# Multiplicative stochastic processes in statistical physics

A. Schenzle and H. Brand
*Universität Essen Gesamthochschule, Fachbereich Physik, 43 Essen, Germany*
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For a large class of nonlinear stochastic processes with pure multiplicative fluctuations the corresponding time-dependent Fokker-Planck-equation is solved exactly by analytic methods. A universal eigenvalue spectrum and the corresponding set of eigenfunctions are obtained in closed form. The eigenvalue spectrum consists of a discrete as well as a continuous part. To emphasize the significance of the model proposed for the description of more-general stochastic processes the authors investigate its stability with respect to the inclusion of weak additive fluctuations. A discussion of the differences in the static as well as the dynamic behavior of multiplicative and additive stochastic processes is given in detail. It is shown explicitly how internal as well as externally imposed fluctuations can lead to multiplicative stochastic processes. The applications of the results to various fields such as nonlinear optics—subharmonic generation, parametric three-wave mixing, Raman scattering—electronic devices, autocatalytic chemical reactions, and population dynamics are given. In particular, a comparison with recent experiments by S. Kabashima *et al.*, who investigated the statistical properties of electronic parametric oscillators driven by external noise, is carried out.

## I. INTRODUCTION

One can often describe the physics of macroscopic systems very well by a small number of collective variables and their deterministic time evolution, disregarding thereby the many-particle aspects of their microscopic structure. If the system is in a globally stable state, the statistical nature of the macroscopic dynamics is of minor importance and can safely be neglected. When, however, by changing, e.g., some external parameters the state of the system approaches the limit of stability, large excursions about the deterministically described values occur and fluctuations are enhanced to a degree where they play a role important to any understanding of the macroscopic evolution. The equilibrium phase transitions are one class of examples—another class of phenomena consists of the various phase-transition analogs that have been found in such nonequilibrium systems as the laser and many systems in nonlinear optics in the threshold region,[1-3] hydrodynamic instabilities,[3-6] instabilities in the spatial or temporal homogeneity of autocatalytic chemical reactions,[7-12] current instabilities (e.g., tunnel diodes, Gunn effect),[13,14] and self-excited electronic circuits with noise.[15] In order to analyse these systems a deterministic macroscopic description is no longer adequate, and one must go one step further towards the many-particle picture by including fluctuations.

In a variety of problems these fluctuations can be taken into account by adding a "fluctuating force" to the deterministic equations of motion. Starting from a microscopic description, these fluctuations arise from the elimination of the irrelevant degrees of freedom in favor of a small number of macroscopic variables. Probably the most familiar example of this type of *additive fluctuations* are the vacuum fluctuations of the electromagnetic field that trigger spontaneous emission of atoms, allowing them to relax to the equilibrium population. The most important characteristic of additive fluctuations is the fact that they do not depend on the values of the collective variables of the system—the fluctuations jiggle the particles about irrespective of their position.

But there also exist processes where the fluctuations do depend on the values of the macroscopic variables. This can be seen very easily by considering an example from chemical reaction dynamics. In an autocatalytic chemical reaction the production of a molecule of some type is enhanced by the presence of other molecules of the same type that have been produced already. However, the probability of the spontaneous formation of these molecules in question is so extremely low that "vacuum fluctuations" do not play an important role. Therefore the only possible reaction channel is the autocatalytic reproduction of the molecules according to the "blueprint" provided by the molecules of the same kind already present. In these types of processes the fluctuations of the number of molecules must die away when the concentration of the autocatalytic molecules approaches zero. As a result, the fluctuations in this case *do* depend on the state of the system. If this dependence can be described by a function of the macroscopic variables multiplying the "fluctuating stochastic forces," we call such process a "multiplicative stochastic process."

We will now give a precise definition of what we summarized qualitatively above in two of the basic languages used in the theory of stochastic processes.

### A. Langevin formalism

Let us consider a system of many degrees of freedom defined by the Hamiltonian $H$, the macroscopic behavior of which can be characterized by a set of collective variables $\{x_i\}$. Nonlinear internal interactions as well as linear couplings to other degrees of freedom like external reservoirs create dissipative as well as fluctuation processes which make the time evolution of the variables $\{x_i\}$ irreversible on the one hand, and call for a statistical description on the other hand. The resulting equations of motion are of the Langevin form and can be derived in a straightforward way by several methods.[16-18] A typical equation of this kind reads

$$\frac{d}{dt} x_i = \Gamma_{ij}^{(L)} x_j + \Gamma_{ijk}^{(NL)} x_j x_k + \cdots + F_i^{(0)} + x_j F_j^{(i)}, \quad i = 1, 2, 3, \ldots, \tag{1.1}$$

where the $\Gamma$'s are time-independent matrices and the $F$'s are the fluctuating forces defined by their statistical properties, e.g.,

$$\begin{aligned} &\langle F_j^{(i)}(t) F_k^{(i)}(t')\rangle = G_{jk}^{(i,l)}(t-t') \\ &\langle F_j^{(i)}(t)\rangle = 0 \ \forall j . \end{aligned} \tag{1.2}$$

When the fluctuations stem from a reservoir close to equilibrium, $G_{jk}$ depends only on the time difference $(t - t')$. If it is sufficient for the dynamical description of the collective variables $\{x_i(t)\}$ to consider a coarse-grained time scale, large compared to the correlation time of the fluctuations themselves, $G$ can be reasonably approximated by a $\delta$ correlation

$$\langle F_j^{(i)}(t+\tau) F_k^{(l)}(t)\rangle = Q_{jk}^{(i,l)} \delta(\tau), \tag{1.3}$$

where $Q$ is a measure of the fluctuations, independent of the variables $\{x_i\}$. In this limit Eq. (1.1) describes a multidimensional Markovian process. In the case where $F_j^{(i)} = 0$, $i = 1, 2, 3, \ldots, F_j^{(0)} \neq 0$, we call (1.1) an additive stochastic process

$$\dot{x}_i(t) = L_i(\{x_j\}) + F_i , \tag{1.4}$$

while for vanishing $F_j^{(0)}$ the process is called multiplicative[9,10,19-21]

$$\begin{aligned} &\dot{x}_i(t) = L_i(\{x_j\}) + G_{ij}(\{x_k\}) F_j , \\ &G_{ij} \neq \text{const} . \end{aligned} \tag{1.5}$$

These equations can hardly ever be solved exactly when $L(\{x_i\})$ is a nonlinear function; and one must then resort to approximation methods. So far, however, the fluctuating forces are not characterized sufficiently by Eq. (1.2) or (1.3). A complete definition of a stochastic variable must include correlations to all orders. If higher-order correlations allow a factorization according to a Gaussian assumption, we can resort to another description of this process which is equivalent to (1.1) or (1.4) and (1.5), but one having a mathematical formalism which is linear in nature.

### B. Fokker-Planck formalism

In the case of Gaussian white noise Eq. (1.5) is stochastically equivalent to the following Fokker-Planck equation for the probability density[22,16] $P(\{x_i\}, t)$:

$$\frac{\partial}{\partial t} P(\{x_k\}, t) = -\frac{\partial}{\partial x_i} (k_i^{(1)}(\{x_k\}) P) + \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} (k_{ij}^{(2)}(\{x_k\}) P) , \tag{1.6}$$

where the coefficients $k_i^{(1)}$ and $k_{ij}^{(2)}$ are related to the coefficients of the Langevin Eq. (1.5) by the following relations:

$$k_i^{(1)}(\{x_k\}) = L_i(\{x_k\}) + \frac{1}{2} \frac{\partial G_{il}}{\partial x_l} G_{lj} \tag{1.7}$$

and

$$k_{ij}^{(2)}(\{x_k\}) = G_{il}(\{x_k\}) G_{jl}(\{x_k\}) . \tag{1.8}$$

Summation over repeated indices is always implied if not stated otherwise. Without loss of generality we have assumed that the forces $F_i$ are not crosscorrelated. For an additive stochastic process $k_{ij}^{(2)}$ is a constant positive semidefinite matrix independent of the variables $\{x_k\}$, while for multiplicative processes $k_{ij}^{(2)}$ is an explicit matrix function of the $\{x_k\}$. A Langevin description emerging directly from the microscopic formalism of a specific problem always allows an intuitive physical interpretation of the process and the role of the fluctuations. In this paper we will therefore characterize a given example from physics or chemistry in the language of the Langevin description first and then we will switch over to the Fokker-Planck picture for the explicit solution of the dynamics by the correspondence (1.7)–(1.8).

In Sec. II we will give several examples from physics and chemistry and derive the corresponding statistical equations of motion in order to show how multiplicative fluctuations come about. In Sec. III some formal tools useful in dealing with stochastic processes are collected, with special emphasis on multiplicative fluctuations. Light is shed on the similarities as well as on the differences of additive and multiplicative processes in Sec. IV. Sections V–VII are the heart of the paper where we solve some relevant problems in one and more dimensions and discuss the stationary as well as the time-dependent properties in detail. What makes these parts especially interesting is the fact that we can derive exact analytical solutions without resorting to numerical computations for a whole class of stochastic processes which are of significant importance for realistic physical and chemical problems. Besides gaining a complete understanding of these special processes, the existence of exact solutions is always of great importance because those processes can serve as standard models on which the discussion of more complicated problems can be based.

In many problems with dominant multiplicative fluctuations additive ones are still inevitable and cannot be neglected entirely, even when by some physical arguments we can consider them to be very small. The influence of weak additive fluctuations on the eigenvalue spectrum of the Fokker-Planck equation is discussed in Sec. VIII. Finally, comparison with experimental results, especially with some very ingenious and detailed experiments by the Kabashima group[15] on fluctuations of parametric oscillators, is made.

## II. PHYSICAL MOTIVATION

Before going into the computational details of nonlinear stochastic processes with multiplicative fluctuations it seems appropriate first to give examples of physical processes in which these fluctuations emerge in a natural way. It will be seen that in a realistic model we must always deal with various sources for fluctuations acting upon the collective variables, resulting in very complex stochastic processes.

However, the different physical origin of the fluctuations allows the discrimination of various limiting cases where one type of fluctuations dominates the others. For a number of different processes taken from the field of nonlinear optics as well as chemical reaction dynamics, we will derive in the limit of large external fluctuations a fundamental type of multiplicative process which is the motivation for the investigation of this class of processes in detail in the subsequent paragraphs.

All the examples discussed here start from a system of nonlinear processes with additive fluctuations, which can be derived from first principles by standard methods.[16-18] Eliminating all but one or two degrees of freedom by, e.g., an adiabatic-approximation argument, we are in general left with a mixed stochastic process. Depending on the parameters of the problem, the additive or the multiplicative part will play the dominant role. In this paper we will focus our attention on the latter because these processes, having attracted very little attention so far, seem to be rather interesting.

### A. Maxwell-Bloch equations

An ensemble of homogeneous two-level atoms interacting with a single traveling mode of the electromagnetic field is described by the well-known system of Maxwell-Bloch equations. Introducing quantum as well as various thermodynamic fluctuations and the corresponding relaxation processes, we may write these equations[18,23,24]

$$\dot{P}^+ = (i\Delta - 1/T_2)P^+ - 2igWE^+ + \Gamma^+ , \qquad (2.1)$$

$$\dot{W} = -1/T_1\,(W - W_0) - ig(P^+E^- - E^+P^-) + \Gamma^0 , \qquad (2.2)$$

$$\dot{E}^+ = -\text{æ}E^+ + i\bar{g}P^+ + \text{æ}E_0^+ + F^+ , \qquad (2.3)$$

where $\Delta$ characterizes the frequency mismatch of the polarization and the field mode. The coupling constants $g$ and $\bar{g}$ are proportional to the dipole matrix element between the two levels, while $\bar{g}$ contains, besides some fundamental constants, the additional intensive factor $N/V$, the density of the atoms. Here $P^\pm$ describes the collective polarization of the ensemble of atoms and $W$, $|W| \leq 1$ its inversion; $E^\pm$ is the complex slowly varying amplitude of the electromagnetic field. The dissipative processes resulting in longitudinal as well as transverse relaxation are characterized by the damping constants $T_1^{-1}, T_2^{-1}$ and the corresponding fluctuating forces $\Gamma^0$, $\Gamma^\pm$, while the finite lifetime of the photons in the optical cavity is characterized by æ and $F^\pm$. The connection between the damping constant and the forces is given by the fluctuation-dissipation theorem. For the properties of the field fluctuations, e.g., we have

$$\langle F^+(t)F^-(t')\rangle = 2\text{æ}(e^{\hbar\omega/k_BT} - 1)^{-1}\delta(t-t') . \qquad (2.4)$$

The details of the statistical properties have been derived by a number of authors[24] and will therefore not be rederived here.

