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Population probabilities of the excited levels of ions in a steady-state plasma. I. Basic equations and results

David Salzmann*
Department of Plasma Physics, Soreq Nuclear Research Center, Yavneh, Israel
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In this work an explicit formula is obtained for the population probabilities of the excited levels of the various charge states in an optically thin plasma. The plasma is assumed to be in a complete steady state, i.e., time independent and homogeneous. First, a recursive expression is obtained for the partial densities of the various charge states in the plasma. This expression is used to get an implicit algebraic expression for the population probabilities of the excited states, which is finally solved for low electron densities to yield an explicit, and simple, expression for the population probabilities. Simplified formulas for H-like ions and a few numerical examples are also presented.

I. INTRODUCTION

The experimental evidence for high x-ray radiation rates from hot dense plasmas produced by laser-matter interaction led to a need for reliable calculations of these radiation rates. These rates are generally calculated by assuming either corona equilibrium (CE) or local thermodynamic equilibrium (LTE) throughout the plasma. These two models are certainly valid in low-electron-density (CE) and high-electron-density (LTE) plasma, respectively, but none of them can cover the whole range of temperature and density variations in laser-produced plasmas. In fact, it was found[1] that when an aluminum target is irradiated by a 3×10^{13} W/cm^2 Nd: glass laser beam only ~7% of the total x-ray radiation is emitted from the low-density high-temperature periphery, where CE is valid. About the same percentage of the radiation comes from the hot high-density focal region, where LTE prevails, but the rest, i.e., about 86% of the x-rays are emitted from the intermediate portion, out of the validity regimes of these two major plasma models. For higher intensity laser beams one would expect more radiation from the CE region, but the influence of the intermediate-density region on the x-ray emission is negligible only at extremely high laser intensities.

The intermediate-density model is called sometimes the collisional-radiative model,[2] we prefer the name complete steady state[3-5] (CSS). This model is characterized by the fact that the ionization state densities, as well as the level population probabilities are time independent, i.e., $dN_j/dt=0$ and $dn_{j,m}/dt=0$ (N_j is the partial density of ions with charge state j and $n_{j,m}$ is the partial densities of the ions excited to state m). These conditions do not necessarily imply a state of equilibrium, but rather a steady state only. In this case, the ionizing processes exactly compensate the recombination processes thereby producing a steady state in the plasma. However, as the energy-dissipating radiative processes are not negligible relative to the energy-conserving collisional processes, the plasma is supposed to cool down, but this cooling follows a thermodynamic path which ensures that the changes in the internal ionization energy will be minimum. This is the thermodynamic meaning of CSS.

In CE steady state occurs when the collisional ionization is exactly compensated by the energy-dissipating radiative recombination. At the high-density extreme, LTE, the steady state is approached when the ionization is compensated by the energy-conserving collisional recombination. In CSS both dissipative and conservative recombinations should be accounted for. This greatly complicates the corresponding formulas so that their use in plasma simulating codes may consume long computer times. That is the reason why CE or LTE are preferred for use in computer codes in spite of their serious limitations. In fact, in some cases orders of magnitude inaccuracies can be caused by using one of these models out of its validity domain.

In this paper it will be shown that the complicated formulas of CSS can be solved at the low-electron-density limit and an explicit formula will be derived for the population probabilities of the ionic levels. When these probabilities are known they can be used to solve a recursive equation for the charge-state distributions and, finally, when the composition of the plasma is known, the radiation rates can be calculated. As it sometimes happens, the solution of this very complicated problem turns out to be a very simple (in fact, linear) expression.

The calculation presented here assumes an optically thin plasma with no photon-plasma interaction. This approximation is generally correct for the greater part of laser-produced plasmas. In cases where this approximation does not hold a radiation-transport problem must be applied. This is out of the scope of the present work. In Sec. II the basic equation and the general formalism are set up. This equation is then reduced to a recursive equation from which the charge-state distribution can be calculated. In Sec. III an implicit equation is derived for the population probabilities. In Sec. IV we finally obtain the explicit expression for the population probabilities at low electron density. In Sec. V an approximate expression for H-like ions is derived and numerical examples are given to illustrate the results. Further consequences are presented in a subsequent paper.

II. BASIC EQUATION AND THE TOTAL IONIZATION DENSITY

The rate of population change of a particular level m of the ion j is affected by three groups of processes capable of populating or depopulating this level. These processes are (i) populating and depopulating ionization processes, (ii) populating and depopulating recombinations, and (iii) excitations and deexcitations. The first two processes produce transitions to the next ionization states, whereas the third causes transitions within the same charge state, affecting only the population distribution of the excited levels; see Fig. 1 for a graphical representation.

