Relativistic Glauber amplitudes for elastic electron and positron scattering by hydrogen atoms and hydrogenlike ions in the ground state

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A relativistic amplitude for elastic scattering of electrons and positrons by hydrogen atoms or hydrogenlike ions in the ground state is derived in the straight-line Glauber approximation without exchange terms. A small-angle approximation valid to first order in the fine-structure constant is used for the relativistic corrections to the input amplitudes describing the scattering of the projectile by the proton and the electron of the target, and spin-flip terms are included. The results are given in closed form in terms of hypergeometric functions.

I. INTRODUCTION

In recent years, a number of papers on the application of the Glauber approximation to electron-atom collisions have appeared in the literature. The reader is referred to the review articles by Gerjuoy and Thomas[1] and Byron and Joachain[2] for references to and a discussion of such calculations through 1976. Since that time, a major interest in this field has been in the treatment of the exchange effects, which become increasingly important as the energy decreases and must be taken quite seriously below, say, 100 eV. The various methods by which the exchange effects have been treated in the literature are discussed, e.g., in Ref. 3.

The present paper is concerned with scattering at higher energies, where the relativistic corrections to the Coulomb-scattering amplitude may become appreciable. To study the influence of these effects on the atomic scattering process we use a formulation of the Glauber theory where the amplitude for the scattering of the projectile on the individual scatterers in the target is used as input, rather than the potential between the projectile and the scatterers. In doing this, we use an approximate form of the relativistic Coulomb amplitude, corresponding to small-angle scattering in the second Born approximation, including spin-flip terms. We limit ourselves to elastic scattering in a one-electron system in the ground state, considering then the hydrogenlike ions in addition to the hydrogen atom. It has previously been shown[4] that the nonrelativistic Glauber amplitudes for the hydrogen atom can be expressed very simply in terms of hypergeometric functions. We find that for elastic scattering this is the case also for other one-electron systems and that similar hypergeometric functions appear in the relativistic case as well.

Thomas and Franco[5] have already calculated the Glauber amplitudes for nonrelativistic inelastic scattering in hydrogenlike ions, expressing the results in terms of Meijer's G functions. Their calculations are, however, done in such a way that the resulting amplitudes are not applicable to the elastic case.

It should be pointed out that relativistic effects within the target have not been included in the present calculation, i.e., the electron wave function is used in its nonrelativistic form. Further, the version of the Glauber approximation which is used here is the one which in this area of physics is generally called the "restricted Glauber approximation." This means that the projectile trajectory (the z axis) is taken to be a straight line perpendicular to the momentum-transfer vector $\vec{q}=\vec{k}_i-\vec{k}_f$ ($k_i=k_f=k$); as is then usually done, we also use the exact value $q=2k\sin(\frac{1}{2}\theta)$ for the magnitude of the momentum transfer in the resulting formulas (the "wide-angle approximation").

II. SPIN-DEPENDENT GLAUBER PROFILE FUNCTION

We consider the scattering of a projectile with charge Z_1e ($Z_1=+1$ for positrons, $e>0$) in the Coulomb field from a point charge Z_2e. The scattering operator, acting between the Pauli spinors describing the initial and final spin states of the projectile, can be written

$$\mathfrak{F}_C(\eta,\vec{q})=f(\eta,q)+ig(\eta,q)\vec{\sigma}\cdot\vec{n}\,, \tag{1}$$

where $\vec{\sigma}$ is the Pauli spin operator, $\vec{n}=(\vec{k}_f\times\vec{k}_i)/|\vec{k}_f\times\vec{k}_i|$ is the normal to the scattering plane, and $\eta=Z_1Z_2\alpha/\beta$, with $\alpha=\frac{1}{137}$, $\beta=(\epsilon^2-1)^{1/2}/\epsilon$, $\epsilon=E/m=(m^2+k^2)^{1/2}/m$, $|\vec{k}_i|=|\vec{k}_f|=k$, and m being the electron mass. We use units for which $\hbar=c=1$. By solving the Dirac equation for a Coulomb field[6] and retaining terms to the order α relative to the Rutherford amplitude we find

