity as J goes to infinity. Then elements of $\mathcal{K}_A\to\mathcal{K}_B$ should split into $\mathcal{K}_{A_1}\otimes\mathcal{K}_{A_2}\to\mathcal{K}_{B_1}\otimes\mathcal{K}_{B_2}$. But there remains the intrinsic spin label $j=J_1\otimes J_2$. In the next section it will be shown that L and j can be replaced by σ_1 and σ_2, even in the limit when L gets large. With such a change of variables, further Kronecker δ's occur in the kernel, Eq. (2.14); these δ functions (in $s_2J_2\sigma_2$) express, in partial-wave variables, the fact that the direction and energy of the particle in the A_1 cluster does not change in the overall reaction.

A similar result holds for the $2\to N$ disconnected term; as $J\to\infty$, $S_{B,A}\to S_{B_1A_1}\otimes S_{B_2A_2}$, where $A_1\to B_1$ is a $2\to N$ reaction. The action of the tensor product is given by

$$(S_{B_1A_1}\otimes S_{B_2A_2})f(sJ\sigma Lj,s_1J_1B_1,s_2J_2B_2)=\sum_{A_2}\!\!\!\!\!\!\int \mathfrak{A}^{2\to N}(s_1J_1,B_1)S(s_2J_2,B_2,A_2)\times f(sJ\sigma Lj,s_1J_1,s_2J_2A_2) \quad (2.15)$$

where $\mathfrak{A}^{2\to N}(s_1J_1,B_1)$ is the partial-wave amplitude for the $2\to N(A_1\to B_1)$ reaction. Again the Lj variables can be converted to $\sigma_1\sigma_2$ variables.

The results that have been obtained in this section are of course not restricted to relativistic systems. Since we have chosen canonical spin as a spin-projection variable (see Appendix A), it is straightforward to carry out a nonrelativistic limit of the above relativistic variables. But in a nonrelativistic theory one has a Hamiltonian that includes potential terms. These potential terms are usually functions of magnitudes of differences of position variables, $V_{ij}|\vec{x}_i-\vec{x}_j|$. So to derive cluster properties for nonrelativistic systems means dealing with position, rather than momentum variables, and then checking strong operator limits of the potentials with respect to the total angular momentum.[11]

For example, in a three-body cluster decomposition the relevant matrix elements are $\langle\vec{p}_1\vec{p}_2\vec{p}_3|V_{ij}|\vec{p}_1'\vec{p}_2'\vec{p}_3'\rangle$. For V_{12} there is no difficulty in separating off particle 3. However, the proper limits become important when particle 1 (or 2) is to be separated. Then it is necessary to go to a (free-particle) coupling scheme of the form 1 (23). If the matrix element is written as

$$\langle\vec{p}_1(\vec{p}_2,\vec{p}_3)|V_{12}|\vec{p}_1'(\vec{p}_2',\vec{p}_3')\rangle=\int d^3x_1\,d^3x_2\,d^3x_3\left[e^{i\sum_{j=1}^{3}(\vec{p}_j'-\vec{p}_j)\cdot\vec{x}_j}\right]V_{12}|\vec{x}_1-\vec{x}_2| \quad (2.16)$$

and both momentum and position variables changed to relative momenta and positions, the same sorts of Bessel functions and Clebsch-Gordan coefficients appear as in Eq. (2.10). Only in the case of potential scattering, whether or not the matrix element (2.16) goes to zero strongly as the angular momentum goes to infinity, depends on the nature of $V_{12}(|\vec{x}_1-\vec{x}_2|)$, rather than on the assumed limit, Eq. (2.3).

III. INELASTIC UNITARITY AND CLUSTER PROPERTIES

In this section constraints imposed by unitarity on the scattering operators will be combined with cluster properties involving the angular momentum limit discussed in Sec. II. Now $S_{B,A}=\Lambda_B S\Lambda_A$, the projected or channel scattering operator, does not by itself satisfy an operator relation of the form

$SS^\dagger = S^\dagger S = I$. But, though $S_{B,A}$ is not even a normal operator, it is, as shown in Ref. 9, closely associated with a normal operator and hence can be spectrally represented.