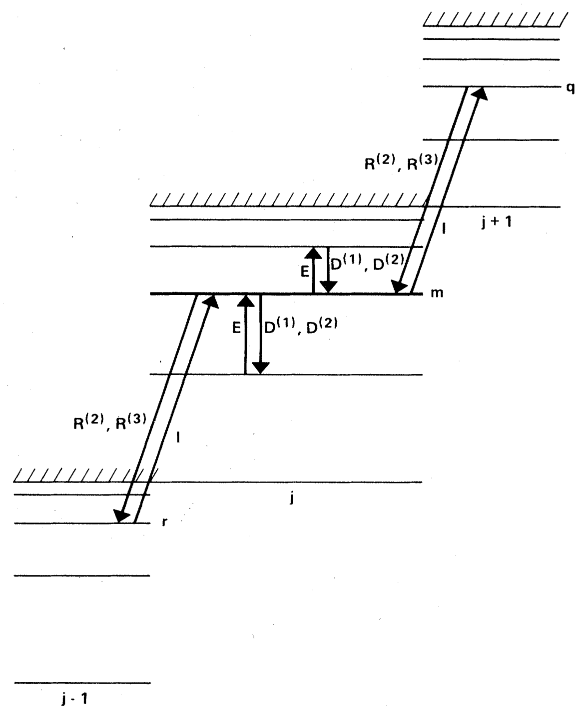

FIG. 1. Various processes which populate or depopulate an atomic level.

The rate equation for a particular level in the ion j is

$$\frac{dn_{jm}}{dt} = -\sum_{q} I_{j,m:j+1,q}n_{j,m}n_e - \sum_{r}[R^{(2)}_{j,m:j-1,r} + n_e R^{(3)}_{j,m:j-1,r}]n_{j,m}n_e + \sum_{r} I_{j-1,r:j,m}n_{j-1,r}n_e$$
$$+ \sum_{q}[R^{(2)}_{j+1,q:j,m} + n_e R^{(3)}_{j+1,q:j,m}]n_{j+1,q}n_e - \sum_{s>m} E_{j,m:j,s}n_{j,m}n_e - \sum_{u<m}[D^{(1)}_{j,m:j,u} + n_e D^{(2)}_{j,m:j,u}]n_{j,m}$$
$$+ \sum_{u<m} E_{j,u:j,m}n_{j,u}n_e + \sum_{s>m}[D^{(1)}_{j,s:j,m} + n_e D^{(2)}_{j,s:j,m}]n_{j,s}. \tag{2.1}$$

Here n_{jm} and n_e are the level population density and electron density, respectively. For the rate coefficients we used the following notation: I is rate coefficient for electron-impact ionization; $R^{(2)}$ and $R^{(3)}$ are the rate coefficients for two-body recombination (radiative and dielectronic) and three-body recombination, respectively; E denotes the rate coefficients for electron-impact excitation; $D^{(1)}$ and $D^{(2)}$ are the rate coefficients for spontaneous decay ($D^{(1)}$ is the Einstein A coefficient) and for the electron deexcitation. Every rate coefficient is assigned by two couples of indexes, the first couple denote the ionization state and the excited-level indexes of the initial ion, the second couple has the same meaning for the final state. For example, $I_{j,m:j+1,q}$ is the rate coefficient for electron-impact ionization of level m of ionization state j, resulting in ion $j+1$ excited to level q. The ionization state index j may have values between 0 and Z, the level index m counts the bound levels only. In real plasmas, there is an ionization potential reduction due to the local microfields in the plasma. The main effect of this potential reduction is that the highest atomic levels become unbound, thereby reducing the number of bound levels of every atom to a finite one. Consequently, the m index may take only a finite number of values which depend on the electron density in the plasma. This fact limits both the number of equations in (2.1) and the sums in (2.1) to a finite number only.

The partial densities of the atomic states are related to the total partial density of the ionization state N_j and the populations probability P_{jm} by

$$n_{j,m}=N_j P_{j,m}, \tag{2.2}$$

$$N_j=\sum_m n_{j,m}, \tag{2.3}$$

$$\sum_m P_{j,m}=1. \tag{2.4}$$

Generally, for every plasma the total ion density N_t is a known quantity, therefore the sum of the N_j's must be equal to

$$\sum_{j=0}^{Z} N_j=N_t, \tag{2.5}$$

whereas the electron density is given by

$$n_e=\sum_{j=0}^{Z} jN_j. \tag{2.6}$$

We shall first obtain a recursive formula for the partial densities of the ionization states in a stationary plasma. Summing (2.1) over all states m, the left-hand side becomes dN_j/dt, i.e., the equation now describes the rate of change of the density of the whole charge state without specifying the state of excitation. On the right-hand side of (2.1) the excitation and deexcitation (or decay) parts cancel exactly, reflecting the fact that these processes induce transitions only within the same ion and cannot, therefore, affect the change in N_j. The resulting equation for N_j is