$$
\begin{aligned}
f(\eta,q) &= c(\eta,x)[1-\tfrac{1}{4}\epsilon^{-1}(\epsilon-1)x^2-\tfrac{1}{4}\beta^2\eta x d(x)],\\
g(\eta,q) &= \tfrac{1}{2}c(\eta,x)x[\epsilon^{-1}(\epsilon-1)(1-\tfrac{1}{4}x^2)^{1/2}\\
&\qquad -\tfrac{1}{4}\beta^2\eta x(1-\tfrac{1}{4}x^2)^{-1/2}d(x)],\\
c(\eta,x) &= -2k^{-1}\eta\exp[2i\sigma(\eta)](\tfrac{1}{2}x)^{-2i\eta x^{-2}},\\
d(x) &= \pi(1-\tfrac{1}{2}x)+ix\ln(\tfrac{1}{2}x),\\
x &= q/k = 2\sin(\tfrac{1}{2}\theta),\\
\exp[2i\sigma(\eta)] &= [\Gamma(1+i\eta)]/[\Gamma(1-i\eta)].
\end{aligned}
\tag{2}
$$

To terms of the first order in α and all orders in x the cross section, averaged over initial and summed over final polarizations, corresponding to this amplitude is

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= |f|^2+|g|^2\\
&= \tfrac{1}{4}\eta^2k^{-2}\sin^{-4}(\tfrac{1}{2}\theta)\{1-\beta^2\sin^2(\tfrac{1}{2}\theta)-\pi\beta^2\eta\sin(\tfrac{1}{2}\theta)\\
&\qquad\times[1-\sin(\tfrac{1}{2}\theta)]\},
\end{aligned}
\tag{3}
$$

which is the McKinley-Feshbach[7,8] cross section obtained in the second Born approximation.

Keeping now terms to the order x^2 in the amplitudes (2), which corresponds to a small-angle expansion in the spirit of the Glauber approximation, we write

$$
\begin{aligned}
f(\eta,q) &= c(\eta,x)[f^{00}(\eta,x)+f^{02}(\eta,x)\\
&\qquad +f^{11}(\eta,x)+f^{12}(\eta,x)],\\
g(\eta,q) &= c(\eta,x)[g^{01}(\eta,x)+g^{12}(\eta,x)],\\
f^{00} &= 1,\quad f^{02}=-\tfrac{1}{4}\epsilon^{-1}(\epsilon-1)x^2,\\
f^{11} &= -\tfrac{1}{4}\pi\beta^2\eta x,\quad f^{12}=\tfrac{1}{8}\pi\beta^2\eta x^2,\\
g^{01} &= \tfrac{1}{2}\epsilon^{-1}(\epsilon-1)x,\quad g^{12}=-\tfrac{1}{8}\pi\beta^2\eta x^2,
\end{aligned}
\tag{4}
$$

where the indices indicate the power of α and power of x, respectively. This then reproduces the cross section (3) to terms of first order in α and second order in x.

As is well known,[9] there are difficulties connected with the application of the Glauber approximation to the pure Coulomb potential, due to the infinite range of the potential. We consider instead the more realistic case of a screened Coulomb potential (using, e.g., a sharp cutoff or a Yukawa-type screening factor), which in the limit of an infinite screening radius is described[9] by the scattering amplitude

$$
\mathfrak{F}_{sc}(\vec{q}) = i\delta(q^2/2k)+\exp(i\Lambda)\mathfrak{F}_C(\vec{q}),
\tag{5}
$$

where Λ is an infinite phase constant. We write the corresponding profile function

$$
P(\eta,\vec{b}) = (2\pi ik)^{-1}\int d^2q\,\exp(-i\vec{q}\cdot\vec{b})\mathfrak{F}_{sc}(\vec{q})
\tag{6}
$$

as

$$
\begin{aligned}
P(\eta,\vec{b}) &= 1-\exp[i\chi(\eta,\vec{b})],\\
\exp[i\chi(\eta,\vec{b})] &= \exp(i\Lambda)S(\eta,\vec{b}),
\end{aligned}
\tag{7}
$$

where the phase-shift function (in the usual terminology actually the exponential of the phase-shift function)

$$
S(\eta,\vec{b}) = S_0(\eta,b)+(\vec{\sigma}\times\vec{e}_z)\cdot\hat{b}S_1(\eta,b)
\tag{8}
$$

for the Coulomb potential is an operator in spin space; we have here

$$
\begin{aligned}
S_0(\eta,b) &= \frac{i}{k}\int_0^\infty q\,dq\,f(\eta,q)J_0(bq),\\
S_1(\eta,b) &= \frac{i}{k}\int_0^\infty q\,dq\,g(\eta,q)J_1(bq).
\end{aligned}
\tag{9}
$$

The impact parameter $\vec{b}$ is by definition a vector perpendicular to the z axis, which by choice is in the direction $\frac{1}{2}(\vec{k}_i+\vec{k}_f)$; the fact that $\vec{q}\perp\vec{e}_z$ is then a consequence of our choice of z axis, and not an additional assumption.