Before discussing the spectral representation for $S_{B,A}$ it is necessary to be somewhat more precise about the variables used in elements of the Hilbert spaces. Thus far $\mathcal{H}_A$ has denoted the tensor product of single-particle Hilbert spaces, the tensor product being taken with respect to all the particles in the A cluster. In partial-wave Hilbert spaces elements are labeled by parameters that include the invariant mass $\sqrt{s}$, the total momentum $\vec{P}$, the angular momentum J, and spin projection σ of the cluster, along with other variables needed to specify the internal configuration of the cluster. Now if $\mathcal{H}_{\vec{P}sJ\sigma,A}$ is split into two subspaces corresponding to two subclusters of particles in $A, A_1 \cap A_2 = 0, A_1 \cup A_2 = A$, then each of these subspaces should have a similar set of partial-wave labels of the form s_i, $\vec{P}_i$, J_i, and σ_i, along with other variables needed to specify the internal configurations of the subclusters. But $\vec{P}_1$ and $\vec{P}_2$ can be replaced by the total momentum $\vec{P}$ and $\vec{p}$ [see Eq. (2.2)]; then $\vec{p}$ can be replaced by s, L, and σ_L, the relative orbital angular momentum and orbital-angular-momentum projection of the two subclusters. When L is coupled to j, (the spin angular momentum of the two subclusters), the variables J and σ result, which, along with s and $\vec{P}$, indeed give the partial-wave variables needed in $\mathcal{H}_A$. In obtaining the variables of $\mathcal{H}_{\vec{P}sJ\sigma,A}$ in this way, the variables s_1, s_2, J_1, J_2, L, and j become part of the set of internal variables. But when the limit for large J is taken and $\mathcal{H}_A$ splits into $\mathcal{H}_{A_1}$ and $\mathcal{H}_{A_2}$, two of the partial-wave variables, namely σ_1 and σ_2, do not occur as labels in $\mathcal{H}_{s_1 J_1 \sigma_1, A_1}$ and $\mathcal{H}_{s_2 J_2 \sigma_2, A_2}$, respectively. What is needed is a change of variables from L, j to σ_1, σ_2. But, because of the Clebsch-Gordan coefficients connecting J, L, and j, it is not clear what happens to the change of variables when J, and hence L, gets large.

We will show that, even for large values of J, the change of variables for L, j to σ_1, σ_2 is well defined, and in particular the Clebsch-Gordan coefficients remain finite. This is most easily seen by first making a change of variables, $J(Lj) \to J(lj)$, defined by $J = j + L - l$ with $l = 0, \ldots, 2j$. Then, according to Eq. (A10), it is possible to write

$$f(sJ\sigma, s_1 J_1 \sigma_1 A_1, s_2 J_2 \sigma_2 A_2) = \sum_{j=|J_1-J_2|}^{J_1+J_2} \sum_{l=0}^{j} \langle jm | J_1 \sigma_1 J_2 \sigma_2 \rangle \langle J\sigma | L\,0\,jm \rangle \times f(sJ\sigma, Lj, s_1 J_1 A_1, s_2 J_2 A_2), \tag{3.1}$$

where $L = J - j + l$. For fixed J_1 and J_2 the sums here are clearly finite. What must be investigated is the limit for large J of $\langle Jm | J - j + l\,0, jm \rangle$. This limit is given in Appendix B, Eq. (B7), and is clearly well behaved. Thus we conclude that the spin-projection variables σ_1 and σ_2 can replace L and j and the limit in J still makes sense. This means that the coupling scheme, and not the use of the variable L, determines which clusters will be separated when J gets large.

For the remainder of this section—when discussing inelastic unitarity—it will be assumed that spin-projection variables are used. The reason is to see how eigenvectors of $S_{B,A}$ break up into eigenvectors of S_{B_1,A_1} and S_{B_2,A_2} and the variables used in each of these eigenvectors should be of the same type, namely of the form $sJ\sigma$, needed as labels for the partial-wave spaces.

Reference 9 shows how the unitarity equations $SS^\dagger = S^\dagger S = I$ can be written in terms of the channel scattering operators, Eq. (2.1), to give a spectral representation of the form

$$S_{B,A} = \int d\mu(\gamma) \eta_\gamma e_\gamma^B \otimes d_\gamma^{A\dagger}, \tag{3.2}$$