$$\begin{aligned}\frac{dN_j}{dt}=&-N_j n_e\sum_{m,q} I_{j,m:j+1,q}P_{j,m}\\&-N_j n_e\sum_{m,r}[R^{(2)}_{j,m:j-1,r}+n_e R^{(3)}_{j,m:j-1,r}]P_{j,m}\\&+N_{j-1}n_e\sum_{m,s}P_{j-1,s}I_{j-1,s:j,m}\\&+N_{j+1}n_e\sum_{m,q}[R^{(2)}_{j+1,q:j,m}+n_e R^{(3)}_{j+1,q:j,m}]P_{j+1,q}.\end{aligned} \tag{2.7}$$

We introduce the assumption that the plasma is in a complete steady state (CSS), defined by the condition

$$\frac{dN_j}{dt}=0. \tag{2.8}$$

The meaning of this assumption is that the plasma is either at equilibrium, or the thermalization processes are fast enough to adjust the population densities rapidly to any change in the total ion density or temperature.

Under condition (2.8), Eq. (2.7) reduces to a simple recursive expression:

$$N_{j-1}i_{j-1}=N_j(r_{2j}+n_e r_{3j}), \tag{2.9}$$

$$\frac{N_j}{N_{j-1}}=\frac{i_{j-1}}{r_{2j}+n_e r_{3j}}, \tag{2.10}$$

where for brevity, we introduced the following notation:

$$i_j=\sum_{m,q}P_{j,m}I_{j,m:j+1,q}, \tag{2.11}$$

$$r_{2j}=\sum_{m,s}P_{j,m}R^{(2)}_{j,m:j-1,s}, \tag{2.12}$$

$$r_{3j}=\sum_{m,s}P_{j,m}R^{(3)}_{j,m:j-1,s}. \tag{2.13}$$

The explicit appearance of the electron density n_e in the denominator of Eq. (2.10) divides the density dependence of the results into two asymptotic regions:

(a) The high-density region is where

$$n_e r_{3j}\gg r_{2j} \tag{2.14}$$

i.e., the three-body collisional recombination is the main recombining effect. In this region Eq. (2.10) reduces to

$$n_e N_j/N_{j-1}=i_{j-1}/r_{3j} \tag{2.15}$$

which is the Saha equation. This equation is valid at high densities where local thermodynamic equilibrium (LTE) dominates, and from general principles it can be shown to be equal to

$$\frac{N_j}{N_{j-1}}=\frac{U_j(T_e)}{U_{j-1}(T_e)}\exp(-\mu_e/T_e)\exp\left(-\frac{X_{j-1}-\Delta X_{j-1}}{T_e}\right) \tag{2.16}$$

(T_e in eV). U_j is the partition function, ΔX_{j-1} is the lowering of the ionization potential due to the electrostatic fields in high-density plasmas, and μ_e is the chemical potential of the continuum electrons. When the electrons are nondegenerate, this formula reduces to the ordinary Saha equation.

(b) In the low-density region the corona equilibrium (CE) model is applicable. This is approached when

$$r_{2j}\gg n_e r_{3j}. \tag{2.17}$$

In this limit (2.9) can be rewritten as

$$N_j/N_{j-1}=i_{j-1}/r_{2j} \tag{2.18}$$

which is the well-known formula for CE.

Equations (2.15) and (2.18) indicate that the CSS model incorporates both LTE and CE in its asymptotic regions, and has, therefore, a very wide range of applicability.

III. LEVEL POPULATION PROBABILITY

Equation (2.10) is still not in an appropriate form for solution, since its right-hand side is dependent on the level population probabilities, $P_{j,m}$. To obtain values for $P_{j,m}$ it is necessary to solve the full system of differential equations (2.1). However, at extreme density conditions this is not a prerequisite since estimates can be obtained from basic physical principles.

At the low-electron-density extreme, the excitation rate by collisions is so low relative to the spontaneous decay rate of the excited levels that the ions are mainly in their ground state:

$$P_{j,m}=\begin{cases}1 & \text{ground state}\\ 0 & \text{excited states}.\end{cases} \tag{3.1}$$

At the other extreme, a high-density plasma, the collision rate is high enough to produce thermodynamic equilibrium between the ground state and all excited states. The level density probability approaches a Boltzmann-type distribution, characteristic for thermal equilibrium,

$$P_{j,m}=\beta_j g_{j,m}\exp(-E_{j,m}/T_e), \tag{3.2}$$

where β_j is a normalization factor, g is the statistical weight, and T is in energy units.

The main aim of this work is to find the correct low-density asymptotic behavior apart from the limit given by (3.1) and to propose an approximate formula which can give reasonably accurate results for intermediate densities.