The phase-shift functions S_0 and S_1 for scattering without and with spin flip can now be calculated by using the amplitudes [Eq. (4)]. Admittedly, one encounters certain mathematical problems in doing this, as well as in the inverse Bessel transformations from S_0 and S_1 back to f and g. These difficulties are again connected with the infinite range of the Coulomb potential and could be remedied by retaining a finite screening radius until a later stage in the calculation than we have done. In practice, the same purpose is served by the introduction of an artificial damping at small q in the f^{00} term of Eq. (4), e.g., replacing x^{-2} by $x^{-2+\delta}$ in $c(\eta,x)$ and afterwards letting $\delta\to 0$, and similarly employing a convergence factor $\exp(-\nu x^2)$, where $\nu\to 0$, in the terms that otherwise make the integrals (9) undefined due to the large-q behavior. What this amounts to is simply that the phase-shift functions [Eq. (9)] can be evaluated from the amplitudes [Eq. (4)] and, inversely, the amplitudes can be evaluated from the phase-shift functions by taking[10]

$$
\begin{aligned}
&\int_0^\infty t^{m\pm 2i\eta}\,dt\,J_n(yt)\\
&\quad = 2^{m\pm 2i\eta}\left(\Gamma[\tfrac{1}{2}(n+m+1)\pm i\eta]/\Gamma[\tfrac{1}{2}(n-m+1)\mp i\eta]\right)\\
&\qquad\times y^{-m-1\mp 2i\eta}
\end{aligned}
\tag{10}
$$

for all m and n of interest, the actual requirements for the validity of this expression being $-n-1<m<0$. With upper indices corresponding to those of Eq. (4) the contributions to the phase-shift functions S_j are then

$$
\begin{aligned}
S_0^{00}(\eta, b) &= (kb)^{2i\eta}, \\
S_0^{02}(\eta, b) &= -\eta^2\epsilon^{-1}(\epsilon-1)(kb)^{2i\eta-2}, \\
S_0^{11}(\eta, b) &= \tfrac{1}{2}i\pi\beta^2\eta^2 \exp\{2i[\sigma(\eta)-\sigma'(\eta)]\} \\
&\quad\times(kb)^{2i\eta-1}, \\
S_0^{12}(\eta, b) &= \tfrac{1}{2}\pi\beta^2\eta^3(kb)^{2i\eta-2}, \\
S_1^{01}(\eta, b) &= -i\eta\epsilon^{-1}(\epsilon-1)(kb)^{2i\eta-1}, \\
S_1^{12}(\eta, b) &= \tfrac{1}{4}i\pi\beta^2\eta^2(1-2i\eta) \\
&\quad\times\exp\{2i[\sigma(\eta)-\sigma'(\eta)]\}(kb)^{2i\eta-2}, \\
\exp[2i\sigma'(\eta)] &= [\Gamma(\tfrac{1}{2}+i\eta)/\Gamma(\tfrac{1}{2}-i\eta)].
\end{aligned} \tag{11}
$$

III. SCATTERING AMPLITUDE FOR THE COMPOSITE SYSTEM

Let now the charge Z_2 appearing in Sec. II be that of the nucleus, so that the phase-shift function defined there corresponds to projectile-nucleus scattering. The relevant charge-velocity parameter for the scattering of the projectile on the target electron is $-\eta_1$, where $\eta_1 = Z_1\alpha/\beta$, and the corresponding phase-shift function should be evaluated at $\vec{b}' = \vec{b} - \vec{s}$, in which $\vec{s}$ is the component $\perp\vec{e}_z$ of the electron coordinate $\vec{r} = (\vec{s}, z)$. Making the Glauber assumption about the additivity of the phases, we then write the scattering operator for the composite system, screened at infinity, as