where the spectral measure $d\mu(\gamma)$ is unknown except that $\gamma = 1$ corresponds to a true eigenvector e_1^B, which is proportional to the $2 \to B$ partial-wave amplitude. The proportionality factor is $1/(1 - \eta^2)^{1/2}$, where η is the inelasticity parameter. When the angular momentum of the $2 \to B$ partial-wave amplitude gets large, $\eta \to 1$ so that the $2 \to B$ partial-wave amplitude goes to zero, as is to be expected for a connected amplitude. $\eta_\gamma = 1$ for $\gamma \neq 1$ and equals η for $\gamma = 1$. Including $\gamma = 1$, $\{e_\gamma^B\}$ spans $\mathcal{H}_B$ but the e_γ^B are not in general orthogonal in $\mathcal{H}_B$. That $S_{B,A}$ is not a normal operator is reflected in the fact that the set $\{d_\gamma^B\}$ also spans $\mathcal{H}_B$ but does not coincide with $\{e_\gamma^B\}$; also d_1^B is proportional to the $B \to 2$ partial-wave amplitude, again with the proportionality factor $1/(1 - \eta^2)^{1/2}$. All of the statements made for the B cluster also, of course, hold for the particles in the A cluster.

We now ask whether the convergence of $S_{B,A}$ to $S_{B_1 A_1} \otimes S_{B_2 A_2}$ for large J implies any conditions on the eigenvectors of $S_{B,A}$ converging to the eigenvectors of $S_{B_1 A_1} \otimes S_{B_2 A_2}$. That is, if a sequence of normal (bounded) operators (labeled by J) converge to a limit normal operator, under what circumstances will the eigenvectors of the sequence of normal operators converge to the corresponding eigenvectors of the limit operator? Reference 12 states some conditions under which eigenvector convergence takes place. In our problem, however, very little is known about the spectral measure $d\mu(\gamma)$, so these theorems are of

little help.

But the converse is of considerably more interest as far as physical applications are concerned. For in trying to build an operator-S-matrix theory in which the eigenvectors are the fundamental theoretical objects, what is of interest is to find conditions on the eigenvectors that will guarantee that the cluster requirements are satisfied. The relevant theorem here states that if the eigenvectors of a sequence of normal operators converge to the corresponding eigenvectors of the limit operator, then the operators themselves converge to the limit operator. Thus, if we assume that

$$e^B_{\gamma_1\gamma_2}(sJ; s_1J_1\sigma_1B_1, s_2J_2\sigma_2B_2) \to e^{B_1}_{\gamma_1}(s_1J_1;B_1)e^{B_2}_{\gamma_2}(s_2J_2;B_2),$$

$$d^B_{\gamma_1\gamma_2}(sJ; s_1J_1\sigma_1B_1, s_2J_2\sigma_2B_2) \to e^{B_1}_{\gamma_1}(s_1J_1;B_1)e^{B_2}_{\gamma_2}(s_2J_2;B_2) \tag{3.3}$$

as $J\to\infty$, then $S_{B,A}\to S_{B_1A_1}\otimes S_{B_2A_2}$. $\gamma=\gamma_1\gamma_2$ means that the spectral measure can be written as a product measure so that as J gets large $e^B_{\gamma_1\gamma_2}$ splits into $e^{B_1}_{\gamma_1}e^{B_2}_{\gamma_2}$. The splitting into products is assumed to hold for all subclusters so there will in general be many different ways in which the spectral measure γ breaks into a product measure, depending on the scheme used to couple two subclusters together to form the variables for the entire cluster. Note that the e_γ and d_γ are elements of partial-wave spaces only, since as J gets large they do not tend to zero.

Two cases of particular interest involve the splitting of a cluster into one particle plus the remainder, which generates the one-line disconnected form of Eq. (2.14), and into two particles plus the remainder, which generates the $2\to B$ disconnected term of Eq. (2.15). For the one-line disconnected term, $\sqrt{s_1}$ becomes the mass of the noninteracting particle, while $J_1=0$, since we are considering only spinless particles in this paper. According to Eq. (3.3), this means $e^B_\gamma(sJ; s_2J_2B_2)\to e^{B_2}_\gamma(s_2,J_2;B_2)$ as $J\to\infty$, so there is no splitting of the spectral measure for the one-line term.