CSS is defined by the condition that the partial densities of the ionic states are time independent,

$$\frac{dn_{j,m}}{dt}=0. \tag{3.3}$$

In principle, this condition seems to be more restrictive than the usual condition (2.8) since it implies equilibrium among the excited states of the ion, whereas (2.8) requires a steady state only for the ionization state as a whole. In practice, however, the two conditions describe the same physical situation [see (2.2)], and only in extreme cases can one produce a real physical system where (2.8) holds but not (3.3). These will be generally nonequilibrium systems.

Inserting (3.3) in (2.1), one can isolate $n_{j,m}$ in the right-hand side of (2.1), yielding the following equation for $n_{j,m}$,

$$\begin{aligned}n_{j,m}\Big(n_e\sum_q I_{j,m:j+1,q}+n_e\sum_r R^{(2)}_{j,m:j-1,r}+n_e\sum_{s>m}E_{j,m:j,s}+n_e\sum_{u<m}D^{(2)}_{j,m:j,u}+n_e^2\sum_r R^{(3)}_{j,m:j-1,r}+\sum_{u<m}D^{(1)}_{j,m:j,u}\Big)\\ =n_e\sum_r I_{j-1,r:j,m}n_{j-1,r}+n_e\sum_q R^{(2)}_{j+1,q:j,m}n_{j+1,q}+n_e\sum_{u<m}E_{j,u:j,m}n_{j,u}+n_e\sum_{s>m}D^{(2)}_{j,s:j,m}n_{j,s}\\ +n_e^2\sum_q R^{(3)}_{j+1,q:j,m}n_{j+1,q}+\sum_{s>m}D^{(1)}_{j,s:j,m}n_{j,s}.\end{aligned} \tag{3.4}$$

Equation (2.2) is inserted into this equation to replace $n_{j,m}$, $n_{j-1,r}$, and $n_{j+1,q}$ by the appropriate population probabilities and the partial ionization state densities, N_j, N_{j-1}, and N_{j+1}. The recursive equation (2.9) is then used to replace N_{j+1} and N_{j-1} by N_j, which appears after these manipulations as a constant multiplicative factor on the right-hand side of the equation. Dividing the equation by N_j, one gets the final result for the population probability,

$$P_{jm}=\frac{1}{i_{j-1}}\Big(\sum_{k=0}^{3}\alpha_k n_e^k\Big)\Big/\Big(\sum_{k=0}^{3}\beta_k n_e^k\Big) \tag{3.5}$$

with,

$$\alpha_0=i_{j-1}r_{2,j+1}\sum_{s>m}P_{j,s}D^{(1)}_{j,s:j,m}; \tag{3.6}$$

$$\begin{aligned}\alpha_1=r_{2,j}r_{2,j+1}\sum_r P_{j-1,r}I_{j-1,r:j,m}+i_j i_{j-1}\sum_q P_{j+1,q}R^{(2)}_{j+1,q:j,m}+i_{j-1}r_{2,j+1}\sum_{u<m}P_{j,u}E_{j,u:j,m}\\ +i_{j-1}r_{2,j+1}\sum_{s>m}P_{j,s}D^{(2)}_{j,s:j,m}+i_{j-1}r_{3,j+1}\sum_{s>m}P_{j,s}D^{(1)}_{j,s:j,m};\end{aligned} \tag{3.7}$$

$$\alpha_2 = r_{2,j} r_{3,j+1} \sum_r P_{j-1,r} I_{j-1,r:j,m} + r_{2,j+1} r_{3,j} \sum_r P_{j-1,r} I_{j-1,r:j,m} + i_j i_{j-1} \sum_q P_{j+1,q} R^{(3)}_{j+1,q:j,m}$$
$$+ i_{j-1} r_{3,j+1} \sum_{u<m} P_{j,u} E_{j,u:j,m} + i_{j-1} r_{3,j+1} \sum_{s>m} P_{j,s} D^{(2)}_{j,s:j,m}; \tag{3.8}$$

$$\alpha_3 = r_{3,j} r_{3,j+1} \sum_r P_{j-1,r} I_{j-1,r:j,m}; \tag{3.9}$$

$$\beta_0 = r_{2,j+1} \sum_{u<m} D^{(1)}_{j,m:j,u}; \tag{3.10}$$

$$\beta_1 = r_{3,j+1} \sum_{u<m} D^{(1)}_{j,m:j,u} + r_{2,j+1}\left(\sum_q I_{j,m:j+1,q} + \sum_r R^{(2)}_{j,m:j-1,r} + \sum_{u>m} E_{j,m:j,u} + \sum_{s<m} D^{(2)}_{j,m:j,s}\right); \tag{3.11}$$

$$\beta_2 = r_{3,j+1}\left(\sum_q I_{j,m:j+1,q} + \sum_r R^{(2)}_{j,m:j-1,r} + \sum_{u>m} E_{j,m:j,u} + \sum_{s<m} D^{(2)}_{j,m:j,s}\right) + r_{2,j+1} \sum_r R^{(3)}_{j,m:j-1,r}; \tag{3.12}$$

$$\beta_3 = r_{3,j+1} \sum_r R^{(3)}_{j,m:j-1,r}. \tag{3.13}$$

Equation (3.5) together with the definitions (3.6)–(3.13) are the basic formulas used hereafter.