In order to cover several fields of application, we have included two fundamental pumping processes: $W_0$ characterizes incoherent pumping utilizing real or virtual transition levels, while $E_0^\pm$ accounts for resonant coherent pumping from a monochromatic outside source introducing external fluctuations.

To derive the statistical properties of the electromagnetic field it is a tedious but straightforward procedure to eliminate the atomic variables step by step, and we end up with a third-order equation for the complex field amplitudes. The resulting equation looks rather unwieldy and it is difficult if not impossible to draw any physical information from it intuitively. Restricting ourselves to the good-cavity limit, we gain a small parameter in the model $\mathcal{E} = æT_2 \ll 1$, which allows a drastic reduction of the complexity of the equations. Keeping only the lowest-order terms in $\mathcal{E}$, we arrive at the following first-order equation of motion:

$$\frac{\partial \overline{E}^+}{\partial \tau}(i\Delta T_2 - 1) - \tfrac{1}{2}\overline{E}^+ \frac{\partial}{\partial \tau}\,|\overline{E}|^2$$
$$= \overline{E}^+(|\overline{E}|^2 - \tfrac{1}{2}\overline{E}_0^+\overline{E}^- - \tfrac{1}{2}\overline{E}_0^-\overline{E}^+ + 1 + \Gamma^2 - i\Delta T_2)$$
$$+ \overline{E}_0^+(i\Delta T_2 - 1) + K_{t1}\,, \qquad (2.5)$$

where we have made use of the normalized variables

$$æt = \tau\,,$$
$$\overline{E}^{\pm} = E^{\pm}(4g^2T_1T_2)^{1/2}\,, \qquad (2.6)$$
$$\overline{E}_0^{\pm} = E_0^{\pm}(4g^2T_1T_2)^{1/2}\,;$$

$\Gamma^2$ is an abbreviation for

$$\Gamma^2 = -2g\bar{g}T_2W_0/æ\,, \quad W_0 \gtrless 0\,.$$

The fluctuating terms $K_{t1}$ contain the following contributions: (i) Fluctuations of the field,

$$4g^2T_1T_2æ^{-1}[F^+(1 + \tfrac{1}{2}\,|\overline{E}|^2) + \tfrac{1}{2}F^-(\overline{E}^+)^2]\,. \qquad (2.7)$$

(ii) A totally identical term is contributed by the fluctuations of the external, partially coherent source where only $F^{\pm}$ has to be replaced by the external fluctuations $F_0^{\pm}$. (iii) The inversion fluctuations contribute a simple multiplicative term of the form

$$-(\Gamma^2T_1/W_0)\overline{E}^+\Gamma_0^+\,, \qquad (2.8)$$

while the fluctuations of the polarization bring along

$$-i[\Gamma^2(T_1T_2)^{1/2}/W_0]\Gamma^+\,. \qquad (2.9)$$

Neglecting higher-order terms in the field amplitude, we can condense the fluctuating forces formally into

$$F_1^+ + \overline{E}^+F_2^+\,,$$

where $F_1^+$ is the collection of all additive and $F_2^+$ is the sum of the leading multiplicative fluctuations.

This equation is the starting point for a statistical model of the laser as well as the recently discussed problem of optical bistability,[25-29] containing additive and multiplicative processes. Investigating the formal properties of Eq. (2.5), one can easily see that the multiplicative terms cannot be neglected without causing serious inconsistencies, resulting in the unphysical divergence of the field amplitude. The most elementary multiplicative process can be derived from (2.5) by dropping the coherent pumping term $E_0^{\pm}$ and neglecting the fluctuations of the polarization and of the field $F^{\pm}$. Keeping, e.g., only the inversion fluctuations, we find

$$\frac{\partial \overline{E}^+}{\partial \tau} = -(1 + \Gamma^2)\overline{E}^+ + \Gamma^2|\overline{E}|^2\overline{E}^+ + \overline{E}^+F_2\,, \qquad (2.10)$$

where we have expanded the saturation term up to first order in the field intensity. This is a model for the laser transition with pure multiplicative inversion fluctuations. The laser threshold is designated by $\Gamma^2 = -1$. For the comparison with results to be derived subsequently we want to emphasize that Eq. (2.10) is a special case of the more general process

$$\frac{\partial}{\partial t}X^+(t) = dX^+ - b\,|X|^{2\gamma}X^+ + X^+F\,, \qquad (2.11)$$

where $X^+$ is a complex variable and $\gamma$ a real positive exponent.

### B. Subharmonic generation

The interaction of electromagnetic field modes in a nonlinear dielectric medium can be described by a set of nonlinear field equations assuming that the complex dynamics of the nonresonant medium can be eliminated adiabatically. The generation of the subharmonic frequency in a nonlinear crystal which lacks inversion symmetry leads by means of the rotating-wave approximation to the following set of equations[30-36]:

$$\frac{d}{dt}A_1^+(t) = -æ_1A_1^+ + 2gA_2^+A_1^- + F_1^+\,, \qquad (2.12)$$

$$\frac{d}{dt}A_2^+(t) = -æ_2A_2^+ - g(A_1^+)^2 + P^+ + F_2^+\,; \qquad (2.13)$$

$A_1^+$ and $A_2^+$ characterize the complex field amplitudes of frequency $\omega$ and $2\omega$, respectively. We assume that the nonlinear crystal is located within an optical cavity with different optical qualities for the two frequencies indicated by the two damping constants $æ_1$ and $æ_2$. These inverse lifetimes are connected with the reflection coefficients $R_i$ of the mirrors in the following way:

$$æ_i = (c_i/L)(1 - R_i)\,,$$

where $c_i/L$ is the transit time of the light beam through the cavity—the corresponding fluctuations are indicated by $F_i^+$. The pump force $P^+$ couples the field inside the cavity to an outside coherent pump source. The field equations for the generation of the subharmonic and the generation of the second-harmonic field are the same, the only difference that distinguishes these two processes is the presence of the driving term $P^+$ in Eq. (2.13).

Assuming that the quality of the cavity for the field $A_2$ is much lower than for the subharmonic field $A_1$, we can eliminate $A_2$ adiabatically under the condition

$$\frac{dA_2^+}{dt} \ll \varkappa_2 A_2^+ , \tag{2.14}$$

and end up with a closed equation for the subharmonic wave $A_1$ alone:

$$\frac{d}{dt} A_1^+ = -\varkappa_1 A_1^+ + 2gA_1^- P^+ - 2g^2 |A_1|^2 A_1^+ + F_1^0 + 2gA_1^- F_2^+ . \tag{2.15}$$

In a realistic situation, however, the pump field is provided by a light source which is itself subject to fluctuations, and $P^+$ can be separated into a coherent and a fluctuating part:

$$P^+ = P_0^+ + F_0^+ (2g)^{-1} .$$

If the fluctuations of the external field dominate the internal thermodynamic and quantum fluctuations, we arrive at the simplified equation

$$\frac{d}{dt} A_1 = (2gP_0 - \varkappa_1) A_1 - 2g^2 A_1^3 + A_1 F_0 , \tag{2.16}$$

where the phase fluctuations have been neglected—an assumption which is very reasonable owing to the breaking of phase symmetry by the external coherent field. The Eq. (2.16) describing the amplitude fluctuations of the subharmonic field driven by a partially coherent pump source is again a special case of the class of multiplicative processes characterized above by Eq. (2.11).

### C. Parametric three-wave mixing

In the more general case where the subharmonic photons are not degenerate we describe the parametric generation of two partially coherent fields with frequencies $\omega_1$ and $\omega_2$ driven by an incoming laser field of frequency $2\omega$. At resonance, the frequencies obey the conservation law

$$2\omega = \omega_1 + \omega_2 .$$

Including the additional degree of freedom—the so-called idler mode—in the field equations, we arrive at the following description[30,31,36]:

$$\frac{dA_1^+}{dt} = -\varkappa_1 A_1^+ + gA_3^+ A_2^- + F_1^+ , \tag{2.17}$$

$$\frac{dA_2^+}{dt} = -\varkappa_2 A_2^+ + gA_3^+ A_1^- + F_2^+ , \tag{2.18}$$

$$\frac{dA_3^+}{dt} = -\varkappa_3 A_3^+ - gA_1^+ A_2^+ + P^+ + F_3^+ , \tag{2.19}$$

which is an obvious generalization of Eqs. (2.12) and (2.13).

Assuming now again that the optical quality of the cavity containing the nonlinear medium is only high for the signal mode $A_1$, we can eliminate the fields $A_2$ and $A_3$ adiabatically and find, after expanding the saturation term up to lowest order in the field intensities, the Langevin equation

$$\frac{dA_1^+}{dt} = \left( |P_0|^2 \frac{g^2}{\varkappa_3^2 \varkappa_2} - \varkappa_1 \right) A_1^+ - \frac{g^4}{\varkappa_2^2 \varkappa_3^2} |P_0|^2 |A_1|^2 A_1^+ + \frac{g^2 P_0}{\varkappa_2 \varkappa_3} A_1^+ F_0 . \tag{2.20}$$

In order to arrive at this simplified form we have assumed as above that the quantum fluctuations can be neglected compared with the fluctuations $F_0$ of the pump source

$$P = P_0 + F_0 .$$

With the definition of the generalized potential

$$U(A_1^+, A_1^-) = -\left( |P_0|^2 \frac{g^2}{\varkappa_3^2 \varkappa_2} - \varkappa_1 \right) |A_1|^2 + \frac{g^4 |P_0|^2}{\varkappa_2^2 \varkappa_3^2} |A_1|^4 , \tag{2.21}$$

we can write (2.20) in the form

$$\frac{\partial A_1^+}{\partial t} = -\frac{\partial}{\partial A_1^-} U(A_1^+, A_1^-) + \frac{g^2 P_0}{\varkappa_2 \varkappa_3^2} A_1^+ F , \tag{2.22}$$

and find again an example of a multiplicative process of the formal structure of Eq. (2.11).

### D. Raman scattering

In a crystal with inversion symmetry, in liquid as well as in gaseous samples, three-wave mixing is forbidden by symmetry arguments, whereas four-wave mixing processes with considerably smaller cross sections are compatible with the symmetry conditions. Because the variety of possible four-wave processes is quite large, we want to restrict ourselves here only to one typical mechanism, the generation of Stokes-shifted light in Raman-type processes.

Using a formalism analogous to the description of the parametric mixing above, we can write the following field equations[37-39]:

$$\dot{A}_S^+ = -\varkappa_S A_S^+ + g^2 T_2 |A_L|^2 A_S^+ + F_S^+ , \tag{2.23}$$

$$\dot{A}_L^+ = -\varkappa_L A_L^+ - g^2 T_2 |A_S|^2 A_L^+ + F_L^+ + P^+ ; \tag{2.24}$$

$A_S^\pm$ describes the Stokes-scattered light amplitude, while $A_L^\pm$ characterizes the laser mode inside the cavity coupled to the resonant external source $P^+$. Again eliminating the laser field adiabatically, we find the following equation of motion for the Stokes field $A_S^+$:

$$\dot{A}_s^+ = \left(\frac{g^2 T_2}{æ_L^2} |P|^2 - æ_S\right) A_S^+ - 2\frac{g^4 T_2^2}{æ_L^3} \times |P|^2 |A_S|^2 A_S^+ + A_S^+ F^+ , \qquad (2.25)$$

where we considered only the fluctuations of the pump field, neglecting quantum fluctuations entirely. This equation is obviously again a multiplicative nonlinear process of the above-mentioned category (2.11).