For the two-line disconnected term, in which the reaction $A\to B$ is cluster separated into the $A_1\to B_1$ and $A_2\to B_2$ reactions, with A_1 a two-particle cluster, Eq. (3.3) implies that

$$d^A_{\gamma_1\gamma_2}(sJ; s_1J_1\sigma_1, s_2J_2\sigma_2A_2) \to \text{const}\times\delta_{\gamma_1,1}d^{A_2}_{\gamma_2}(s_2J_2;A_2),$$

as $J\to\infty$. For e^B_1 is proportional to the $2\to B_1$ partial-wave amplitude, and forcing $d^A_{\gamma_1\gamma_2}$ to pick out the term in γ_1 equal to 1 guarantees that the operator associated with the $A\to B$ reaction will cluster decompose into the tensor product $S_{B_1A_1}\otimes S_{B_2A_2}$, where the kernel of the $S_{B_1A_1}$ operator is the $2\to B_1$ partial-wave amplitude, as given in Eq. (2.15).

IV. CONCLUSION

In building an operator-S-matrix theory it is important that the theory satisfy cluster requirements. For an operator-S-matrix theory in which the basic theoretical objects are the eigenvectors appearing in the spectral representation of the channel scattering operators $S_{B,A}$, this means that the eigenvectors must satisfy certain conditions that will guarantee that the cluster requirements hold. In this paper these conditions were obtained in two steps. First, starting with a more or less intuitive notion of separability in which a separation operator $U^\tau_{\vec{a}}$ moves a subcluster of particles away from other particles in the cluster, a cluster property was formulated in Eq. (2.3) as a strong operator limit as $|\vec{a}|$ gets large. Section II then showed how the strong operator limit in the separation variable $|\vec{a}|$ was equivalent to a strong operator limit in the angular variable $\mathcal{L}$ [Eq. (2.10)]. From this it was shown that cluster properties could be expressed as a limit as J, the angular momentum, gets large. This limit cannot be a strong operator limit since J is a relativistic invariant and does not single out one subcluster of particles over another. Rather, a definite coupling scheme, involving certain variables that describe the subclusters, must be used in conjunction with J to correctly formulate the cluster property in terms of an angular momentum limit.

The second step introduced unitarity in the form of a spectral representation for the channel scattering operators. It was argued that if the eigenvectors appearing in the spectral representation split into products of eigenvectors corresponding to the eigenvectors of separated channel scattering operators, that the cluster requirements formulated in terms of large angular momentum would be guaranteed to hold. And such a formulation can only be made in partial-wave Hilbert spaces, where elements, such as those eigenvectors, do not go to zero as the angular momentum gets large.

In a previous paper[6] we have shown how crossing can be formulated as an operator requirement on the channel scattering operators. Other physical requirements that are being investigated include causality and time-reversal invariance. The goal in these papers is to express physical requirements as operator conditions on the channel scattering operators and then translate these operator conditions into conditions that must be

satisfied by the eigenvectors. For example, there is reason to believe that causality can be translated into conditions that the spectral measures must satisfy. In any event, representations for the channel scattering operators that automatically satisfy physical requirements should provide a convenient starting point for both phenomenological investigations of multiparticle reactions and investigations into a fundamental-particle theory.

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APPENDIX A: MULTIPARTICLE POINCARÉ STATES

In this Appendix the various one- and two-particle states that are needed for the cluster decomposition will be defined. We will generally follow the discussion given by Werle,[13] except that canonical spin rather than helicity states will be used.

A one-particle state $|\vec{p}\sigma\rangle$, of momentum $\vec{p}$ and spin projection σ, transforms under a Lorentz transformation Λ as

$$U_\Lambda|\vec{p},\sigma\rangle = \sum_{\sigma'} D^j_{\sigma'\sigma}(p,\Lambda)|\Lambda\vec{p},\sigma'\rangle\,, \tag{A1}$$

where (p,Λ) is a Wigner rotation defined by $(p,\Lambda) = B^{-1}(\Lambda p)\Lambda B(p)$ and p is a four-vector satisfying $p\circ p = s$, the invariant mass of the particle (system). $B(p)$ is a boost which we choose to write as $B(p) = R(\hat{p})\Lambda_z(|\vec{p}|)R^{-1}(\hat{p})$, where $\Lambda_z(|\vec{p}|)$ is a pure Lorentz transformation along the z axis such that the rest-frame four-vector $(\sqrt{s},\vec{0})$ becomes $(E,0,0,p)$. $R(\hat{p})$ is a rotation $R(\varphi,\theta,0)$, in which the Euler angles (φ,θ) are specified by the momentum direction $\hat{p}$. The boost $B(p)$ provides the meaning of the spin index σ of the one-particle state and, from the definition given here, indicates that σ is a canonical spin index. This can be seen by restricting Λ, a general Lorentz transformation, to R, an arbitrary rotation. Then the Wigner rotation becomes

$$\begin{aligned}(p,R) &= B^{-1}(Rp)RB(p)\\ &= [R(R\hat{p})\Lambda_z(|\vec{p}|)R^{-1}(R\hat{p})]^{-1}\\ &\quad\times RR(\hat{p})\Lambda_z(|\vec{p}|)R^{-1}(\hat{p})\\ &= R\,.\end{aligned} \tag{A2}$$