It should be noted that the α coefficients are still dependent on the population probabilities of all the excited levels except, of course, the j,m level which is under consideration. Even if seemingly (3.5) is very complex, it has far-reaching consequences, some of which will be derived here.

The main advantage of (3.5) is that the electron density dependence of $P_{j,m}$ is explicitly shown. This enables one to see the complete functional behavior of $P_{j,m}$ versus electron density. There still remains some implicit dependence on n_e through the upper limits of the summations, but this dependence is generally very weak relative to the explicit powers of n_e in (3.5).

It can be shown, that (3.5) implies both (3.1) and (3.2) in the low- and high-electron-density limits, respectively; however, it facilitates the deduction of the population probabilities at intermediate regions as well.

IV. LOW-ELECTRON-DENSITY ASYMPTOTIC BEHAVIOR

We shall now prove the main result of the present work, namely, that at low electron densities the population probabilities vary linearly with the electron density. An explicit formula will be found for the slope of this linear function, and a few conclusions will be derived.

For low electron density, Eq. (3.5) reduces to

$$P_{j,m} \cong \frac{\alpha_1 n_e + \alpha_0}{i_{j-1}(\beta_1 n_e + \beta_0)}. \tag{4.1}$$

As we are concerned with nonzero electron density every atom in the plasma may have only a finite number of bound states. Let us denote the highest bound state by M. As there are no higher levels, for this highest level $\alpha_0 = 0$, because the sum in (3.6) contains no elements. Neglecting $\beta_1 n_e$ in the denominator of (4.1) relative to β_0, one obtains for the highest bound state

$$P_{j,M} \cong C_{j,M} n_e \tag{4.2}$$

with $C_{j,M}$ a constant independent on the electron density. Inserting this last result into the equation of the $(M-1)$th level, one gets that the population probability of this state is also linear with the electron density,

$$P_{j,M-1} \cong \frac{\alpha_1 n_e + C_{j,M} D^{(1)}_{j,M:j,M-1} n_e}{i_{j-1}\beta_0} = C_{j,M-1} n_e. \tag{4.3}$$

Continuing the same reasoning, one finds that for all excited states $P_{j,m}$ is proportional to n_e. The ground state is an exception, as for $m=1$ the denominator is zero, $\beta_0 = 0$, so that $P_{j,1}$ should be determined from the normalization condition (2.4).

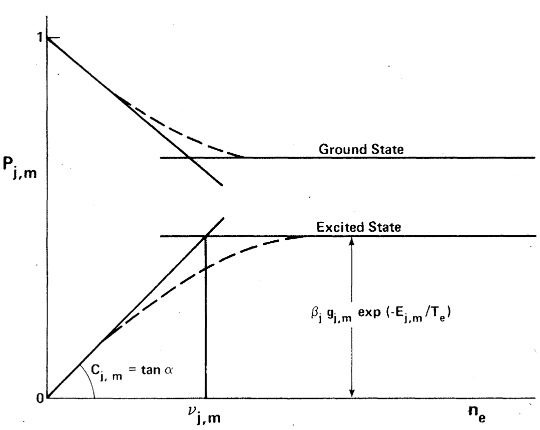

FIG. 2. Schematic representation of the population probability variation vs electron density.

It is preferable to replace $C_{j,m}$ by another constant $\nu_{j,m}$, defined by

$$P_{j,m}=\frac{n_e}{\nu_{j,m}}\,\beta_j g_{j,m}\exp(-E_{j,m}/T_e)=\frac{n_e}{\nu_{j,m}}\,P_{j,m}(\mathrm{LTE})\,. \tag{4.4}$$

The main advantage of this definition is that ν_{jm} has a simple physical meaning. In fact, ν_{jm} is the electron density at which the low-density asymptote intersects the high-density asymptote; see Fig. 2. Therefore, it may be more meaningful to refer to this quantity rather than C_{jm}. We shall refer hereafter to $\nu_{j,m}$ as the "characteristic density" of level m of the jth ionization state.

$C_{j,m}$ is almost independent on the electronic density n_e. The $\nu_{j,m}$, however, is dependent explicitly on $\beta_j(T)$, which at high enough temperatures is a function of the number of the bound levels and the electron density in the plasma. In spite of this drawback, its simple physical meaning makes the characteristic density a very useful quantity.