$$
\hat{\mathfrak{F}}_{sc}(\vec{q}) = (2\pi)^{-1}ik\int d^2b \exp(i\vec{q}\cdot\vec{b})\, d^3r\, \psi_f^\dagger(\vec{r})(1-\exp\{i[\chi(\eta,\vec{b})+\chi(-\eta_1,\vec{b}')]\})\psi_i(\vec{r}), \tag{12}
$$

where ψ_i and ψ_f are the initial- and final-state electron wave functions. It is to be understood that the product involving the phase-shift functions should be symmetrized in $(\eta, \vec{b})$ and $(-\eta_1, \vec{b}')$. Denoting the phase factor connected with the scattering against the electron by Λ_1 we can then write Eq. (12) as

$$
\hat{\mathfrak{F}}_{sc}(\vec{q}) = i\delta_{fi}\delta(q^2/2k) + \exp[i(\Lambda+\Lambda_1)]\mathfrak{F}(\vec{q}), \tag{13}
$$

where the intensity away from the forward direction is determined by the operator

$$
\mathfrak{F}(\vec{q}) = -(2\pi)^{-1}ik\int d^2b\exp(i\vec{q}\cdot\vec{b})\, d^3r\, \psi_f^\dagger(\vec{r})\tfrac{1}{2}[S(\eta,\vec{b})S(-\eta_1,\vec{b}') + S(-\eta_1,\vec{b}')S(\eta,\vec{b})]\psi_i(\vec{r}). \tag{14}
$$

We consider now elastic scattering in the $1s$ state, using a nonrelativistic wave function for the electron. In doing the integrations in Eq. (14) we have been inspired by the methods developed by Thomas and Gerjuoy[4] for the lowest-order non-spin-flip term in the case $\eta_1 = \eta$. It is convenient to write the result of the integration over the electron z coordinate as

$$
\begin{aligned}
&\int_{-\infty}^{\infty} dz\, |\psi(\vec{r})|^2 = -(4\pi)^{-1}\lambda^3\frac{\partial}{\partial\lambda}K_0(\lambda s), \\
&\lambda = 2/a, \quad a = a_0/Z_2,
\end{aligned} \tag{15}
$$

where a_0 and a are the Bohr radii for the hydrogen atom and the hydrogenlike ion in question, and K_0 as usual denotes the zeroth-order modified Bessel function of the second kind. After integration over the directional angle for the impact parameter $\vec{b}$ we are left with a three-dimensional integral for $\mathfrak{F}$, the variables being b, s and the angle ϕ between $\vec{s}$ and $\vec{b}$. The scattering amplitude for the composite system takes the same form as that of Eq. (1), $\vec{\sigma}$ being as before the spin operator for the projectile (we have not considered the target spin). We write

$$
\begin{aligned}
&\mathfrak{F}(\vec{q}) = \bar{F}(q) + i\bar{G}(q)\vec{\sigma}\cdot\vec{n}, \\
&\frac{d\sigma}{d\Omega} = |\bar{F}|^2 + |\bar{G}|^2,
\end{aligned} \tag{16}
$$

where $d\sigma/d\Omega$ is the cross section summed and averaged over spin directions. Polarization effects can of course also be considered when $\bar{F}$ and $\bar{G}$ are given separately. The amplitude $\bar{F}$ contains a contribution from the nonflip amplitude for scattering against the nucleus combined with the corresponding amplitude for scattering against the electron $[f(\eta)\times f(-\eta_1)]$, plus a double-spin-flip contribution $[g(\eta)\times g(-\eta_1)]$. Similarly, $\bar{G}$ is due to the spin-flip amplitude against the nucleus combined with the nonflip amplitude against the electron $[g(\eta)\times f(-\eta_1)]$, and the other way around $[f(\eta)\times g(-\eta_1)]$. Explicitly,

$$
\begin{aligned}
&\bar{F}(q) = \bar{F}_0(q) + \bar{F}_1(q), \quad \bar{G}(q) = \bar{G}_0(q) + \bar{G}_1(q), \\
&\bar{F}_j(q) = ik\int_0^\infty b\, db\, J_0(qb)P_{Fj}(b), \\
&\bar{G}_j(q) = ik\int_0^\infty b\, db\, J_1(qb)P_{Gj}(b), \\
&P_{\gamma j}(b) = (4\pi)^{-1}\lambda^3\frac{\partial}{\partial\lambda}\int_0^\infty s\, ds\, K_0(\lambda s) \\
&\qquad\times\int_0^{2\pi} d\phi\, T_{\gamma j}(b, s, \cos\phi), \\
&T_{FO} = S_0(\eta, b)S_0(-\eta_1, b'),
\end{aligned}
$$