Here use has been made of the fact that $R^{-1}(R\hat{p})RR(\hat{p})$ is some rotation R_z about the z axis and hence commutes with $\Lambda_z(|\vec{p}|)$. Thus for canonical spin we have the well-known transformation

$$U_R|\vec{p}\sigma\rangle = D^j_{\sigma'\sigma}(R)|R\vec{p}\sigma'\rangle\,.$$

Clearly, under a Lorentz translation a the momentum state transforms as

$$U_a|\vec{p}\sigma\rangle = e^{-ip\cdot a}|\vec{p}\sigma\rangle\,.$$

To find the action of a and Λ on a square-integrable wave function we write (following Werle, p. 213) a general element of the Hilbert space as

$$|\varphi\rangle = \sum_{\sigma=-j}^{+j}\int\frac{d^3p}{E}\,\varphi(\vec{p},\sigma)|\vec{p}\sigma\rangle\,,$$

$$\begin{aligned}U_\Lambda|\varphi\rangle &= \sum_{\sigma,\sigma'}\int\frac{d^3p}{E}\,\varphi(\vec{p},\sigma)D^j_{\sigma'\sigma}(p,\Lambda)|\Lambda\vec{p}\sigma'\rangle\\ &= \sum_{\sigma,\sigma'}\int\frac{d^3p'}{E'}\,\varphi(\Lambda^{-1}\vec{p}',\sigma)D^j_{\sigma'\sigma}(\Lambda^{-1}p',\Lambda)\\ &\quad\times|\vec{p}'\sigma'\rangle\,,\end{aligned}$$

so that

$$(U_\Lambda\varphi)(\vec{p},\sigma) = \sum_{\sigma'} D^j_{\sigma\sigma'}(\Lambda^{-1}p,\Lambda)\varphi(\Lambda^{-1}p,\sigma')\,, \tag{A3}$$

with φ an element of the one-particle Hilbert space satisfying

$$\|\varphi\|^2 = \sum_\sigma\int\frac{d^3p}{E}|\varphi(\vec{p},\sigma)|^2 < \infty\,.$$

Similarly, $(U_a\varphi)(\vec{p},\sigma) = e^{ip\cdot a}\varphi(\vec{p},\sigma)$. The normalization of the states $|\vec{p}\sigma\rangle$ is given by $\langle\vec{p}'\sigma'|\vec{p},\sigma\rangle = E\delta^3(\vec{p}'-\vec{p})\delta_{\sigma'\sigma}$.

Two-particle states are defined by going to the overall cm frame of the two-particle system. That is, if $\vec{P}\equiv\vec{p}_1+\vec{p}_2$ and $p = B^{-1}(P)p_1$, then the two-particle state $|\vec{P}=\vec{0},\vec{p},\sigma_1\sigma_2\rangle$ is defined as

$$\begin{aligned}&|\vec{P}=\vec{0},\vec{p},\sigma_1\sigma_2\rangle = N|\vec{p}\sigma_1\rangle|-\vec{p}\sigma_2\rangle\,,\\ &|\vec{P},\vec{p},\sigma_1,\sigma_2\rangle \equiv U_{B(P)}|\vec{P}=\vec{0},\vec{p},\sigma_1\sigma_2\rangle\,,\end{aligned} \tag{A4}$$

where N is a normalization factor. N is fixed by defining an equivalent two-particle state $|\vec{P}s\hat{p}\sigma_1\sigma_2\rangle$, where s, the invariant mass of the two-particle system, is given by $s = (p_1+p_2)^2 = (s_1+\vec{p}^{\,2})^{1/2} + (s_2+\vec{p}^{\,2})^{1/2}$. We want $|\vec{P}s\hat{p}\sigma_1\sigma_2\rangle$ to be normalized like a one-particle state of momentum $\vec{P}$ and energy $E = (\vec{P}^2+s)^{1/2}$; that is,

$$\langle\vec{P}'s'\hat{p}'\sigma_1'\sigma_2'|\vec{P}s\hat{p}\sigma_1\sigma_2\rangle = \delta^4(P'-P)\delta^2(\hat{p}-\hat{p}')\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}\,.$$

This then fixes the normalization factor N; it is given explicitly by Werle in Ref. 13.