### E. Autocatalytic reactions

To show that multiplicative fluctuations are not restricted to the field of quantum optics, we would like to given an elementary example from chemical reaction kinetics. As a basic model of an autocatalytic reaction we think of the production of a chemical substance $X$ in an autocatalytic step like[7-10,21]

$$A + X \rightleftharpoons NX , \quad B + X \rightarrow C .$$

By chemical means, the concentrations $n_A$ and $n_B$ are kept constant on the average, leaving inevitable fluctuations around these fixed average values

$$n_A = n_A^0 + \delta_A , \quad \langle \delta_A \rangle = 0 , \quad n_B = n_B^0 + \delta_B , \quad \langle \delta_B \rangle = 0 . \qquad (2.26)$$

Introducing the chemical rate constants $\vec{k}_1$, $\overleftarrow{k}_1$, $\vec{k}_2$ that control the velocity of the individual reaction channels above, we find the following rate equation for the concentration $n_x$:

$$\frac{d}{dt} n_x = \vec{k}_1 n_A n_x - \overleftarrow{k}_1 (n_x)^N - \vec{k}_2 n_B n_x . \qquad (2.27)$$

Separating the deterministic from the fluctuating terms and using the abbreviation

$$F = \vec{k}_1 \delta_A - \vec{k}_2 \delta_B ,$$

we find the pure multiplicative process

$$\frac{d}{dt} n_x = (\vec{k}_1 n_A^0 - \vec{k}_2 n_B^0) n_x - \overleftarrow{k}_1 (n_x)^N + n_x F \qquad (2.28)$$

with the deterministic threshold condition

$$d = \vec{k}_1 n_A^0 - \vec{k}_2 n_B^0 \geq 0 .$$

With these examples we feel that we have shown that the standard nonlinear multiplicative process described by Eq. (2.11) plays a fundamental role in the description of systems driven by external fluctuations. The list of examples is certainly not complete and we can think of many fields of application where these processes play an important role. To mention just one more example, we want to recall that the mathematical description of electronic devices like self-excited circuits and many other applications first called for a stochastic picture[22] to allow the proper treatment of signals with noise. The reader who is not primarily interested in the formal aspects may skip the next three sections and go directly to Sec. VI.

## III. FORMAL CONCEPT

For the mathematical description of stochastic processes there exist two fundamental concepts: the method of the stochastic differential equation as introduced by Itô in the 1950's and which has been generalized and developed further by a number of authors (cf. Refs. 40–42) and the Stratonovich interpretation of stochastic differential equations. While the Itô interpretation has a number of valuable advantages, the rules of differential and integral calculus must be redefined. The advantage of Stratonovich's method, however, is that it allows us to retain the usual rules of differential calculus, and we will therefore use this interpretation throughout the present paper.

The general solution $P(x,t)$ of the Fokker-Planck equation (1.6) subject to natural boundary conditions, and the arbitrary initial condition $P(x, t=0)$ describes completely the dynamical evolution of the stochastic process. The only systematic way to derive an exact analytical solution utilizes the eigenfunction expansion. With the ansatz $P(\{x_k\}, t) = P(\{x_k\}) e^{-\lambda t}$ the problem consists in solving the following eigenvalue equation:

$$L P_n(\{x_k\}) = -\lambda_n P_n(\{x_k\}) , \qquad (3.1)$$

where the eigenvalues $\lambda_n$ can form a discrete as well as a continuous spectrum.

### A. Properties of the eigenfunctions

In general the Fokker-Planck operator $L$ is not self-adjoint. However, when $L$ satisfies the condition of detailed balance[43] we can always find a transformation[44] $T$ which brings $L$ into self-adjoint form:

$$H(\{x_k\}) = T^{-1}(\{x_k\}) L T(\{x_k\}) . \qquad (3.2)$$

With the definition

$$W(\{x_k\}) = T^{-1}(\{x_k\}) P(\{x_k\}) \qquad (3.3)$$

we can write the eigenvalue problem in the equivalent form

$$H W_n = \lambda_n W_n . \qquad (3.4)$$

The so-far unspecified transformation function $T(\{x_k\})$ can now be defined by requiring the operator $H$ to be self-adjoint: $H = H^+$, leaving the following equations [for the notation see Eq. (1.6)]:

$$k_{ij}^{(2)}(\{x_k\})\,\frac{\partial T}{\partial x_j}=\frac{1}{2}\left(k_i^{(1)}-\frac{\partial k_{ij}^{(2)}}{\partial x_j}\right)T\,. \qquad (3.5)$$

This set of equations for the scalar potential $\ln T$ has a solution only when the compatibility equations are satisfied.

In one dimension this problem has a unique and general solution given up to a quadrature by

$$T^2(x)=P_0(x)$$

$$=k_2^{-1}(x)\exp\left(\frac{2}{Q}\int_0^x \frac{k_1(x')}{k_2^2(x')}\,dx'\right), \qquad (3.6)$$

where $P_0(x)$ is the steady-state solution of (3.1) for $\lambda_0=0$. The self-adjointness of Eq. (3.4) guarantees that (i) $\lambda_n$ is real. (ii) The eigenfunctions $W_n$ form an orthonormal set

$$\int W_n(x)W_{n'}(x)dx=\delta_{nn'}\,, \qquad (3.7)$$

whereas the eigenfunctions of the Fokker-Planck operator $L$ itself satisfy the relation

$$\int P_0^{-1}(x)P_n(x)P_{n'}(x)dx=\delta_{nn'}\,. \qquad (3.8)$$

For $n=n'$ we have the normalization condition

$$\int P_0^{-1}(x)P_n^2(x)dx=1 \qquad (3.9)$$

and for $n'=0$

$$\int P_n(x)dx=\delta_{n0}\,.$$

(iii) Assuming the completeness of the set of eigenfunctions of the self-adjoint problem (3.4) we conclude that the Fokker-Planck eigenfunctions $P_n$ will satisfy the completeness relation

$$\sum_n\!\!\!\!\!\!\int\; P_0^{-1/2}(x)P_n(x)P_{n'}(x')P_0^{-1/2}(x')=\delta(x-x')\,. \qquad (3.10)$$

### B. Correlation functions

Assuming that the eigenfunctions $P_n(x)$ form a complete set according to the relation (3.10), we can expand the general solution $P(x,t)$ in the form

$$P(x,t)=\sum_n\!\!\!\!\!\!\int\;\int dx' P_0^{-1}(x')P_n(x')$$
$$\times P(x',t=t_0)P_n(x)e^{-\lambda_n(t-t_0)}\,, \qquad (3.11)$$

satisfying the arbitrary initial condition $P(x,t=t_0)$. With the aid of the conditional probability $P(x_2t_2/x_1t_1)$ satisfying the special initial condition $P(x,t')=\delta(x-x')$, we can write the general two-time probability density in the form

$$P(x_2t_2,x_1t_1)=P(x_2t_2/x_1t_1)P(x_1t_1) \qquad (3.12)$$

and expand it into the eigenfunctions of $L$

$$P(x_2t_2,x_1t_1)=\sum_{n,m}\int dx'\,\frac{P(x',t=t_0)P_m(x')}{P_0(x')P_0(x_1)}$$
$$\times P_n(x_1)P_m(x_1)P_n(x_2)$$
$$\times e^{-\lambda_n t_2}e^{-(\lambda_m-\lambda_n)t_1}\,. \qquad (3.13)$$

With the system initially (at $t_0=0$) in the stationary state $P_0(x)$ according to (3.9), Eq. (3.13) gives the stationary two-time distribution $P_0$

$$P_0(x_2t_2,x_1t_1)=\sum_n P_n(x_2)P_n(x_1)e^{-\lambda_n(t_2-t_1)}\,, \qquad (3.14)$$

depending only on the time difference $t_2-t_1$.

These results allow the explicit calculation of all dynamical statistical properties of the Gaussian Markov process. The most fundamental and experimentally important characterization of the statistical properties is given by the stationary two-time correlation function

$$G_2(t)=\langle x(t+t')x(t')\rangle\,, \qquad (3.15)$$

which by using (3.14) can be written in the general form

$$G_2(t)=\sum\!\!\!\!\!\!\int\; g_n^2e^{-\lambda_n t}\,, \qquad (3.16)$$

where $g_n$ is the first moment of the stochastic variable $x$ evaluated with the $n$th eigenfunction $P_n(x)$

$$g_n=\int xP_n(x)dx\,. \qquad (3.17)$$

If the eigenvalue spectrum is partly discrete and partly continuous, special care must be taken with the definitions above and the question of completeness. In this case the correlation function (3.16) reads

$$G_2(t)=\sum_{n=1}^N g_n^2e^{-\lambda_n t}+\int_{\gamma_1}^{\gamma_2} g^2(\lambda)e^{-\lambda t}\,d\lambda\,. \qquad (3.18)$$

The analytic behavior of $G_2(t)$ for large times will show a simple exponential time dependence if

$$\gamma_1 \gg \lambda_1\,,$$

whereas in the opposite case $\gamma_1 \ll \lambda_1$ or even if $\gamma_1=0$, the analytic properties of the asymptotic dependence will be given by the second term on the right-hand side of Eq. (3.18) and may be rather complicated.

We will see that the multiplicative processes do have continuous branches in their eigenvalue spectrum even when the corresponding deterministic problem is globally stable.

### C. Transformation properties of the Langevin and Fokker-Planck equations

For practical purposes it is important to know how the solutions of two Fokker-Planck equations transform into each other when we know the transformation that connects the two corresponding Langevin equations. A given Langevin equation

$$\dot{X}=L(x)+G(x)F \tag{3.19}$$

may be transformed by the definition of a new variable $y$, $x=g(y)$ into the equivalent equation

$$\dot{y}=\hat{L}(y)+\hat{G}(y)F\,, \tag{3.20}$$

which in general is again a multiplicative process. The transformed functions $\hat{L}(y)$ and $\hat{G}(y)$ are given by

$$\hat{L}(y)=L(g(y))\left(\frac{dg(y)}{dy}\right)^{-1}, \tag{3.21}$$

$$\hat{G}(y)=G(g(y))\left(\frac{dg(y)}{dy}\right)^{-1}. \tag{3.22}$$

From these transformation properties it is obvious that in general a multiplicative process remains multiplicative under the transformation of the stochastic variable $x$, while the additive processes with $G(x)=1$ remain additive only under linear transformations.

