Photon statistics of partially polarized Gaussian light

Tadashi Aoki and Katsumi Sakurai
Department of Pure and Applied Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153 Japan
(Received 24 April 1979)

Statistical properties of partially polarized Gaussian light were studied experimentally for degrees of polarization P from 0.03 to 1 by a standard technique of photon counting. The Gaussian light with an arbitrary degree of polarization was obtained from Rayleigh scattering of orthogonally polarized laser beams. The factorial moments up to sixth order of the photoelectron distribution were measured with an arbitrary temporal and spatial coherence. After corrections were taken for the experimental conditions, the theoretical values of these factorial moments were compared with the experimental results. It is shown that the theoretical predictions were quantitatively well verified by the present measurements. The microscopic process is discussed from the standpoint of the light scattering system used in the present measurements, and it is shown that the orthogonally polarized components of the scattered light observed are statistically independent.

I. INTRODUCTION

The statistical and temporal properties of photoelectrons reflect the fluctuation properties of light illuminating a photodetector. On the basis of this correspondence various experimental and theoretical works have been carried out on the statistical properties of Gaussian light.[1] Most of these works have until now been concerned with linearly polarized light. For a partially polarized beam of thermal light Mandel has derived the ensemble distribution for the number of photons[2] and Jaiswal and Mehta have obtained an expression for the cumulants of the integrated intensity.[3] Martienssen and Spiller studied experimentally the probability distributions of photoelectron pulses for depolarized light and showed a substantial departure from those of polarized light.[4]

We had studied experimentally the statistical properties of Gaussian light with a Lorentzian spectral shape[5,6] and those of a mixture of Gaussian and coherent light[7] for the case of linear polarization. Here we report the measurement of the statistical properties of Gaussian light with an arbitrary degree of polarization and present a quantitative comparison of the experimental results with the theoretical predictions. The scattering process is discussed and it is shown that the orthogonal components of scattered light can be regarded as statistically independent in our experimental scheme.

II. EXPERIMENT

Figure 1 shows the experimental setup. A vertically polarized light beam from a 5145-Å argon ion laser was focused into a sample cell which contained latex polystyrene spheres of diameter 4810 Å with a standard deviation of 18 Å. The light beam which exitted the cell in the forward direction was reflected back into the cell along the same path. A $\frac{1}{4}\lambda$ Babinet compensator was placed between the cell and a reflecting mirror and adjusted so that the reflected beam reentering the cell was horizontally polarized. The scatterer was therefore illuminated by the two incident beams the polarizations of which were mutually orthogonal and propagation vectors of which were opposite. Since the scattered light was observed in a horizontal plane, the scattering efficiency was much smaller for horizontally polarized light than for vertically polarized light. The scattering angle θ was therefore chosen such that the same intensities were obtained for both scattered components. The light beam scattered from the cell entered on the photocathode of a Hamamatsu TV R300 photomultiplier through two irises the diameters of which were 60 and 120 μm. The second iris was 30 cm from the first. Photoelectron pulses from the photomultiplier were standardized to a Transistor-Transistor Logic level after being discriminated by a EG& G T-105/N dual discriminator. The count-processing system used is the same as that employed in our previous work.[5-7]

With average counting rates $\langle n_\perp\rangle$ and $\langle n_\parallel\rangle$ of photoelectrons contributed from the vertical and the horizontal polarization components of the scattered light integrated during a count interval T, the degree of polarization P is defined by[8]

$$P=|\langle n_\perp\rangle-\langle n_\parallel\rangle/\langle n_\perp\rangle+\langle n_\parallel\rangle|\,. \tag{1}$$

When we chose the scattering angle θ to be 63°, the average count rates $\langle n_\perp\rangle$ and $\langle n_\parallel\rangle$ were 7.8×10^4 pulses/sec. For these count rates the average contribution from each component was 3.3 pulses during a sampling interval $T=4.2\times10^{-5}$ sec. The degree of polarization of the light was changed from 0.03 to 1 by reducing the intensity of the horizontal component of the incident light beams, while the