It is now possible to define the various angular momentum states that are needed for the cluster decomposition. We begin with the orbital angular momentum state, defined as

$$|\vec{P}=\vec{0}\,sL\sigma_L\sigma_1\sigma_2\rangle \equiv \left(\frac{2L+1}{4\pi}\right)^{1/2}\int d\hat{p}\, D^{L\,*}_{\sigma_L 0}(\hat{p})|\vec{P}$$
$$=\vec{0}\,s\hat{p}\sigma_1\sigma_2\rangle\,,$$

$$|\vec{P},sL\sigma_L\sigma_1\sigma_2\rangle \equiv U_{B(P)}|\vec{P}=\vec{0}\,sL\sigma_L\sigma_1\sigma_2\rangle\,, \tag{A5}$$

which satisfies

$$U_R|\vec{P}=\vec{0}\,sL\sigma_L\sigma_1\sigma_2\rangle$$
$$=\sum D^{L}_{\sigma'_L\sigma_L}(R)D^{J_1}_{\sigma'_1\sigma_1}(R)D^{J_2}_{\sigma'_2\sigma_2}(R)$$
$$\times|\vec{P}=0\,sL\sigma'_L\sigma'_1\sigma'_2\rangle\,. \tag{A6}$$

The boosted orbital angular momentum state has the same rotational properties as the $\vec{P}=\vec{0}$ state because of the fact that $RB(P)=B(RP)(P,R)$ $=B(RP)R$ for canonical spin boosts, and is normalized so that $\langle\vec{P}'s'L'\sigma'_L\sigma'_1\sigma'_2|\vec{P}sL\sigma_L\sigma_1\sigma_2\rangle$ $=\delta^4(P'-P)\delta_{L'L}\delta_{\sigma'_L\sigma_L}\delta_{\sigma'_1\sigma_1}\delta_{\sigma'_2\sigma_2}$. From the rotational properties of the orbital angular momentum states it is clear that a total angular momentum state can be defined by coupling together the intrinsic spins J_1 and J_2 and then taking the resulting spin j and coupling it to L to form the overall angular momentum J:

$$|\vec{P}sJ\sigma Lj\rangle = \sum_{\substack{\sigma_L m\\ \sigma_1\sigma_2}}\langle J\sigma|L\sigma_L jm\rangle\langle jm|J_1\sigma_1 J_2\sigma_2\rangle\,. \tag{A7}$$
$$\times|\vec{P},sL\sigma_L\sigma_1\sigma_2\rangle$$

One other total angular momentum state is needed for the cluster decomposition, in which the degeneracy parameters L and j are replaced by the spin projection σ_1 and σ_2. Such a state is defined by

$$|\vec{P}=\vec{0}\,sJ\sigma\sigma_1\sigma_2\rangle \equiv \left(\frac{2J+1}{4\pi}\right)^{1/2}\int d\hat{p}\, D^{J\,*}_{\sigma\sigma_1+\sigma_2}(\hat{p})$$
$$\times U_{R(\hat{p})}|\vec{P}=\vec{0}(00p)\sigma_1\sigma_2\rangle$$
$$=\left(\frac{2J+1}{4\pi}\right)^{1/2}\int d\hat{p}\, D^{J\,*}_{\sigma\sigma_1+\sigma_2}(\hat{p})D^{J_1}_{\sigma'_1\sigma_2}(\hat{p})$$
$$\times D^{J_2}_{\sigma_2\sigma_2}(\hat{p})|\vec{P}=\vec{0}s\hat{p}\sigma'_1\sigma'_2\rangle\,, \tag{A8}$$

which transforms under rotations as

$$U_R|\vec{P}=\vec{0}\,sJ\sigma\sigma_1\sigma_2\rangle = \sum_{\sigma'} D^{J}_{\sigma'\sigma}(R)|\vec{P}=\vec{0}\,sJ\sigma'\sigma_1\sigma_2\rangle\,.$$

As with the orbital angular momentum states, $|\vec{P}sJ\sigma\sigma_1\sigma_2\rangle = U_{B(P)}|\vec{P}=\vec{0}\,sJ\sigma\sigma_1\sigma_2\rangle$ and has the same rotational properties as the $\vec{P}=\vec{0}$ state.