Given a general multiplicative process we can always find an associated additive one by the implicit transformation

$$x=g(y)\,,\quad y=\int_0^g \frac{dg'}{G(g')}\,.$$

In all physically motivated problems the Langevin equation will be the starting point. Therefore, if we know the transformation connecting two Langevin equations, we would like to know the solution of the Fokker-Planck equation corresponding to Eq. (3.20), provided that the solution of the Fokker-Planck equation of the process (3.19) has been established. The Fokker-Planck equation corresponding to the process (3.19) is

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}\left[\left(L(x)+\frac{Q}{2}\frac{\partial G^2}{\partial x}\right)P\right]+\frac{Q}{2}\frac{\partial^2}{\partial x^2}(G^2P)\,, \tag{3.23}$$

whereas the process (3.20) is described by an analoguous equation for the probability distribution $P(y)$. By just comparing the equations of motion it is straightforward to prove the transformation rule

$$\hat{P}(y)=P(g(y))g'(y)\,,\quad g'=\frac{dg}{dy}\,, \tag{3.24}$$

or, schematically,

$$\begin{array}{ccc}
\dot{x}=L(x)+G(x)F & \xrightarrow{x=g(y)} & \dot{y}=\hat{L}(y)+\hat{G}(y)F\\
\downarrow & & \downarrow\\
\dfrac{\partial P}{\partial t}=D(x)P(x) & & \dfrac{\partial \hat{P}}{\partial t}=\hat{D}(y)\hat{P}(y)\\
\downarrow & & \downarrow\\
P(x)\rightarrow P(g(y))=\tilde{P}(y)=\hat{P}(y)/g'(y) & \leftarrow & \hat{P}(y)\,.
\end{array}$$

We will utilize these results in Sec. VI to generalize the solutions of one specific problem to a whole class of processes which are equivalent to each other under a group of transformations $g_n(y)$.

## IV. ADDITIVE VERSUS MULTIPLICATIVE PROCESSES

Additive and multiplicative processes have a number of common features as well as some striking differences. In this section we want to emphasize some properties that are typical for multiplicative processes. In order to have the same terminology for both types of stochastic processes, we use the general definitions of Eqs. (1.5) and (1.6), which in principle include additive processes as well when we allow $G_{ij}$ and $K_{ij}^{(2)}$ also to be independent of $x$. In order not to overburden the formalism we restrict ourselves for the moment to one-dimensional problems.

### A. Stationary points

In the deterministic limit with fluctuations neglected for a moment the Langevin equation (1.5) has the stationary solutions $x_0^j$ given by the roots of the equation

$$L(x_0^j)=0\,. \tag{4.1}$$

Depending on the local and global stability of these points the fluctuations will either smear the distribution out over the neighborhood of the stationary points or cause large macroscopic excursions to a stable state. This is a common feature of additive as well as multiplicative processes, provided that

$$G(x_0^j)\neq 0\,. \tag{4.2}$$

In the case where $L(x)$ and $G(x)$ have a zero point $x_0$ in common, this point plays a somewhat singular role because if the system is initially prepared to be at $x=x_0$ it will stay there for all times, in spite of fluctuations. The value $x=0$ in Eq. (2.11), e.g., is one such stationary point and $P(x)=\delta(x)$ is a stationary solution. As will be shown in Sec. V, we can also find another, more general, steady-state solution $P_0(x)$ for this process so that the lowest eigenvalue is continuously degenerate:

$$P_{st}=c_1P_0(x)+c_2\delta(x-x_0)\,. \tag{4.3}$$

For the special processes to be discussed here in detail this degeneracy is of no importance at all, because the δ-function distribution does not take part in the dynamics of the system, owing to the fact that the eigenfunctions of the Fokker-Planck equation vanish at $x=x_0$, excluding this point from the time evolution. This property guarantees especially that the probability distribution is not piling up at $x=0$ in the course of time.

### B. Most-probable values

In one dimension, the steady-state distribution of the general multiplicative processes is given up to a quadrature by the expression

$$P_0(x)=G^{-1}(x)\exp\left(\frac{2}{Q}\int^x \frac{L(x')}{G^2(x')}\,dx'\right). \tag{4.4}$$

The peaks of this distribution are the values most likely to be observed in a trial experiment. From

$$\left.\frac{dP(x)}{dx}\right|_{x=\hat{x}_0}=0,\quad \left.\frac{d^2P(x)}{dx^2}\right|_{x=\hat{x}_0}<0 \tag{4.5}$$

we have the necessary condition

$$L(x_0)-\frac{Q}{4}\frac{d}{dx}G^2(x)\bigg|_{x=\hat{x}_0}=0. \tag{4.6}$$

Here lies a drastic difference between additive and multiplicative processes.[9,10] While the most probable value of an additive random process coincides with the deterministic steady-state value

$$\frac{dG}{dx}\equiv 0\Rightarrow L(\hat{x}_0)=0, \tag{4.7}$$

they *do* depend explicitly on the strength of the fluctuations in a multiplicative process

$$\hat{x}_0=\hat{x}_0(Q), \tag{4.8}$$

approaching the deterministic value only in the limit $Q\rightarrow 0$. This is a striking new feature of multiplicative processes which allows to change the properties of the system even qualitatively by changing the strength of the fluctuations, while for additive processes $Q$ affects only the quantitative properties, i.e., additive processes:

$$P_0(x)=\left[\exp\left(2\int^x L(x')dx'\right)\right]^{1/Q},$$

multiplicative processes:

$$P_0(x)=G^{-1}(x)\left[\exp\left(2\int^x \frac{L(x')}{G^2(x')}dx'\right)\right]^{1/Q}.$$

As an example, we may consider a system undergoing a bifurcation and define the threshold by an order parameter $X$ approaching zero at the transition point. If we identify the order parameter with the most probable value of the stochastic process $\hat{X}_0$, we recognize that the threshold itself depends on the fluctuations in the case of the multiplicative process, while additive fluctuations do not enter the threshold condition. This observation predicts the remarkable effect that a multiplicative random process can be driven through the threshold region by only changing the strength of the fluctuations.

### C. Stability

An additive stochastic process is already stable when the associated deterministic problem has a globally stable steady state with respect to arbitrarily large fluctuations. For a one-dimensional system we can formulate this statement quite generally: The additive process $\dot{x}=L(x)+F$ has a stable stationary solution and all its moments $\langle x^n\rangle$ exist up to the $n$th order when $L(x)$ satisfies the inequality

$$-\frac{2}{Q}\int^x L(x')dx'>(n+1)\ln x+\text{const} \tag{4.9}$$

in the asymptotic limit $x>x_0$ where $x_0$ is arbitrarily large but finite. For a multiplicative process, however, the proof of the stability of the deterministic problem is not enough to guarantee stability when fluctuations are present.

The most simple multiplicative process we can think of makes this point quite obvious. Consider, e.g., the linear Gaussian Markov process

$$\dot{x}=-dx+xF, \tag{4.10}$$

for which we can derive all statistical properties immediately. The moments are given by the relation

$$\langle x^n\rangle_t=\langle x^n\rangle_{t=0}\exp[-nt(d-\tfrac{1}{2}nQ)], \tag{4.11}$$

while the the stationary distribution reads

$$P_0(x)=x^{-1-2d/Q}. \tag{4.12}$$

Unfortunately this distribution is not normalizable for any value of the parameter $d$.

For this process only those moments $\langle x^n\rangle$ for which $n\leq 2d/Q$ remain bounded as $t\rightarrow\infty$. If $d<\frac{1}{2}Q$ the multiplicative fluctuations overcome the restoring force $-dx$ and distribute the random variable over the whole range of definition. The corresponding deterministic problem, however, has a stable steady-state solution.

For a multiplicative process we must replace the condition (4.9) by the more general relation

$$-\frac{2}{Q}\int^x \frac{L(x')}{G^2(x')}dx'+\ln G(x)>(n+1)\ln x+\text{const},$$

for $x>x_0$, with $x_0$ arbitrarily large but finite. If we can replace $L$ and $G$ asymptotically by a single power law

$$\lim_{x\rightarrow\infty}L(x)=ax^l,\quad \lim_{x\rightarrow\infty}G(x)=bx^g,$$

then the system is stable and all its stationary moments exist when the following inequality is met:

$$l > 2g - 1 .$$

With these remarks we would like to close the discussion of the general properties of multiplicative stochastic processes and now turn to the detailed description of some special model systems.

## V. STEADY-STATE RESULTS

After having derived some general properties and tools in the preceding sections, we will now start to present the explicit analytic solutions of some model systems, the most important of which—with respect to applications in physics—is the process (2.11). We will therefore discuss here the corresponding Fokker-Planck equations and their solutions in detail and add the results of some other exactly solvable processes without further detailed derivations. In Sec. V we will present the stationary-state or equilibrium properties, while the explicit dynamics will be developed in subsequent sections.

### A.

The process $\dot{x} = dx - bx^{1+\gamma} + xF$, with $F$ representing a $\delta$-correlated Gaussian process, corresponds to the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left[(dx - bx^{1+\gamma} + \tfrac{1}{2}Qx)P\right] + \frac{Q}{2}\frac{\partial^2}{\partial x^2}(x^2 P) . \tag{5.1}$$

The steady-state solution $\partial P_0/\partial t = 0$ is given after performing a straightforward integration by the following expression[15,22] :

$$P_0(x) = N x^{-1+2d/Q} \exp\left(-\frac{2b}{Q}\frac{x^\gamma}{\gamma}\right), \tag{5.2}$$

where $N$ is a suitably defined normalization constant. We will see in Sec. VI that the probability current $j$ at the origin $x=0$ vanishes not only for the steady-state solution, but also for all higher eigenfunctions of the Fokker-Planck operator. Therefore, if the variable $x$ is found at one time to be positive, it will remain in this half space for all times. The proper normalization constant $N$ is therefore given by

$$N = \gamma(2b/\gamma Q)^{2d/\gamma Q}\,\Gamma^{-1}(2d/\gamma Q) . \tag{5.3}$$

#### *1. Most probable values*

The most probable value $x_0$ is given by

$$\hat{x}_0 = \begin{cases} 0, & d \le \tfrac{1}{2}Q , \\ [(1/b)(d - \tfrac{1}{2}Q)]^{1/\gamma} , & d \ge \tfrac{1}{2}Q . \end{cases} \tag{5.4}$$

This behavior is reminiscent of an equilibrium phase transition when we interpret $\hat{x}_0$ as some kind of order parameter. The parameters $d$ and $\frac{1}{2}Q$ will then play the role of an inverse temperature and an inverse critical temperature $Q = 2/T_c$. The nonequilibrium phase-transition analogies have been pointed out for many interesting systems using additive stochastic processes. Probably the classical example of this kind is the laser, where $d$ plays the role of the pump parameter, and $x$ characterizes the amplitude of the coherent laser field.[1-3,18,45] Phase-transition models with multiplicative fluctuations have recently attracted considerable attention and are under investigation theoretically[9,21] as well as experimentally.[15] The relation (5.4) demonstrates explicitly the dependence of $\hat{x}_0$ on the strength of the fluctuations, while in the deterministic problem without fluctuations the threshold is reached when the stationary solution approaches zero

$$d_{\text{th}} = 0 .$$

A suitable definition of the threshold with fluctuations is given by $\hat{x}_0 = 0$, resulting in the threshold condition

$$d_{\text{th}} = \tfrac{1}{2}Q .$$

This is one of the remarkable specific properties of multiplicative processes which is not known from processes with additive fluctuations.

A demonstration of this behavior is given in Fig. 1, where we show how the stationary distribution changes qualitatively with the strength $Q$ of the fluctuations. The observation that the probability density becomes narrower when the fluctuations are reduced is common to all statistical processes.

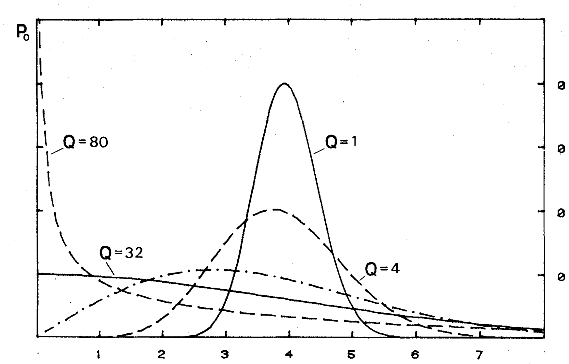


FIG. 1. Stationary solution $P_0(x)$ plotted for $d = 16$, $\rho \equiv \frac{1}{2}\gamma$ as a function of the strength of the fluctuations. For $Q=1$ the distribution peaks at $x=4$. When $Q$ is increased ($Q=4$, $Q=6$) the distribution shifts to smaller values and broadens, until the threshold is reached at $Q=32$. If $Q$ is increased further, the coherent motion is finally suppressed ($Q=80$).