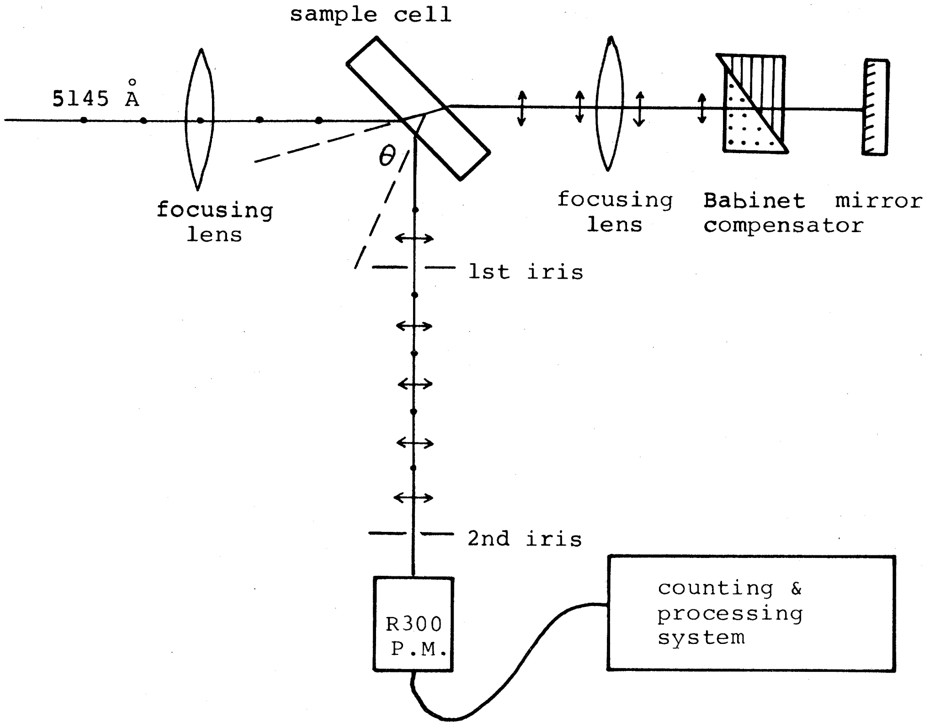

FIG. 1. Experimental setup.

vertical component was unchanged. From the values of θ and the diameter of the spheres the correlation times of fluctuations of the scattered light were evaluated to be 1.00×10^{-3} and 3.48×10^{-3} sec for the vertical and horizontal components, respectively. The number of counting samples was 2×10^6 for each fixed value of P. From the distribution of photoelectrons counted normalized factorial moments were calculated for each value of P and plotted in Fig. 2. The experimental dispersion of the plot is shown by a bar when greater than the size of dots. Typical plots of the photoelectron distribution are shown for the cases of $P=0.03$ and $P=1$ in Fig. 3.

III. DISCUSSION

We define the kth factorial moment $F(k)$ of the distribution of photoelectrons by

$$F(k)=\frac{\langle n(n-1)(n-2)\cdots(n-k+1)\rangle}{\langle n\rangle^k}-1\,. \tag{2}$$

$F(k)$ is equal to the normalized moment M_k of the distribution of the light intensity W integrated during a count interval T,

$$F(k)=M_k\equiv\langle W^k\rangle/\langle W\rangle^k-1\,. \tag{3}$$

The normalized moments can be written up to sixth order from the kth and lower-order normalized cumulants C_k's of W as[9]