Equation (4.4) shows that all P_{jm}, $m\geq 2$, are first-order quantities in n_e, except the ground state which is a zero-order quantity, $P_{j,1}\cong 1-O(n_e)$. When these results are substituted into (4.1) and terms are retained to first order only, one gets

$$\alpha_0=i_{j-1}r_{2,j+1}\sum_{s>m}C_{j,s}D^{(1)}_{j,s:j,m}n_e\,, \tag{4.5}$$

$$\alpha_1=r_{2,j}r_{2,j+1}I_{j-1,1:j,m}+i_j i_{j-1}R^{(2)}_{j+1,1:j,m}+i_{j-1}r_{2,j+1}E_{j,1:j,m}\,, \tag{4.6}$$

$$i_{j-1}\beta_0=i_{j-1}r_{2,j+1}\sum_{u<m}D^{(1)}_{j,m:j,u}\,. \tag{4.7}$$

These last equations are inserted into Eq. (4.1) to give

$$P_{j,m}=C_{j,m}n_e=\frac{\alpha_1 n_e+\alpha_0}{i_{j-1}\beta_0}=\left(\frac{r_{2,j}r_{2,j+1}I_{j-1,1:j,m}+i_j i_{j-1}R^{(2)}_{j+1,1:j,m}+i_{j-1}r_{2,j+1}E_{j,1:j,m}+i_{j-1}r_{2,j+1}\sum_{s>m}C_{j,s}D^{(1)}_{j,s:j,m}}{i_{j-1}r_{2,j+1}\sum_{u<m}D^{(1)}_{j,m:j,u}}\right)n_e\,. \tag{4.8}$$

The only term which is still dependent on the population probabilities of other atomic states, except the mth, is the last term in the nominator. This term can be rewritten as $\langle C_{j,s}\rangle\sum_{s>m}D^{(1)}_{j,s:j,m}$ where $\langle C_{j,s}\rangle$ is the average value of the C's for the levels above the mth with the $D^{(1)}$'s as weighting functions. As the main contribution to the sum comes from the $m+1$ term one would expect that $\langle C_{j,s}\rangle$ will be close to $C_{j,m+1}$. Furthermore, it will be shown that the C's vary not too rapidly with m, so one can approximate $\langle C_{j,s}\rangle\approx C_{j,m}$. Anyway, this last term contributes not more than 10%–15% to $P_{j,m}$, so that the above approximation can cause at most a few percent inaccuracy to $P_{j,m}$. With these approximations (4.8) can be solved for $C_{j,m}$,

$$C_{j,m}=\frac{(r_{2,j}/i_{j-1})I_{j-1,1:j,m}+(i_j/r_{2,j+1})R^{(2)}_{j+1,1:j,m}+E_{j,1:j,m}}{\sum_{u<m}D^{(1)}_{j,m:j,u}-\sum_{s>m}D^{(1)}_{j,s:j,m}} \tag{4.9}$$

and for $\nu_{j,m}$ we get,

$$\nu_{j,m}=\beta_j g_{j,m}\frac{\sum_{u<m}D^{(1)}_{j,m:j,u}-\sum_{s>m}D^{(1)}_{j,s:j,m}}{e^{E_{j,m}/T_e}[E_{j,1:j,m}+(r_{2,j}/i_{j-1})I_{j-1,1:j,m}+(i_j/r_{2,j+1})R^{(2)}_{j+1,1:j,m}]}\,. \tag{4.10}$$

All the rate coefficients in the denominator contain a term of the form $\exp(-E_{jm}/T_e)$, which cancels the first exponent in the denominator. The remaining terms are only slightly dependent on the temperature. As the nominator is independent of the temperature, we find that $\nu_{j,m}$ is only a slowly varying function of T_e. Equation (4.10) is our final result, as the right-hand side does not contain unknown quantities anymore. (In the r_2 and i_j terms only first-order quantities, i.e., ground-state terms, are to be taken).

To summarize, we repeat the recipe for the calculation of the population probabilities and charge-state distributions for a low-electron-density plasma with a given electron temperature: (a) from Eq. (4.10) derive the characteristic electron density $\nu_{j,m}$ for all excited levels of every ionization state; (b) calculate the population probabilities of the excited levels by Eq. (4.4); (c) calculate the ground-state population probability using the normalization condition (2.4); (d) solve the recursive formula (2.10) with the auxiliary condition (2.5) to get the ionization state distribution in the plasma.

This procedure may seem to be lengthy and in some simple cases it may be easier to solve Eq. (3.4) numerically. The real importance of the analytical expressions (4.4) and (4.10) is that they provide a better insight into the underlying physics.