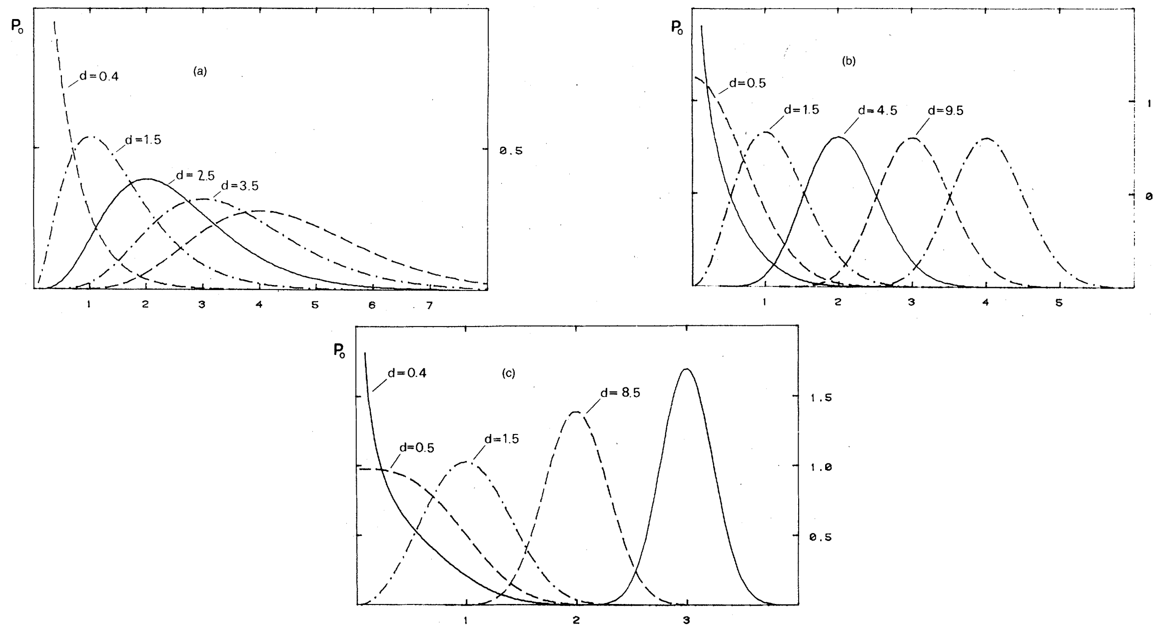


FIG. 2. Stationary distribution $P_0(x)$ as defined in Eq. (5.2) plotted for three different processes with the exponents: (a). $\rho=\frac{1}{2}$, (b). $\rho=1$, (c). $\rho=\frac{3}{2}$. The pump parameter $d$ has been varied from below threshold to well above. The strength of the fluctuations $Q$ has been kept constant at the value $Q=1$. It is obvious that the processes fall into three classes according to their asymptotic behavior: with increasing pump parameter $d$ the variance increases for $\rho<1$, decreases for $\rho>1$, and approaches a constant value for $\rho=1$.

The shift of the peak, however, is characteristic of multiplicative processes. This result has been discussed theoretically by several authors[10,21] and has been demonstrated in a series of experiments by S. Kabashima *et al.*[15]

*2. Moments*

The equilibrium properties can be characterized as well by the hierarchy of stationary moments which, for the class of probability densities described above, can be given in closed analytical form for arbitrary $\gamma$:

$$M_n=\langle x^n\rangle=\left(\frac{2b}{\gamma Q}\right)^{-n/\gamma}\Gamma^{-1}\left(\frac{2d}{\gamma Q}\right)\Gamma\left(\frac{2d}{\gamma Q}+\frac{n}{\gamma}\right). \quad (5.5)$$

For rational exponents $\gamma$ the moments of the order $n=m\gamma$, with $m$ a natural number, assume the simple algebraic form

$$M_n=\left(\frac{2b}{\gamma Q}\right)^{-m}\left(\frac{2d}{\gamma Q}+m-1\right)\cdots\left(\frac{2d}{\gamma Q}\right). \quad (5.6)$$

*3. Asymptotic behavior*

Close to the transition region $d/Q<2$ the distribution undergoes rapid changes, while for large pump parameters $d\to\infty$, $P_0(x)$ may approach a limiting configuration. In this respect the processes with different exponents $\gamma$ behave quite differently, and it is interesting to note that the exponent $\gamma=2$ which, according to the examples in Sec. II is found in many physical applications, plays a special role.

The stationary distribution $P_0(x)$ in the limit $d/Q\to\infty$ can be approximated by the following Gaussian[45]:

$$\lim_{d\to\infty}P_0(x)=N\exp[-(1/\sigma^2)(x-\langle x\rangle)^2] \quad (5.7)$$

located around

$$\langle x\rangle=(d/b-Q/2b)^{1/\gamma}, \quad (5.8)$$

with a variance given by the relation

$$\sigma^2=\frac{1}{\gamma+2}\left(\frac{b}{Q}\right)^{-2/\gamma}\left(\frac{d}{Q}-\frac{1}{2}\right)^{(2-\gamma)/\gamma}. \quad (5.9)$$

While the center of the distribution shifts to the right irrespective of the exponent $\gamma$, the width $\sigma^2$ shows a pronounced dependence on this parameter. In the limit $d\to\infty$ we find

$$\lim_{d\to\infty}\sigma^2=\begin{cases}\infty, & \gamma<2,\\ \frac{1}{4}(b/Q)^{-1}, & \gamma=2,\\ 0, & \gamma>2.\end{cases}$$

This behavior is shown for three values of $\gamma$ in Fig. 2.

B.

The process $\dot{x}=-d/x+b/x^3+(1/x)F$ is statistically equivalent to the Fokker-Planck equation

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial x}\left[\left(\frac{d}{x}-\frac{b}{x^3}+\frac{Q}{2x^3}\right)P\right]+\frac{Q}{2}\frac{\partial^2}{\partial x^2}\left(\frac{P}{x^2}\right). \tag{5.10}$$

We describe here the stationary properties of this process because it is one of the few examples which, as we will demonstrate in Sec. VII, allows an exact analytical solution of its time-dependent Fokker-Planck equation. The stationary distribution is given by

$$P_0(x)=2\left(\frac{d}{Q}\right)^{b/Q+1}\Gamma^{-1}\left(\frac{b}{Q}+1\right)x^{2b/Q+1}\times\exp\left(-\frac{d}{Q}x^2\right). \tag{5.11}$$

Because of the similarity of the two stationary solutions (5.3) and (5.11) we can skip further detailed discussions of the properties. One remarkable difference should be mentioned, however: The most probable value $\hat{x}_0$ of this distribution,

$$\hat{x}_0=d^{-1/2}(b+\tfrac{1}{2}Q)^{1/2},$$

shifts to the right with increasing fluctuations, while in the previous examples we found the opposite tendency. In the dynamic behavior, however, as will be shown in Sec. VII, we will find no further similarities between the processes (5.1) and (5.10).

## VI. SOLUTION OF THE DYNAMICAL EVOLUTION OF THE PROCESS $\dot{x}=dx-bx^3+xF$

After discussion of the steady-state properties we are now prepared to solve the time-dependent Fokker-Planck equation by analytical methods, expressing the general solution in terms of the eigenfunction expansion (3.11). We will see that the eigenvalue spectrum consists of a discrete as well as a continuous branch,[46] which we will discuss separately in the following sections.

### A. Discrete eigenvalue spectrum

The eigenvalue Eq. (3.1) for the special exponent $\gamma=2$ assumes the form

$$\frac{Q}{2}\frac{\partial^2}{\partial x^2}[x^2P(x)]-\frac{\partial}{\partial x}\{[(d+\tfrac{1}{2}Q)x-bx^3]P\}=-\lambda P, \tag{6.1}$$

supplemented by the normalization condition (3.9)

$$\int_0^\infty \frac{P_n^2(x)}{P_0(x)}dx=1, \tag{6.2}$$

where we have restricted ourselves to the positive half space. With the product transformation

$$P(x)=W(x)S(x), \tag{6.3}$$

where $S$ is given by

$$S(x)=x^{-3/2+d/Q}\exp(-(b/2Q)x^2), \tag{6.4}$$

we can transform Eq. (6.1) into normal form:

$$0=W''(x)+W(x)\left[-\frac{(d/Q)^2-\frac{1}{4}-2\lambda/Q}{x^2}+2\frac{b}{Q}\left(\frac{d}{Q}+1\right)-\left(\frac{b}{Q}\right)^2x^2\right]. \tag{6.5}$$

With the additional transformations of the dependent as well as the independent variable

$$W=z^{-1/4}u(z),\quad x=z^{1/2}, \tag{6.6}$$

we arrive at the Whittaker differential equation[47]

$$u''(z)+\tfrac{1}{4}u(z)\left[\frac{1}{z^2}\left(-\frac{d^2}{Q^2}+\frac{2\lambda}{Q}+1\right)+\frac{2b}{Q}\frac{d/Q+1}{z}-\frac{b^2}{Q^2}\right]=0, \tag{6.7}$$

which can be solved in terms of the confluent hypergeometric functions

$$u(z)=z^v\exp\left(-\frac{bz}{2Q}\right){}_1F_1\left[v-\frac{1}{2}\left(\frac{d}{Q}+1\right),2v,\frac{b}{Q}z\right], \tag{6.8}$$

with

$$v_{1,2}=\frac{1}{2}\pm\frac{1}{2}\left[\left(\frac{d}{Q}\right)^2-\frac{2\lambda}{Q}\right]^{1/2}. \tag{6.9}$$

After inserting definitions (6.3) and (6.6) into (6.8) we find the two linearly independent solutions

$$P^{1,2}(x)=x^{-2+d/Q+2v_{1,2}}\exp\left(-\frac{bx^2}{Q}\right)\times{}_1F_1\left[v_{1,2}-\frac{1}{2}\left(1+\frac{d}{Q}\right),2v_{1,2},\frac{b}{Q}x^2\right]. \tag{6.10}$$

From this continuous set of solutions only a finite number of functions satisfies the normalization condition (6.2). The convergence in the asymptotic region $x\to\infty$ can only be guaranteed for the discrete set of eigenvalues

$$\lambda_m=2mQ(d/Q-m), \tag{6.11}$$

while the convergence of the normalization integral (6.2) at the origin requires the inequality

$$d/Q\geq 2m. \tag{6.12}$$

Both conditions can only be fulfilled simultaneously by the eigenfunction $P^1(x)$. An illustration of the lowest eigenfunctions $P_n(x)$ is given in Fig. 3(b). The eigenvalue spectrum consists according to (6.11) and (6.12) of a finite number of discrete values that become more and more numerous with increasing "pump parameter" $d$.