$$M_2=C_2\,, \tag{4}$$

$$M_3=C_3+3C_2\,, \tag{5}$$

$$M_4=C_4+4C_3+6C_2+3C_2^2\,, \tag{6}$$

$$M_5=C_5+5C_4+10C_3+10C_2+10C_3C_2+15C_2^2\,, \tag{7}$$

$$M_6=C_6+6C_5+15C_4+20C_3+15C_2+15C_4C_2+10C_3^2+60C_3C_2+15C_2^3+45C_2^2\,. \tag{8}$$

The integrated total intensity W is given by the sum of $W_\perp$ and $W_\parallel$, which are the contributions from the vertically and horizontally polarized components of scattered light. It is shown later that these orthogonally polarized components are statistically independent of each other. In this case the normalized cumulants of the light intensity are given by the weighted sum of the normalized cumulants of each polarization component,[3]

$$C_k=[\tfrac{1}{2}(1+P)]^k C_{k,\perp}+\tfrac{1}{2}[(1-P)]^k C_{k,\parallel}\,. \tag{9}$$

Each cumulant $C_{k,\perp}$ and $C_{k,\parallel}$ can be calculated from the values of the correlation times $\tau_\perp$ and $\tau_\parallel$ and a count interval T,[5]

$$C_2=\frac{2}{a^2}\,[a+(e^{-a}+1)]\,, \tag{10}$$

$$C_3=\frac{12}{a^2}\left(e^{-a}+1+\frac{2}{a}(e^{-a}-1)\right)\,, \tag{11}$$

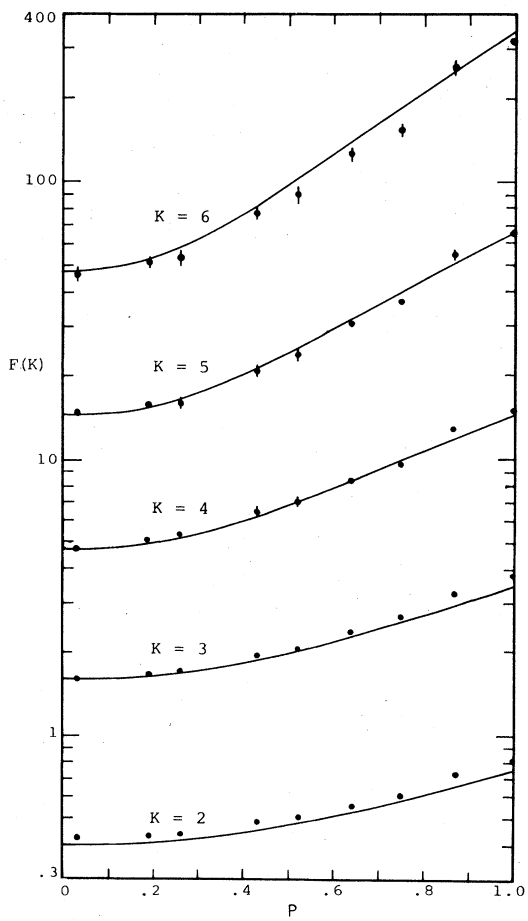

FIG. 2. Normalized factorial moment vs degree of polarization. Count interval is 4.2×10^{-5} sec. Each point consists of 2×10^{6} samples. Correlation times are 1.00×10^{-3} and 3.48×10^{-3} sec for the vertically and horizontally polarized components, respectively. Solid lines show the theoretical values.

$$C_4=\frac{12}{a^2}\left(4e^{-a}+\frac{10}{a}(2e^{-a}+1)+\frac{1}{a^2}(e^{-2a}+28e^{-a}-29)\right), \tag{12}$$

$$C_5=\frac{20}{a^2}\left(8e^{-a}+\frac{72}{a}e^{-a}+\frac{12}{a^2}(e^{-2a}+20e^{-a}+7)+\frac{24}{a^3}(e^{-2a}+12e^{-a}-13)\right), \tag{13}$$

$$C_6=\frac{60}{a^2}\left(8e^{-a}+\frac{112}{a^2}e^{-a}+\frac{24}{a^2}(2e^{-2a}+27e^{-a})+\frac{72}{a^3}(3e^{-2a}+25e^{-a}+7)+\frac{4}{a^4}(e^{-3a}+66e^{-2a}+495e^{-a}-562)\right), \tag{14}$$