The connection between the two angular momentum states can be obtained from Eqs. (A7) and (A8):

$$|\vec{P}=\vec{0}\,sJ\sigma\sigma_1\sigma_2\rangle = \sum_{Lj}\left(\frac{2L+1}{2J+1}\right)^{1/2}\langle jm|J_1\sigma_1 J_2\sigma_2\rangle$$
$$\times\langle J\sigma_1+\sigma_2|jmL0\rangle$$
$$\times|\vec{P}=\vec{0}\,sJ\sigma Lj\rangle\,, \tag{A9}$$

so that

$$|\vec{P}\,sJ\sigma\sigma_1\sigma_2\rangle = \sum_{Lj}\left(\frac{2L+1}{2J+1}\right)^{1/2}\langle J\sigma_1+\sigma_2|jmL0\rangle$$
$$\times\langle jm|J_1\sigma_1 J_2\sigma_2\rangle|\vec{P}\,sJ\sigma Lj\rangle\,. \tag{A10}$$

Finally, it is necessary to find the connection between wave functions in different variables in order to compute the action of the separation operator on the total angular momentum. A straightforward calculation following the lines of Eq. (A3) gives

$$f(\vec{P}\,s\hat{p}\sigma_1\sigma_2) = \sum_{L\sigma_L}\left(\frac{2L+1}{4\pi}\right)^{1/2} D^{L\,*}_{\sigma_L 0}(\hat{p})f(\vec{P}\,sL\sigma_L\sigma_1\sigma_2)\,,$$

$$f(\vec{P}\,sL\sigma_L\sigma_1\sigma_2)$$
$$=\sum_{Jj}\langle jmL\sigma_L|J\sigma\rangle\langle J_1\sigma_1 J_2\sigma_2|jm\rangle f(\vec{P}\,sJ\sigma Lj)\,,$$

$$f(\vec{P}\,s\hat{p}\sigma_1\sigma_2) = \sum_{\substack{J\sigma\\ \sigma'_1\sigma'_2}}\left(\frac{2L+1}{4\pi}\right)^{1/2} D^{J\,*}_{\sigma\sigma'_1+\sigma'_2}(\hat{p})D^{J_1}_{\sigma_1\sigma'_1}(\hat{p})$$
$$\times D^{J_2}_{\sigma_2\sigma'_2}(\hat{p})f(\vec{P}\,sJ\sigma\sigma'_1\sigma'_2)\,. \tag{A11}$$

APPENDIX B: VARIOUS LIMITS

In this Appendix various limits will be considered that are needed in the analysis of the cluster decomposition. The first comes from Eq. (2.9) and can be written

$$\lim_{a\to\infty}\sum_{\mathcal{L}=0}^{\infty} j^2_{\mathcal{L}}(pa)|f_{\mathcal{L}}(p)| = 0\,. \tag{B1}$$

We want to show that this limit is equivalent to writing

$$\lim_{\mathcal{L}\to\infty}|f_{\mathcal{L}}(p)| = 0\,. \tag{B2}$$

To show this two properties of the spherical Bessel functions $j_{\mathcal{L}}(pa)$ are needed:

$$\sum_{\mathcal{L}=0}^{\infty} j^2_{\mathcal{L}}(pa) = 1 \quad \text{for all } pa\,, \tag{B3a}$$

$$j_{\mathcal{L}}(pa)\approx \sin(pa-\tfrac{1}{2}\mathcal{L}\pi)/pa \quad \text{for } pa\gg\mathcal{L}\,. \tag{B3b}$$

It is easiest to begin by assuming Eq. (B2) and showing that Eq. (B1) follows. Now Eq. (B2) means there exists an ϵ and an $\mathcal{L}_\epsilon$ such that

$$|f_{\mathcal{L}}(p)| < \tfrac{1}{2}\epsilon \quad \text{when } \mathcal{L} > \mathcal{L}_\epsilon\,. \tag{B4}$$

If M_p denotes the maximum of $|f_{\mathcal{L}}(p)|$ for $0 \leq \mathcal{L} \leq \mathcal{L}_\epsilon$, then we can write

$$\sum_{\mathcal{L}=0}^{\mathcal{L}_\epsilon} j_{\mathcal{L}}^2(pa)|f_{\mathcal{L}}(p)| + \sum_{\mathcal{L}=\mathcal{L}_\epsilon+1}^{\infty} j_{\mathcal{L}}^2(pa)|f_{\mathcal{L}}(p)|$$

$$\leq M_p \sum_{\mathcal{L}=0}^{\mathcal{L}_\epsilon} j_{\mathcal{L}}^2(pa) + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,, \tag{B5}$$

for $pa \gg \mathcal{L} \leq \mathcal{L}_\epsilon$. Here use has been made of property (B3b) of the spherical Bessel function, while the bound on the second term in the sum in Eq. (B5) makes use of property (B3a). Thus, Eq. (B2) implies Eq. (B1).