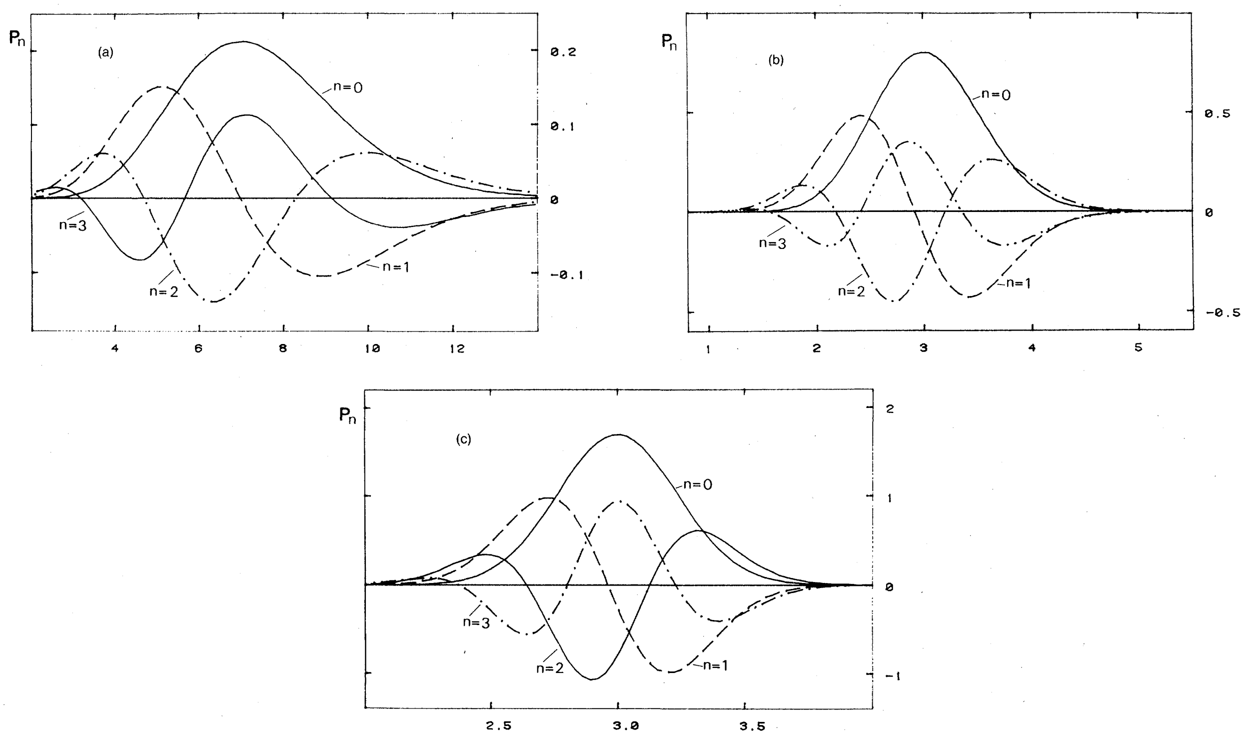


FIG. 3. Steady-state solution $P_0(x)$ and first three eigenfunctions ($n=1, 2, 3$) of the Fokker-Planck Eq. (5.1) are plotted for three different exponents $\rho$: (a) $\rho=\frac{1}{2}$, $d=7.5$, $Q=1$; (b) $\rho=1$, $d=9.5$, $Q=1$; (c) $\rho=\frac{3}{2}$, $d=27.5$, $Q=1$.

Considering the index $m$ for a moment to be continuous, we can find an envelope function, a parabola

$$\tilde{\lambda}=\tfrac{1}{2}d^2/Q\,, \tag{6.13}$$

on which the discrete eigenvalues $\lambda_n$ terminate in a tangential direction as a function of the pump parameter $d$. A plot of the eigenvalue spectrum is given in Fig. 4. In agreement with (6.12), the envelope defined in (6.13) separates the area of real parameters $v_m$ from that of complex ones. The real values of $v$ correspond to eigenvalues below the envelope

$$\lambda_n<\tilde{\lambda}$$

and lead to a system of real eigenfunctions.

So far we have obtained the discrete branch of the eigenvalue spectrum suspecting, however, that it cannot be complete. The plot of the eigenvalues (Fig. 4) already indicates that the curve $\tilde{\lambda}(d)$ not only serves as an envelope of the discrete values $\lambda_n$, but also separates them from the continuous branch of the eigenvalue spectrum.

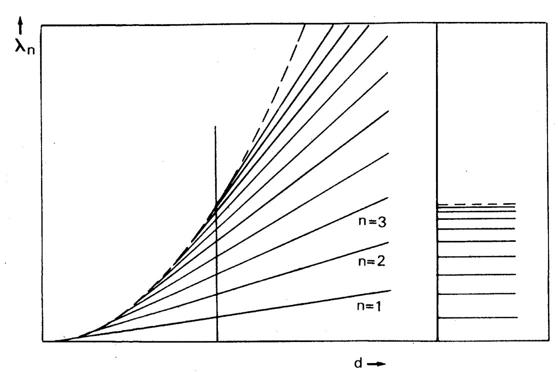


FIG. 4. On the left-hand side, the eigenvalues $\lambda_n$ [see Eq. (6.11)] have been plotted as a function of the pump parameter $d$, while the right-hand side shows a typical eigenvalue spectrum for a given value of $d$.

### B. Continuous branch

The general solution of the differential equation (6.1) is a linear combination of the two fundamental solutions (6.10). To satisfy the normalization condition for the continuous branch,

$$\int \frac{P_\lambda(x)P_{\lambda'}(x)}{P_0(x)}\,dx=\delta(\lambda-\lambda')\,, \tag{6.14}$$

it is not necessary that each partial solution $P^{1,2}(x)$ separately approaches zero in the asymptotic limit. It is sufficient, however, to find a suitable linear combination of $P^1(x)$ and $P^2(x)$ which satisfies this requirement. The linear combination that fulfills the normalization condition for the asymptotic region $x \to \infty$ is the Kummer function[46,47]

$$P(x) = x^{-3/2} \exp\left(-\frac{b}{2Q} x^2\right) \times \left[ \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - æ)} x^{2\mu+1} {}_1F_1\left(\mu - æ + \tfrac{1}{2}, 2\mu + 1, \frac{b}{Q} x^2\right) + (\mu \to -\mu) \right], \quad (6.15)$$

where we have used the abbreviations

$$\mu^2 = \frac{1}{4}\left[\left(\frac{d}{Q}\right)^2 - 2\frac{\lambda}{Q}\right], \quad æ = \frac{1}{2}\left(\frac{d}{Q} + 1\right).$$

To avoid divergencies at the origin $x = 0$ we must require that the eigenvalues have a lower limit

$$\lambda \geq \tfrac{1}{2} d^2/Q. \quad (6.16)$$

All the real values of $\lambda$ that satisfy (6.16) make up the continuous branch of the spectrum. In Fig. 4 this branch lies inside the dashed parabola. The eigenfunctions (6.15) can be properly normalized onto the $\delta$ function by means of the Meijer function.[48]

We want to close this section by summarizing the results: (i) For $\lambda \leq \frac{1}{2} d^2/Q$ we find the eigenfunctions

$$P_n(x) = x^{-1+2d/Q-2n} \exp[-(b/Q)x^2] \times {}_1F_1(-n, 2v_n, (b/Q)x^2), \quad (6.17)$$

with

$$2v_n = 1 + d/Q - 2n > 0$$

and the eigenvalue spectrum

$$\lambda_n = 2nQ(d/Q - n).$$

(ii) For $\lambda > \frac{1}{2} d^2/Q$ we find the eigenfunctions

$$P_\lambda(x) = x^{-1+d/Q} \exp\left(-\frac{b}{Q} x^2\right) \times \left[ \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - æ)} x^{2\mu} {}_1F_1\left(\mu - æ + \tfrac{1}{2}, 2\mu + 1, \frac{b}{Q} x^2\right) + (\mu \to -\mu) \right], \quad (6.18)$$

with

$$\mu = [i/(2Q)^{1/2}](\lambda - d^2/2Q)^{1/2}, \quad æ = \tfrac{1}{2}(d/Q + 1),$$

and the eigenvalue spectrum which consists of all real values of $\lambda$ subject to the restriction

$$\lambda > \tfrac{1}{2} d^2/Q.$$

### C. Correlation functions

For comparison with experimentally observable results we can use the analytical solutions obtained above to calculate all desired correlations and so describe in measurable terms the statistics of the underlying stochastic processes completely. With the general expressions listed in Sec. III we can calculate, e.g., the stationary two-time correlation function from

$$G_2(t) = \langle x(t + t')x(t')\rangle = \sum_{n=0}^{n_{\max}} g_n^2 e^{-\lambda_n t} + \int_{d^2/2Q}^{\infty} g^2(\lambda) e^{-\lambda t} d\lambda, \quad (6.19)$$

with the generalized first moments defined as

$$g_n = \int_0^x x P_n(x) dx. \quad (6.20)$$

The continuous part of the spectrum contributes in general a rather complicated time dependence. In the threshold region $d/Q < 2$, where the correlation function is entirely given by the integral part of (6.19), the analytical structure of $G_2$ is not known exactly, and the integration must be carried out numerically. Apart from this regime, however, in the asymptotic limit $t \to \infty$ the continuous part is dominated by a finite number of exponentials from the discrete spectrum.

In the derivation of (6.20) we assumed that the eigenfunctions are normalized according to (3.9). Because the results of Eqs. (6.10) and (6.17) in the present form have not yet been properly normalized, we must introduce an additional factor $N_n^{-1/2}$ into Eq. (6.20),

$$N_n = 4\left(\frac{Q}{b}\right)^{2d/Q-2n} \Gamma\left(\frac{d}{Q}\right) \sum_{l=0}^{2n} c_l^{(n)} \Gamma\left(\frac{d}{Q} + l - 2n\right), \quad (6.21)$$

with

$$c_l^{(n)} = \sum_{j=0}^{l} \frac{(-n)_j}{(2v_n)_j} \frac{(-n)_{l-j}}{(2v_n)_{l-j}}, \quad (6.22)$$

where

$$(a)_n = a(a+1) \cdots (a+n-1).$$

Substituting the definition form (6.17) into (6.20) and integrating, we find in the regime $d/Q > 2n$

$$g_n = \left(\frac{Q}{b}\right)^{1/2} \times \frac{\sum_{l=0}^{n} d_l^{(n)} \Gamma(d/Q + l - n + \frac{1}{2})}{(\Gamma(d/Q) \sum_{l=0}^{2n} c_l^{(n)} \Gamma(d/Q + l - 2n))^{1/2}}, \quad (6.23)$$

where $d_l^{(n)}$ is an abbreviation for

$$d_l^{(n)} = (-n)_l / (2v_n)_l.$$

It should be recognized that, in spite of its unwieldy appearance, (6.23) is up to a prefactor a simple rational expression containing only powers of $d/Q$. The definition of the coefficient $c_l^{(n)}$ guarantees that when $d/Q$ approaches an even number $2n$, the corresponding moment $g_n$ drops out of the expansion (6.20) quite naturally [cf. also (6.12)]:

$$g_n \to 0 \quad \text{for } d/Q - 2n \to 0^+. \tag{6.24}$$

To illustrate this behavior, we write the first two moments explicitly:

$$n=0\,, \quad g_0 = \langle x\rangle = \left(\frac{Q}{b}\right)^{1/2} \Gamma^{-1}\left(\frac{d}{Q}\right)\Gamma\left(\frac{d}{Q}+\frac{1}{2}\right),$$

$$n=1\,, \quad g_1 = -\frac{1}{2}\left(\frac{Q}{b}\right)^{1/2} \frac{(d/Q-2)^{1/2}}{d/Q-\frac{1}{2}} \frac{\Gamma(d/Q+\frac{1}{2})}{\Gamma(d/Q)}\,.$$

The ratio of these moments is given by

$$\left|\frac{g_1}{g_0}\right|^2 = \frac{1}{4}\,\frac{d/Q-2}{(d/Q-\frac{1}{2})^2}\,, \tag{6.25}$$

and approaches zero for increasing pump parameter. In the long-time limit outside the threshold regime we can express the correlation function by the leading terms

$$G_2(t) = \frac{Q}{b}\,\frac{[\Gamma(d/Q+\frac{1}{2})]^2}{[\Gamma(d/Q)]^2} \times\left(1+\frac{1}{4}\,\frac{d/Q-2}{(d/Q-\frac{1}{2})^2}\,e^{-2(d-Q)t}+\cdots\right), \tag{6.26}$$

and obtain a pure exponential decay. An interesting and remarkable prediction of the model considered is the explicit dependence of the relaxation rates on the strength of the fluctuations $Q$. This linear dependence of $\lambda_n$ on $Q$ as well as on the pump parameter $d$ is open to experimental verification. For an electronic parametric oscillator driven by external multiplicative fluctuations S. Kabashima *et al.* found that the damping constant governing the approach to equilibrium decreased linearly with increasing fluctuations and increased linearly with the pump parameter. This result finds its natural explanation in the eigenvalue spectrum (6.11).