with $a=T/\tau$. In the above equations C_k stands for

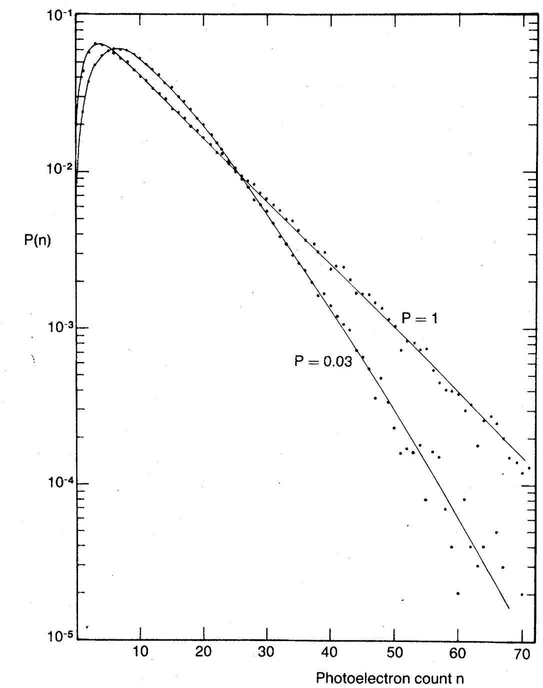

FIG. 3. Typical probability distributions of photoelectrons for linearly polarized light ($P=1$) and depolarized light ($P\cong0$). Solid lines are interpolations of the experimental points. Count duration: 5×10^{-5} sec. Average photoelectron count: 12 counts. Sample number: 10^5.

$C_{k,\perp}$ and $C_{k,\parallel}$, and τ stands for $\tau_\perp$ and $\tau_\parallel$. In our cases, $T=4.2\times10^{-5}$ sec and $\tau_\perp$ and $\tau_\parallel$ are 1.00×10^{-3} and 3.48×10^{-3} sec for the vertical and horizontal components, respectively. The values of C_k are evaluated from these parameters and given in Table I.

Before comparing the theoretical values of the normalized factorial moments with the experimental results a few corrections due to the experimental situations must be considered. First, the laser light used cannot be completely coherent, i.e., the argon ion laser used has a small but wideband residual fluctuation in its output. A correction factor f is introduced to correct the cumulants for this effect.[10] We also make another correction for the influence of spatial coherence. From the work of Cantrell[11] a correction factor S_k of the kth cumulant was calculated for spatial coherence.

TABLE I. Cumulants of the vertically and horizontally polarized components calculated from Eqs. (10)–(14), with $\tau_\perp=1.00\times10^{-3}$ sec and $\tau_\parallel=3.48\times10^{-3}$ sec.

C_k \ k	2	3	4	5	6
$C_{k,\perp}$	0.75	0.64	0.55	0.48	0.41
$C_{k,\parallel}$	0.86	0.79	0.72	0.68	0.62

After these corrections are considered, the kth cumulant C_k is expressed as

$$C_k = S_k f^k \{[\tfrac{1}{2}(1+P)]^k C_{k,\perp} + [\tfrac{1}{2}(1+P)]^k C_{k,\parallel}\} . \qquad (15)$$

The value of the factor f is determined to be 0.96 from an extrapolated value of the autocorrelation curve of photoelectrons at zero time delay after eliminating the effect of spatial coherence.[5] The values of S_k are evaluated to be $S_2 = 0.99$, $S_3 = 0.97$, $S_4 = 0.96$, $S_5 = 0.95$, and $S_6 = 0.94$ for our experimental setup. Theoretical values of the normalized factorial moments are therefore evaluated from Eqs. (3)–(8) and (15) with the values in Table I and plotted in Fig. 2. No systematic disagreement can be seen between the experimental results and the theoretical values.