Going in the other direction is a little more complicated for it might seem possible to pick a subsequence $\mathcal{L}_n$ in Eq. (B1) for which the terms $j_{\mathcal{L}_n}^2(pa)$ are very small, so that $|f_{\mathcal{L}_n}(p)|$ need not go to zero for large $\mathcal{L}_n$. But one other property of the sum can be used to show that such a subsequence cannot exist. Namely, for fixed p as a gets large, the zeros of $j_{\mathcal{L}}(pa)$ change. This can already be seen from Eq. (B3b), where an asymptotic form for $j_{\mathcal{L}}(pa)$ is given. Because the neighborhood in which $j_{\mathcal{L}}(pa)$ is small changes as a changes, it is impossible to find a subsequence $\mathcal{L}_n$ for which $j_{\mathcal{L}_n}(pa)$ remains small as a varies. Hence Eq. (B1) implies Eq. (B2).

The second limit we want to compute involves the Clebsch-Gordan coefficients $(jmL0|Jm)$ in which J and L get large for a fixed value of j and m. The easiest way to compute this limit is to make a change of variables in which $j+L-l=J$ and $l=0,\ldots,2j$, when $L \geq j$. Then

$$\langle j,m;J+l-j,0|Jm\rangle = \sum_{\lambda=l+m-j}^{\min(j+m,l)} \left(\frac{(2J+1)(2J+l-2j)!(J+m)!(J-m)!}{(2J+l+1)!(J-j+\lambda)!(J-j+\lambda)!}\right)^{1/2} \frac{(J+l-j)!}{(J+l-j-\lambda)!}(-1)^{-\lambda}\frac{[(2j-l)!\,l!(j-m)!(j-m)!]^{1/2}}{\lambda!(l-\lambda)!(j-m-\lambda)!(j-l+m+\lambda)!}\,. \tag{B6}$$

The terms in the Clebsch-Gordan coefficients have been so arranged that the first terms all involve J, which will get very large, while the last terms involve only j, l, m, and λ, which remain finite. Further, the first terms are so arranged that their ratios can be readily computed for large J. Thus

$$\frac{(J+l-j)!}{(2J+l-j-\lambda)!} = (J+l-j)\cdots(J+l-\lambda+1) \to J^\lambda\,,$$

$$\left(\frac{(2J+l-2j)!}{(2J+l+1)!}\right)^{1/2} = \left(\frac{1}{(2J+l+1)\cdots(2J+l-2j+1)}\right)^{1/2} \to \left(\frac{1}{(2J)^{+2j+1}}\right)^{1/2},$$

since $\lambda \leq j+m$,

$$\left(\frac{(J+m)!}{(J+\lambda-j)!}\right)^{1/2} = [(J+m)\cdots(J+\lambda-j+1)!]^{1/2} \to (J^{m+j-\lambda})^{1/2},$$

$$\left(\frac{(J-m)!}{(J+\lambda-j)!}\right)^{1/2} = [(J+\lambda-j)\cdots(J-m+1)]^{-1/2} \to \frac{1}{(J^{\lambda-j+m})^{1/2}}\,.$$

Combining these terms gives

$$\langle j,m;J+l-j,0|Jm\rangle \to \left(\frac{(2J-l)!\,l!(j+m)!(j-m)!}{2j}\right)^{1/2} \sum_{\lambda=l+m-j}^{\min(j+m,l)} \frac{(-1)^\lambda}{\lambda!(l-\lambda)!(j-m-\lambda)!(j-l+m+\lambda)!} \tag{B7}$$

as J gets large.

[1]See, for example, the article of Araki in *Proceedings of the International School of Physics, Enrico Fermi*, Course XLV, edited by R. Jost (Academic, New York, 1969), and references cited therein.

[2]See, for example, L. B. Rêdei, Nuovo Cimento A 11, 279 (1972).

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