### D. Generalization of previous results

Utilizing the transformation properties derived in Sec. III C, we can generalize the processes under investigation towards more general nonlinearities. The Langevin equation

$$\dot{x} = dx - bx^3 + xF \tag{6.27}$$

can be transformed into the more general equation

$$\dot{y} = d'y - b'y^{2\rho+1} + yF' \tag{6.28}$$

by using the definition $x = y^\rho$ where the primed terms in (6.28) are equal to the unprimed terms in (6.27) divided by the exponent $\rho$.

Without further calculations we can write immediately the explicit solution of the Fokker-Planck equation corresponding to the whole class of processes (6.28) by using the transformation rules (3.25). For the class of Langevin equations

$$\dot{x} = dx - bx^{2\rho+1} + xF$$

with arbitrary exponent $\rho > 0$ the corresponding Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\{[(d+\tfrac{1}{2}Q)x - bx^{2\rho+1}]P\} + \frac{Q}{2}\,\frac{\partial^2}{\partial x^2}(x^2P) \tag{6.29}$$

can be solved exactly by means of the transformation properties (3.24) and the explicit results (6.10). The stationary solution has been given already in Sec. V. For the eigenfunctions we find

$$P_m^{(\rho)}(x) = x^{-1+2(d/Q-\rho m)}\exp[-(b/\rho Q)x^{2\rho}] \times {}_1F_1[-m, 2v_m^{(\rho)}, (b/\rho Q)x^{2\rho}]\,,$$

$$2v_m^{(\rho)} = 1 - d/\rho Q - 2m\,. \tag{6.30}$$

The corresponding eigenvalues are given by

$$\lambda_m^{(\rho)} = 2m\rho(d/Q - \rho m)\,, \quad d/Q \geq 2\rho m\,.$$

It should be mentioned that the exponent $\rho$ can be chosen quite arbitrarily, but some care has to be taken not to violate normalizability. In many physical systems symmetry arguments determine the possible exponents $\rho$. Inversion symmetry ($x \to -x$), e.g., requires that $\rho$ can assume only integer values, whereas for a system without any internal symmetry $\rho$ can be quite arbitrary. An interesting example is given by the autocatalytic chemical reactions described in Sec. II E,

$$A + x \to 2x\,.$$

This standard reaction step is an example of a fractional exponent, $\rho = 1/2$. The stationary distribution of this process has been plotted in Fig. 2(a).

### E. Amplitude and phase fluctuations

It is possible to generalize the problem above even further by allowing the random variable $x$ to assume complex values. This generalization is important for applications in which the variable $x$ contains amplitude as well as phase information, like the complex field amplitudes in the examples of quantum optics mentioned above. We start from the Langevin description of the following process

$$\dot{z}^* = dz^* - b|z|^{2\gamma} z^* + z^* F^* , \qquad (6.31)$$

assuming $F$ to describe a Gaussian $\delta$-correlated process with vanishing cross correlations characterizing thermodynamical fluctuations; $d$ and $b$ are taken to be real. Decomposing Eq. (6.30) into real and imaginary parts, we obtain the following system of stochastic equations for $z = x_1 + ix_2$:

$$\dot{x}_1 = dx_1 - bx_1(x_1^2 + x_2^2)^{\gamma/2} + x_1 F_1 - x_2 F_2 ,$$

$$\dot{x}_2 = dx_2 - bx_2(x_1^2 + x_2^2)^{\gamma/2} + x_1 F_2 + x_2 F_1 ,$$

where we split the fluctuating force into real and imaginary parts: $F = F_1 + iF_2$. $F$ is characterized by the correlation functions

$$\langle F_{1,2}(t) F_{1,2}(t')\rangle = Q\delta(t-t') , \quad \langle F_{1,2}(t) F_{2,1}(t')\rangle = 0 . \qquad (6.32)$$

It is now straightforward to derive, from the general rules given in Eqs. (1.5)–(1.8) the corresponding Fokker-Planck equation

$$\frac{\partial P(\{x_i\})}{\partial t} = \sum_{i=1}^{2} -\frac{\partial}{\partial x_i}\left(\left\{\left[d - b\left(\sum_{j=1}^{2} x_j^2\right)^{\gamma/2}\right] x_i\right\} P(\{x_i\})\right) + \frac{Q}{2} \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}\left[\left(\sum_{j=1}^{2} x_j^2\right) P(\{x_i\})\right]. \qquad (6.33)$$

The notation used in Eq. (6.33) already indicates that the generalization to arbitrary dimensions can be carried out easily.

The symmetry of the Fokker-Planck equation suggests the use of polar coordinates

$$x_1 = r\cos\varphi , \quad x_2 = r\sin\varphi ,$$

leading to the equation

$$\frac{\partial}{\partial t} P(r,\varphi,t) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(dr - br^{2\gamma+1} - \frac{Q}{2}\frac{\partial}{\partial r} r^2\right) P\right] + \frac{Q}{2}\frac{\partial^2 P}{\partial \varphi^2} . \qquad (6.34)$$

Making use of the rotational invariance of this equation, we find with the ansatz

$$P(r,\varphi,t) = P(r) e^{im\varphi} e^{-\lambda t} \qquad (6.35)$$

the following one-dimensional eigenvalue problem:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(dr - br^{2\gamma+1} - \frac{Q}{2}\frac{\partial}{\partial r} r^2\right) P\right] + (\lambda - \tfrac{1}{2} Qm^2) P = 0 . \qquad (6.36)$$

In the special case $\lambda = 0$, $m = 0$, Eq. (6.36) can be integrated immediately, yielding the steady-state solution

$$P_0^0(r,\varphi) = \frac{\gamma}{\pi}\left(\frac{b}{\gamma Q}\right)^{d/\gamma Q} \Gamma^{-1}\left(\frac{d}{\gamma Q}\right) r^{-2+2d/Q} \times \exp\left(-\frac{b}{\gamma Q} r^{2\gamma}\right) . \qquad (6.37)$$

In analogy to the procedure of Sec. VI A we can solve the time-dependent equation in full generality and find for the discrete branch of the spectrum the eigenfunctions

$$P_n^m(r,\varphi) = r^{-2+2(d/Q-n\gamma)} \exp\left(-\frac{b}{\gamma Q} r^{2\gamma}\right) \times e^{im\varphi} {}_1F_1\left(-n, \frac{d}{\gamma Q} - 2n + 1, \frac{b}{\gamma Q} r^{2\gamma}\right) \qquad (6.38)$$

with the corresponding eigenvalues

$$\lambda_n^m = Q[\tfrac{1}{2} m^2 + 2n\gamma(d/Q - \gamma n)] , \qquad (6.39)$$

subject to the restriction

$$d/Q \geq 2\gamma n .$$

It is interesting to note that the eigenvalues $\lambda_0^m$ do not depend on the pump parameter $d$ at all, but are determined completely by the fluctuations $Q$.

This is one of the few examples in statistical physics where the Fokker-Planck equation of a relevant nonlinear process in more than one dimension can be solved completely by analytical methods. Besides the relevance of these results for the explanation of the physical examples described in Sec. II it is also valuable to have a class of standard statistical processes that allow an exact solution which can be used, e.g., to estimate the power of various approximation methods.

## VII. DYNAMICAL PROPERTIES OF THE PROCESS $\dot{x} = \frac{-d}{x} + \frac{b}{x^3} + \frac{1}{x} F$

In the previously discussed examples the fluctuations increased with the stochastic variable $x$. In contrast to this process here we want to solve another model with multiplicative fluctuations that decrease while $x$ increases.

The equilibrium solution of the associated Fokker-Planck equation has been derived already in Sec. V B [Eq. (5.11)], corresponding to a bound state of the "two-particle" system. Here we will summarize without explicit derivations the properties of the eigenvalue problem. The eigenfunctions of the Fokker-Planck operator (5.16) are given by

$$P_n(x) = x^{1+2b/Q} \exp\left[-\frac{d}{2Q}\left(1 + \frac{1}{1+2Qn/b}\right) x^2\right] \times {}_1F_1\left(-n, \frac{b}{Q}, \frac{dx^2}{Q(1+2Qn/b)}\right) , \qquad (7.1)$$

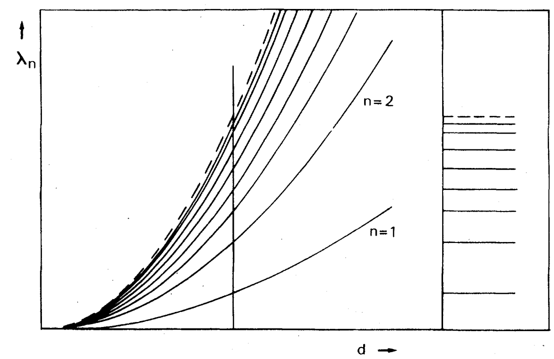


FIG. 5. On the left-hand side, the eigenvalues $\lambda_n$ [see Eq. (7.2)] have been plotted as a function of the pump parameter $d$, while the right-hand side shows a typical eigenvalue spectrum for a given value of $d$.

while the corresponding eigenvalues can be cast into the form

$$\lambda_n = \tfrac{1}{2}(d^2/Q)[1-(1+2Qn/b)^{-2}]. \tag{7.2}$$

The discrete branch of the eigenvalue spectrum consists of infinitely many states with the point of accumulation $\lambda_\infty$, where

$$\lambda_\infty = d^2/2Q, \quad 0 \leq \lambda \leq \lambda_\infty.$$

In Fig. 5 we have plotted the eigenvalue spectrum as a function of the parameter $d/Q$. The continuous branch of the spectrum that can be calculated explicitly by the same arguments as used in Sec. VIB lies above the parabola

$$\lambda = \tfrac{1}{2} d/Q.$$

A cross cut through the spectrum for a fixed parameter $d$ is given on the right-hand side of Fig. 5.

## VIII. ROLE OF WEAK ADDITIVE FLUCTUATIONS

In a real physical system the fluctuations will in general be neither purely additive nor purely multiplicative as in the model systems above. To be of any use, the models must describe a realistic limiting case in which, e.g., the multiplicative fluctuations dominate the additive ones. In this sense we have treated above the extreme limit by neglecting the additive fluctuations entirely.

In this section we now want to go one step further towards a more general and more realistic system by considering the influence of weak additive fluctuations on an otherwise purely multiplicative process. As a model system we choose the process (6.1) and include an additional fluctuating term. In the Langevin picture this generalized process is described by

$$\dot{x} = dx - bx^N + F_1 + xF_2, \tag{8.1}$$

where the forces $F_i$ represent statistically independent Gaussian Markov processes

$$\begin{aligned} \langle F_j(t)F_i(t')\rangle &= Q_j\delta(t-t')\delta_{ij}, \\ \langle F_j(t)\rangle &= 0. \end{aligned} \tag{8.2}$$

By weak additive fluctuations we mean the limit

$$Q_1/Q_2 = \mathcal{E}^2 \ll 1, \quad Q_2 \equiv Q. \tag{8.3}$$

The Fokker-Planck equation corresponding to (8.1), using the definitions of (8.2) and (8.3), assumes the following form;

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}[(dx - bx^N + \tfrac{1}{2}Qx)P] + \frac{Q}{2}\frac{\partial^2}{\partial x^2}[(\mathcal{E}^2 + x^2)P]. \tag{8.4}$$

In order to have stable stationary solutions for the deterministic problem, we must assume $N$ to be an odd integer, that

$$N = 2m+1, \quad m = 1, 2, 3, \ldots.$$

The equilibrium properties are characterized by the stationary distribution which, for arbitrary fluctuations, takes the form

$$P^0_{\mathcal{E}}(x) = N^{-1}(\mathcal{E}^2 + x^2)^{d/Q - 1/2 - (b/Q)(-1)^m \mathcal{E}^{2m}} \times \exp\left(-\frac{b}{Q}\sum_{v=0}^{m-1}(-1)^v \frac{x^{2(m-v)}}{m-v}\mathcal{E}^{2v}\right). \tag{8.5}$$

Comparing this result with the previous distribution (5.2), we notice that outside a threshold region, i.e., for $d/Q > \frac{1}{2}$, no essential changes in the functional dependence of $P(x)$ can be expected in the limit $\mathcal{E} \ll 1$. A considerable effect of the additive fluctuations however is found in the region $d/Q < \frac{1}{2}$, even for small $\mathcal{E}$.