We consider the microscopic process of the scattering of two orthogonally polarized light beams by Brownian particles and show that the orthogonally polarized components of scattered light are not correlated under the present experimental scheme. The analytic signals[8] $\vec{V}(t)$ and $\vec{V}^0_\alpha(t)$ [α $(=\perp, \parallel)$ denotes a polarization state] represent the light field scattered and the two light fields incident on a scattering volume, respectively. When the light beams incident on the sample cell are scattered by N Brownian particles the scattered field $\vec{V}(t)$ on the photodetector is given by the sum of the fields with relative phase differences ϕ which are scattered from individual particles,

$$\vec{V}(t) = \sum_\alpha \sum_k \eta_\alpha \vec{V}^0_\alpha(t) \exp[i\phi_{\alpha k}(t) + \varphi_\alpha] , \qquad (16)$$

$$\phi_{\alpha k}(t) = (\vec{K}_\alpha - \vec{K}) \cdot \vec{r}_k(t) ,$$

where k denotes the kth particle and η represents a scattering efficiency and is independent of the particle and its position. $\vec{K}$ and $\vec{K}_\alpha$ are the wave vectors of the scattered and the incident light beams and $\vec{r}_k(t)$ is the relative position vector of the kth particle. A phase constant φ_α denotes an arbitrary phase difference between the incident fields and is assumed to be stable. Since the polarizations of $\vec{V}^0_\alpha(t)$ and $\vec{V}^0_\beta(t)$ are orthogonal for $\alpha \neq \beta$, the total intensity $I(t)$ of scattered light is given by the sum of the intensity contributed from each polarization component,

$$I(t) = \sum_\alpha \eta^2_\alpha \vec{V}^{0*}_\alpha(t) \cdot \vec{V}^0_\alpha(t) \sum_k e^{-i\phi_{\alpha k}(t)} \times \sum_l e^{-i\phi_{\alpha l}(t)} . \qquad (17)$$

Then the temporal correlation of light intensity is given by

$$\langle I(t)I(t+\tau)\rangle = \sum_\alpha \sum_\beta \eta^2_\alpha \eta^2_\beta \langle I^0_\alpha(t) I^0_\alpha(t+\tau)\rangle_f \times \sum_k \sum_l \sum_m \sum_n \langle \exp\{-i[\phi_{\alpha k}(t) - \phi_{\alpha l}(t) + \phi_{\beta m}(t+\tau) - \phi_{\beta n}(t+\tau)]\}\rangle_p , \qquad (18)$$

$$I^0_\alpha(t) \equiv \vec{V}^{0*}_\alpha(t) \cdot \vec{V}^0_\alpha(t) ,$$

where $\langle\ \rangle_f$ and $\langle\ \rangle_p$ represent ensemble averages over the systems of the incident light fields and Brownian particles. The ensemble average over the system of particles has nonzero values only for the following combinations of indices: $k=l$ and $m=n$, or $k=n$ and $l=m$. Equation (18) is therefore reduced to

$$\langle I(t)I(t+\tau)\rangle = \sum_\alpha \sum_\beta \langle I_\alpha(t) I_\beta(t+\tau)\rangle_f \eta^2_\alpha \eta^2_\beta (N^2 + (N^2 - N)|\langle \exp\{-i[\phi_\alpha(t) - \phi_\beta(t+\tau)]\}\rangle_p|^2) , \qquad (19)$$

where $\phi_\alpha(t)$ and $\phi_\beta(t+\tau)$ are concerned with an arbitrary particle with indices dropped in the ensemble average over the system of particles. The second term in the above equation represents the correlation between the scattering processes of the orthogonal components when $\alpha \neq \beta$. By using the wave vectors of the incident and the scattered light fields this term can be written

$$\langle \exp\{-i[\phi_\alpha(t) - \phi_\beta(t+\tau)]\}\rangle_p = \langle \exp\{i[(\vec{K} - \vec{K}_\alpha) \cdot \Delta\vec{r}(\tau)]\}\rangle_p \langle \exp\{-i[(\vec{K}_\alpha - \vec{K}_\beta) \cdot \vec{r}(t)]\}\rangle_p , \qquad (20)$$