V. APPROXIMATE FORMULA FOR HYDROGENLIKE IONS AND NUMERICAL EXAMPLES

The characteristic density $\nu_{j,m}$ is the ratio between the depopulating one-body spontaneous decay rate coefficients, to the sum of rate coefficients of the populating processes. The first term in the denominator describes electron-impact excitation from the ground state. Generally this is the dominant term in the denominator. The second term corresponds to recombination from the ground state of the jth ion to the $(j-1)$th ion ground state (r_{2j}) which is ionized again to the mth level of the jth ion $(I_{j-1,1:j,m}/i_{j-1})$. This term is nonzero only for level which can be approached by direct ionization of the ground state of the $(j-1)$th ion, e.g., $1s2s$ state of He-like ions produced by inner-shell ionization of a Li-like ion ground state. Except in these specific cases this ionization term can always be neglected.

The last term in the denominator of (4.10) describes the process where the ground state of j is first ionized to the $(j+1)$th ion ground state, which recombines later with a free electron to the mth level. Generally, this term is also very small relative to the excitation term. For certain levels, however, it is the dominant term. An important group of levels having this property are the doubly excited states whose cross section to be excited from ground state is very low, but can be approached with high probability by dielectronic recombination from the next charge state.

For H-like ions fairly accurate expressions can be found in the literature for both the Einstein and the rate coefficients. For these ions the two last terms in the denominator are very small relative to the first. The excitation rate coefficient can be taken from the work of Van Regemorter.[6] His expression for H-like ions, in the temperature range where these ions have appreciable density in the plasma, can be rewritten as

$$E_{z-1,1:z-1,m}=3.71\times10^{-7}e^{-E_m/T_e}T_e^{-1/2}\frac{m^3}{Z^2(m^2-1)^3} \tag{5.1}$$

(T_e in eV), where the Einstein coefficient was approximated by[8] $D^{(1)}_{z-1,m:z-1,1}=1.28\times10^{10}Z^4m^{-5}$, and for Van Regemorter's P function a constant value 0.2 was taken. For the denominator the following expression was used:

$$\begin{aligned}D^*_{z-1,m}&=\sum_{u<m}D^{(1)}_{z-1,m:z-1,u}-\sum_{s>m}D^{(1)}_{z-1,s:z-1,m}\\&=6.65\times10^9Z^4m^{-4}.\end{aligned} \tag{5.2}$$

Inserting (5.1) and (5.2) into (4.10) yields the following formula for hydrogen line ions:

$$\nu_{z-1,m}=3.58\times10^{16}\beta_{z-1}(T)Z^6T^{1/2}m(1-1/m^2)^3. \tag{5.3}$$

In (5.3) the temperature is measured in eV and $\nu_{z-1,m}$ in cm^{-3}. This last expression predicts a very sharp Z dependence of $\nu_{z-1,m}$ and moderate dependence on both the quantum numbers of the excited state and the electron temperature.

A few numerical examples for hydrogenlike and heliumlike ions of aluminum[7] are shown in Table I

TABLE I. Characteristic densities and population probabilities (at a total ion density of 10^{21} cm^{-3}) of the first excited states of H-like and He-like aluminum ions for a few electron temperatures. Units of ν_{jm} are cm^{-3}. In the table 6.17+23 means 6.17×10^{23}, etc.

	$\nu_{jm}(T_e)$			$P_{jm}(T_e)$ at $n_T=10^{21}$ cm^{-3}		
T_e(eV)	100	300	1000	100	300	1000
Al XIII $n=2$	6.17+23	8.71+23	0.58+23	21.1−10	1.71−4	4.50−3
Al XIII $n=3$	18.3 +23	25.0 +23	1.55+23	0.7−10	0.49−4	2.92−3
Al XIII $n=4$	27.7 +23	36.4 +23	2.26+23	0.3−10	0.42−4	3.24−3
Al XIII $n=5$	34.3 +23	43.5 +23	2.64+23	0.2−10	0.47−4	4.13−3
Al XII $1s2p$	5.60+23	9.54+23	6.37+23	60.8−10	2.12−4	4.89−3
Al XII $1s3p$	8.89+23	14.3 +23	8.58+23	2.6−10	0.60−4	2.78−3
Al XII $1s4p$	10.1 +23	15.2 +23	9.27+23	0.9−10	0.42−4	2.34−3

TABLE II. Same as Table I for H-like and He-like carbon.