In the purely multiplicative case the distribution (5.2) is singular at the origin, while in (8.4) additive fluctuations prevent this divergence for any real $x$. The essential discrepancy between the more general result (8.4) and the model calculation (5.2) is, however, confined to the region

$$|x| < \mathcal{E}.$$

The finite value of $P^0_{\mathcal{E}}(0)$ is a measure of the strength of the additive compared to the multiplicative fluctuations. This property allows one to measure even weak additive fluctuations in the presence of strong multiplicative ones by examining the probability distribution in the neighborhood of the origin under weak pumping conditions $d/Q < \frac{1}{2}$. In order to compare this functional dependence with experimental results, we have plotted in Fig. 6 the stationary distribution and the experimental points of Kabashima *et al.*,[15] and obtain very good quantitative agreement.

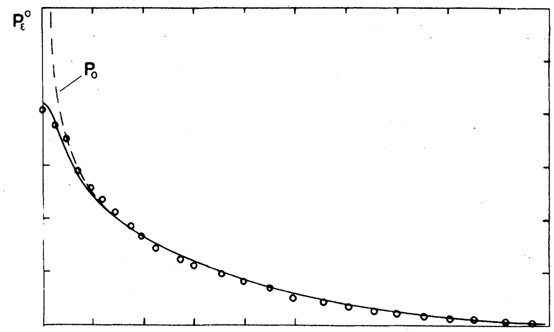


FIG. 6. Stationary probability distribution $P_\mathcal{E}^0(x)$ [Eq. (8.5)], including weak including weak additive fluctuations. Circles indicate the experimental results of Kabashima *et al.* Contrary to the distribution of the pure multiplicative process $P_0(x)$ (dashed line), $P_\mathcal{E}^0(x)$ is nonsingular at $x=0$.

While the stationary properties of the above process can be found for arbitrary fluctuations $Q_1$, $Q_2$, there is no exact solution for the dynamics of the problem when additive as well as multiplicative fluctuations are present. The aim of this section therefore is not to search for an exact solution of Eq. (8.4) but to gain a qualitative understanding of the influence of weak additive fluctuations on multiplicative processes. For the results derived in Secs. V–VII to be of any physical significance it is important that they are not altered completely when small additive fluctuations are included.

Starting from the Fokker-Planck equation (8.4) we find the normal form of the differential equation by the product transformation

$$P=WS\,, \tag{8.6}$$

with

$$S(x)=(\mathcal{E}^2+x^2)^{(a/Q-3/2-(b/Q)(-1)^m\mathcal{E}^{2m})/2}\times\exp\left(-\frac{b}{2Q}\sum_{l=0}^{m-1}(-1)^l x^{2(m-l)}\frac{\mathcal{E}^{2l}}{m-l}\right), \tag{8.7}$$

and obtain after introducing the new independent variable $z=x/\mathcal{E}$ the eigenvalue equation

$$-w''(z)+w\left(\frac{Q^{-2}}{(1+z^2)^2}\left[(d^2-dQ-\tfrac{1}{4}Q^2)z^2+b^2\mathcal{E}^{2(N-1)}z^{2N}+(bQ-2db)\mathcal{E}^{N-1}z^{N+1}+\tfrac{1}{2}Q^2\right]+\frac{Q^{-1}}{1+z^2}(d-Nb\mathcal{E}^{N-1}z^{N-1})\right)=\frac{2\lambda}{Q}\,\frac{W}{1+z^2}\,. \tag{8.8}$$

The appearance of the denominator $1+z^2$ suggests the introduction of a new independent variable $Y$ according to

$$z=\sinh Y\,. \tag{8.9}$$

The resulting differential equation for $\tilde{w}(Y)$ can be transformed to normal form again by the definition

$$\tilde{w}(Y)=R(Y)G(Y)\,, \tag{8.10}$$

with

$$G(Y)=(\cosh Y)^{1/2}\,. \tag{8.11}$$

After some elementary transformations we find the Schrödinger-type equation

$$-R''+V(Y)R=(2/Q)(\lambda-d^2/2Q)R\,, \tag{8.12}$$

with an effective potential given by

$$V(Y)=\frac{1}{Q^2}\left(\frac{1}{\cosh^2 Y}\left[b^2\mathcal{E}^{2(N-1)}\sinh^{2N}Y-b(2d-Q)\mathcal{E}^{N-1}\sinh^{N+1}Y-d^2+dQ\right]-NbQ\mathcal{E}^{N-1}\sinh^{N-1}Y\right). \tag{8.13}$$

The general properties of the potential $V(Y)$ allow a qualitative discussion of the eigenvalue spectrum: (i) In the asymptotic limit $|Y|\to\infty$ the potential can be approximated by the expression

$$V_1(Y)=(1/Q)b^2(\tfrac{1}{2}\mathcal{E})^{2(N-1)}\exp[2(N-1)|Y|]\,. \tag{8.14}$$

(ii) For $Y$ in the neighborhood of $\pm Y_0$

$$Y_0=(1-N)^{-1}\ln[b(\tfrac{1}{2}\mathcal{E})^{N-1}/2d+Q(N-1)]\,, \tag{8.15}$$

the potential assumes the form

$$V_2(Y)=(1/4Q^2)[2d+Q(N-1)]^2\times(e^{2(N-1)|Y\pm Y_0|}-2e^{(N-1)|Y\pm Y_0|})\,, \tag{8.16}$$

with the minimum at $Y=\pm Y_0$. (iii) For small values of $Y$ we find

$$V_3(Y)=-(1/Q^2)d(d-Q)\text{sech}^2 Y\,. \tag{8.17}$$

In order to regain the previous results for pure multiplicative fluctuations, i.e., in the limit $\mathcal{E}\to 0$, we must bare in mind that the transformation $z=x/\mathcal{E}$ rescales the independent variable.

Using the terminology of the Schrödinger equation we will find localized discrete bound states with negative total energies in the potential wells $V_2$ and $V_3$ centered around $Y=0$ and $Y=\pm Y_0$, respectively. Negative total energy means $\lambda\leq d^2/2Q$, the regime where we already found the discrete spectrum in the case of pure multiplicative processes. Utilizing the textbook results for the potentials $V_2(Y)$ and $V_3(Y)$ above, we find the following eigenvalues:

$$\lambda_n^{(2)} = n(N-1)Q[d/Q - \tfrac{1}{2}(N-1)n],$$

$$\lambda_n^{(3)} = (n+1)Q[d/Q - (n+1)].$$

We recognize that the discrete set $\lambda_n^{(2)}$ is identical to (6.30). Transforming the corresponding eigenfunctions $R_n(Y)$ back to the representation $P_n(x)$, we realize that in the limit $\mathcal{E}\to 0$ the functions $P_n^2(x)$ become identical to the solutions (6.30), while the functions $P_n^3(x)$ which are localized around $Y=0$ are squeezed to an infinitesimally narrow range around the origin $x=0$.

One effect of additive fluctuations is therefore to connect the separate stochastic motions in the two half spaces $x<0$ and $x>0$ by allowing the particles to diffuse through the origin. The eigenfunctions corresponding to $\lambda < d^2/2Q$ are made up of the symmetrized eigenfunctions of the pure multiplicative poocesses with an infinitesimally small correction due to the potential $V_3$ confined to a narrow range $|x| < \mathcal{E}$ around the origin. For $\lambda > d^2/2Q$ the particles can diffuse almost freely in the potential well $V_1(Y)$. In the limit $\mathcal{E}\to 0$ the eigenfunctions become strongly delocalized and the difference between neighboring eigenvalues approaches zero: $(\lambda_{n+1}-\lambda_n)\to 0$, forming a quasicontinuous spectrum. This behavior explains in a natural way the appearance of the continuous branch in the spectrum of pure multiplicative processes.

Let us summarize the results of this section: (i) The eigenvalue spectrum contains a discrete and a quasicontinuous branch with a level separation of the order of

$$\Delta\lambda \propto -\sqrt{\lambda}/\ln\mathcal{E}, \quad \lim_{\mathcal{E}\to 0}\Delta\lambda = 0.$$

(ii) The discrete eigenvalues are twofold degenerate owing to the inversion symmetry of the problem. For small but finite $\mathcal{E}$ the degeneracy is lifted. Besides the eigenvalue $\lambda = 0$ another small eigenvalue appears, corresponding to the diffusion between the two stationary states centered at $Y = \pm Y_0$.

## IX. CONCLUSION

The statistical description of macroscopic systems, when derived from microscopic equations, in general consists of a system of coupled Langevin-type equations with additive fluctuations. For many years this approach has been used almost exclusively for the description of fluctuation phenomena in various fields of statistical physics.

We have shown in Sec. II that multiplicative fluctuations arise in a natural way when a system of nonlinear Langevin equations is simplified by using, e.g., the adiabatic principle. While the formal aspects of linearized multiplicative processes have been discussed by several authors, generalization to the nonlinear regime has so far not attracted the attention it deserves.

It was the aim of this paper to study in detail the importance of multiplicative stochastic processes in different fields of statistical physics. Some general properties of multiplicative processes in comparison with additive ones have been reported in Sec. IV. A remarkable difference between these processes was found in the behavior of the most probable values, which no longer coincide with the deterministic stationary points in the case of multiplicative fluctuations. An interesting consequence of this result is the fact, that, e.g., threshold conditions are no longer determined by the deterministic parameters alone but depend explicitly on the strength of the fluctuations.

For a more detailed discussion we formulated a class of nonlinear stochastic equations that can serve as a basic model in statistical physics because all statistical properties characterizing these processes can be calculated exactly by analytic means. The complete description of the dynamics of these systems was possible because the underlying time-dependent Fokker-Planck equations were found to be exactly soluble. The mathematical details of the solution are summarized in Sec. VI. A remarkable property of the eigenvalue spectrum is that it consists of a discrete as well as a continuous branch. A continuous branch has also been found for a second class of multiplicative processes, which we discussed in Sec. VIII. We have been motivated to discuss these classes of multiplicative processes in detail by a number of physical examples from nonlinear optics and chemical reaction dynamics.

The experiments of S. Kabashima *et al.*, who investigated the statistics of electronic parametric oscillators driven by controllable external noise can be described in terms of multiplicative stochastic processes and allow for comparison of the present results with experiments. The experimental results concerning the stationary as well as the dynamical properties are in excellent agreement with the predictions of the model discussed here. In particular, the explicit linear dependence of the relaxation rates on the strength of the fluctuations as reported by S. Kabashima *et al.* finds its natural explanation in the eigenvalue spectrum derived in Sec. VI. We think that the field of multiplicative stochastic processes has exciting new characteristics that have not yet been studied in all their consequences; much work remains to be done both experimentally and theoretically. We hope that the presented analytical results will stimulate further activities in the area of multiplicative stochastic processes, where interesting new features can be expected which are unknown to the well-examined field of additive stochastic processes.

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