$$\Delta\vec{r}(\tau) \equiv \vec{r}(t) - \vec{r}(t+\tau) .$$

For $\alpha \neq \beta$ this term becomes neglegibly small when $|\vec{K}_\alpha - \vec{K}_\beta|^{-1}$ is much smaller than the dimension of the scattering volume. This is the present case since $|\vec{K}_\perp - \vec{K}_\parallel| = 2|\vec{K}_0|$ ($\vec{K}_0$ is the wave vector of the incident light beam) in our experimental scheme. Therefore the cross correlation between the scattered orthogonal components disappears except for that originating from fluctuations in the incident light beams. When the incident light beams are stable, as is the case here, Eq. (19) is rewritten as

$$\langle I(t)I(t+\tau)\rangle = (\eta_\perp \langle I^0_\perp\rangle + \eta_\parallel \langle I^0_\parallel\rangle)^2 N^2 + \eta^2_\perp N^2 \langle I^0_\perp\rangle^2 |\exp\{-i[\phi_\perp(t) - \phi_\perp(t+\tau)]\}|^2 + \eta^2_\parallel N^2 \langle I^0_\parallel\rangle^2 |\exp\{-i[\phi_\parallel(t) - \phi_\parallel(t+\tau)]\}|^2 , \qquad (21)$$

TABLE II. Normalized factorial moments measured for various states of the polarizations and wave vectors of the incident light fields.

$F(k)$ \ k	2	3	4	5	6
$\vec{K}_\perp=\vec{K}_\parallel$, $P\cong 0$	0.890 72	4.115 74	16.5543	70.9073	340.471
$\vec{K}_\perp=-\vec{K}_\parallel$, $P\cong 0$	0.424 43	1.633 56	4.8961	14.5989	46.484
$\vec{K}_\perp$, $P=1$	0.890 97	4.170 35	17.0886	74.3452	354.931

where N is assumed to be much greater than unity. When $\tau=0$ we obtain the second-order moment of the total intensity

$$\langle I^2\rangle=2\langle I_\perp\rangle^2+2\langle I_\parallel\rangle^2+2\langle I_\perp\rangle\langle I_\parallel\rangle \tag{22}$$

from Eq. (21). Here $\langle I_\perp\rangle$ $(=\eta_\perp N\langle I_\perp^0\rangle)$ is the average intensity of the vertical component of the scattered light. On the other hand, the following equation generally holds for the intensity summed over two Gaussian components:

$$\langle I^2\rangle=2\langle I_\perp\rangle^2+2\langle I_\parallel\rangle^2+2\langle I_\perp I_\parallel\rangle\,. \tag{23}$$

We obtain

$$\langle I_\perp I_\parallel\rangle=\langle I_\perp\rangle\langle I_\parallel\rangle \tag{24}$$

from Eqs. (22) and (23). The orthogonal components scattered are therefore not correlated and are thus statistically independent under the experimental scheme of the present measurement.

In this case the normalized second-order cumulant is given from Eq. (22) by

$$\begin{aligned}C_2&\equiv\frac{\langle I^2\rangle}{\langle I\rangle^2}-1=\frac{\langle I_\perp\rangle^2+\langle I_\parallel\rangle^2}{(\langle I_\perp\rangle+\langle I_\parallel\rangle)^2}\\&=\tfrac{1}{2}(1+P)^2+\tfrac{1}{2}(1-P)^2\,,\end{aligned} \tag{25}$$

$$P=\left|\frac{\langle I_\perp\rangle-\langle I_\parallel\rangle}{\langle I_\perp\rangle+\langle I_\parallel\rangle}\right|\,.$$

This expression is consistent with the general form for the normalized cumulant given in Eq. (9).[12]