T_e (eV)	ν_{jm} (T_e) 30	100	300	P_{jm} (T_e) at $n_T=10^{21}$ cm^{-3} 30	100	300
C VI $n=2$	3.35+ 21	1.69+ 21	0.15+ 21	2.42 − 5	2.79 − 2	1.93 − 2
C VI $n=3$	9.87+ 21	4.75+ 21	0.40+ 21	0.30 − 5	2.28 − 2	3.50 − 2
C VI $n=4$	14.8 + 21	6.79+ 21	0.55+ 21	0.18 − 5	2.58 − 2	5.78 − 2
C VI $n=5$	18.1 + 21	8.44+ 21	0.64+ 21	0.16 − 5	3.12 − 2	8.72 − 2
C V $1s2p$	2.15+ 21	1.49+ 21	0.35+ 21	1.45 − 4	5.37 − 2	6.98 − 2
C V $1s3p$	3.55+ 21	2.33+ 21	0.54+ 21	0.26 − 4	3.13 − 2	6.03 − 2

and for the same ions of carbon in Table II. The temperature range was chosen so that these ion species will have significant partial densities in the plasma.

The H-like and He-like ions have about the same features, even the numerical values of $\nu_{j,m}$ for the two species are not very far from each other. As expected from (5.3), $\nu_{j,m}$ increases almost linearly with the principal quantum number m, and a comparison of the two tables reveals a steep Z dependence. The temperature dependence is rather moderate, $T_e^{1/2}$, at the low temperature side of the tables. In this range $\beta_j(T_e)\cong g_{j1}^{-1}=\text{const}$. However, at temperatures comparable to the energy gap of the first excited state, $T_e\cong E_{j,2}$, more terms start to contribute to the sum in $\beta_j(T_e)$ reducing consequently the value of $\nu_{j,m}(T_e)$.

The population probabilities are generally low, 10^{-4} to 10^{-2} in the range of interest for these ions. The probabilities are higher for the high-temperature range, but there a strong reduction of the total densities of these ions occurs in favor of the fully ionized species. The absolute partial densities of the excited states of these ions, therefore, never exceed a total of a few percent.

The characteristic densities of the lower charge states have lower values. For example, for Li-like aluminum, at $T_e=100$ eV, where the density of these ions is maximum, $\nu(j=10,\ m=1s^22p)\cong 2.0\times 10^{22}$ cm^{-3}.

Li-like carbon has maximum abundance at around $T_e=30$ eV with $\nu(j=3,\ m=1s^22p)\cong 5.4\times 10^{18}$ cm^{-3}. The characteristic densities of the lower charge states is much lower.

VI. SUMMARY

In the present paper a consistent solution is given to the population probabilities of the excited states of the various ionic species in a homogeneous, time-independent and optically thin plasma which is in complete steady state, i.e., $dn_{j,m}/dt=0$, and a set of conclusions is derived from these solutions.

First, the equations of the populating and depopulating processes were set up, with an attempt to incorporate into the equations as many processes as practical. Seven processes are already included in the equations, these are the electron-impact ionization, radiative, dielectronic and three-body recombinations, electron-impact excitation and deexcitation, as well as the spontaneous decay of the excited states. Other processes, such as Auger effect, can be incorporated without much difficulty.

From the basic equations, implicit formulas for the charge-state distribution, (2.10), and for the population probabilities, (3.5), were derived. This last one was solved at low electron densities, to yield an analytical formula for $P_{j,m}$ of the excited states (4.4), which is expressed in terms of a characteristic density $\nu_{j,m}$.

It is time to discuss in detail the validity of the various assumptions of these calculations. The assumption of steady state is the most difficult to justify for laboratory plasmas, particularly for laser-produced plasmas. The CSS assumption in these plasmas is valid only if the ionization-state distributions and excited-state populations can adjust rapidly enough to the temperature and density variations in the plasma. This point is not necessarily true in laser-produced plasmas. However, in a comparison of two laser-plasma simulations, one of which used time-dependent equations and the second used a steady-state assumption, it was found that the differences in the final results were surprisingly small, particularly for the x-ray emission rates predicted by the two calculations. The reason for the similarity seems to be the following: intense x-ray emission always originates from those regions of the plasma where the electron density is high enough to produce an appreciable number of excitations, ionizations, and recombinations. As high electron concentration tends to equilibrate the plasma, one would expect that in these regions the plasma will be not too far from steady state. In other words, x rays are emitted mainly from the plasma volume which cannot be very far from a steady state so that for the simulations of x-ray emission CSS is a plausible approximation.

The assumption that the plasma is optically thin holds only for small plasmas. For plasmas having large volumes, or radiation with small mean free path, the photon-plasma interaction with the whole apparatus of radiation transport must be included. This is out of the scope of this work; however, it seems possible that a part of this problem can also be incorporated into the equations.

Finally, Eq. (4.4) was derived with the assumption of low electron density. As we already know the high-electron-density limits of the population probabilities, a reasonably smooth function connecting the two asymptotes can provide an acceptable approximation which is accurate at the low and high density portions and only at intermediate densities would cause moderate inaccuracies.

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