When, on the contrary, two incident light beams have the same wave vectors, i.e., $\vec{K}_\perp=\vec{K}_\parallel$, Eqs. (21) and (22) are written

$$\begin{aligned}&\langle I(t)I(t+\tau)\rangle\\&\quad=\langle I\rangle^2(1+|\langle\exp\{-i[\phi(t)-\phi(t+\tau)]\}\rangle|^2)\,,\end{aligned} \tag{26}$$

$$\langle I^2\rangle=2\langle I\rangle^2\quad(\tau=0)\,. \tag{27}$$

Then one obtains the second-order cumulant as

$$C_2=1\,. \tag{28}$$

These results are independent of the degree of polarization P and the same as those for the case of linearly polarized light ($P=1$). When light beams of orthogonally polarized fields propagate with the same wave vectors, each scattered component can therefore no longer be statistically independent. This was easily verified by a measurement with a slight variation of the original experimental setup. The Babinet compensator and the reflecting mirror were removed from the setup shown in Fig. 1 and the former was placed in front of the sample cell. Adjusting the Babinet compensator as a $\frac{1}{4}\lambda$ phase shifter again we obtained a circularly polarized incident light beam which consisted of two orthogonally polarized waves with a mutual phase difference $|\varphi_\perp-\varphi_\parallel|$ of $\frac{1}{2}\pi$. In order to obtain a randomly polarized ($P\cong 0$) scattered wave we observed a forward scattering ($\theta\cong 180°$ in Fig. 1). The factorial moments measured with this experimental scheme are given in Table II. For comparison, those measured in the original experimental setup are also given in the table for the cases of $P\cong 0$ and $P=1$. As can be seen from the table, the magnitude of a fluctuation for the case of depolarized light ($P\cong 0$) recovers the values of linearly polarized Gaussian light ($P=1$) when $\vec{K}_\perp=\vec{K}_\parallel$.

When $|\vec{K}_\perp-\vec{K}_\parallel|^{-1}$ has the same order of the magnitude as the dimension of a scattering volume the second factor in Eq. (20) does not vanish and the fluctuation properties of scattered light are then characterized by the difference between the wave vectors of orthogonally polarized light beams incident on the Brownian particles.

ACKNOWLEDGMENTS

The authors wish to express their thanks to Professor M. Katayama for making a Babinet compensator available to them. This research was partially supported by a grant in aid from the Ministry of Education.

[1]G. Bedard, Phys. Rev. 151, 1038 (1966); F. T. Arecchi, E. G. Gatti, and A. Sona, Phys. Lett. 20, 27 (1966); E. Jakeman and E. R. Pike, J. Phys. A 1, 128 (1968); C. D. Cantrell, Phys. Rev. A 1, 627 (1970); B. Crosignani, B. Daino, and P. Di Porto, J. Appl. Phys. 42, 399 (1971); E. Jakeman, J. Phys. A 8, 392 (1975).
[2]L. Mandel, Proc. Phys. Soc. London 81, 1104 (1963).
[3]A. K. Jaiswal and C. L. Mehta, Phys. Rev. 186, 1355 (1969).
[4]W. Martienssen and E. Spiller, Phys. Rev. Lett. 16, 531 (1966).
[5]T. Aoki, Y. Okabe, and K. Sakurai, Phys. Rev. A 10, 259 (1974).
[6]Y. Okabe, T. Aoki, and K. Sakurai, J. Phys. Soc. Jpn. 40, 798 (1976).
[7]T. Aoki, Y. Endo, H. Takayanagi, and K. Sakurai, Phys. Rev. A 13, 853 (1976).
[8]M. Born and E. Wolf, *Principles of Optics*, 4th ed. (Pergamon, London, 1970), Chap. X.
[9]M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics* (Hafner, New York, 1963), Vol. 1.
[10]A somewhat more detailed discussion is given in Rev. 5.
[11]C. D. Cantrell and J. R. Fields, Princeton University Department of Physics Technical Report No. 13, 1972 (unpublished).
[12]Here the count duration T is assumed to be zero. $C_{k,\perp}$ and $C_{k,\parallel}$ in Eq. (9) are therefore unity.