$$T_{F1}=S_1(\eta,b)S_1(-\eta_1,b')\hat{b}'\cdot\hat{b}\,,$$
$$T_{G0}=S_1(\eta,b)S_0(-\eta_1,b')\,,$$
$$T_{G1}=S_0(\eta,b)S_1(-\eta_1,b')\hat{b}'\cdot\hat{b}\,,$$
$$b'^2=b^2+s^2-2bs\cos\phi\,,$$
$$\hat{b}'\cdot\hat{b}=(b-s\cos\phi)/b'\,. \qquad (17)$$

In the scattering operator (16) it turns out to be convenient to factor out the Rutherford amplitude $c(\eta,x)$ for scattering against the bare nucleus, a phase factor $\exp[i\Phi(\eta_1)]$, and a Coulomb correction factor $C(\eta_1)$ characteristic of the scattering against the target electron. Introducing the variable $\zeta=(\frac{1}{2}aq)^{-2}$, we then write

$$\begin{aligned}\mathfrak{F}(\vec{q})&=c(\eta,\zeta)\exp[i\Phi(\eta_1)]\\&\quad\times C(\eta_1)[F(\zeta)+iG(\zeta)\vec{\sigma}\cdot\vec{n}]\,,\\c(\eta,\zeta)&=-\tfrac{1}{2}\eta a(ak)^{1+2i\eta}\exp[2i\sigma(\eta)]\zeta^{1+i\eta}\,,\\\exp[i\Phi(\eta_1)]&=(ak)^{-2i\eta_1}\exp[-2i\sigma(\eta_1)]\,,\\C(\eta_1)&=|\Gamma(1\pm i\eta_1)|^2\\&=2\pi\eta_1\exp(-\pi\eta_1)/[1-\exp(-2\pi\eta_1)]\,.\end{aligned} \qquad (18)$$

In the products $T\gamma j$ of individual phase shift functions consistency requires that we keep only those terms that are of the order zero or unity in the fine-structure constant and of second or lower order in the scattering angle, when looked at from the point of view of the original scattering amplitudes, i.e., products

$$S_{n_1}^{l_1m_1}(\eta,b)S_{n_2}^{l_2m_2}(-\eta_1,b')$$

where $l=l_1+l_2\leq 1$ and $m=m_1+m_2\leq 2$. For the contributing terms F_j^{lm} and G_j^{lm}, where

$$\begin{aligned}F_0(\zeta)&=F_0^{00}(\zeta)+F_0^{02}(\zeta)+F_0^{11}(\zeta)+F_0^{12}(\zeta)\,,\\F_1(\zeta)&=F_1^{02}(\zeta)\,,\\G_0(\zeta)&=G_0^{01}(\zeta)+G_0^{12}(\zeta)\,,\\G_1(\zeta)&=G_1^{01}(\zeta)+G_1^{12}(\zeta)\,,\end{aligned} \qquad (19)$$

we then have

$$\begin{aligned}&F_0^{00}=I_{0000},\quad F_0^{02}=-\epsilon^{-1}(\epsilon-1)(\eta^2I_{2000}+\eta_1^2I_{2002})\,,\\&F_0^{11}=\tfrac{1}{2}i\pi\beta^2\{\eta^2\exp[2i\bar{\sigma}(\eta)]I_{1000}+\eta_1^2\exp[-2i\bar{\sigma}(\eta_1)]I_{1001}\}\,,\\&F_0^{12}=\tfrac{1}{2}\pi\beta^2(\eta^3I_{2000}-\eta_1^3I_{2002}),\quad F_1^{02}=\eta\eta_1[\epsilon^{-1}(\epsilon-1)]^2(I_{2002}-I_{2012}),\quad G_0^{01}=-i\eta\epsilon^{-1}(\epsilon-1)I_{1100}\,,\\&G_0^{12}=\tfrac{1}{2}\pi\beta^2\eta\{\tfrac{1}{2}i\eta(1-2i\eta)\exp[2i\bar{\sigma}(\eta)]I_{2100}+\epsilon^{-1}(\epsilon-1)\eta_1^2\exp[-2i\bar{\sigma}(\eta_1)]I_{2101}\}\,,\\&G_1^{01}=i\eta_1\epsilon^{-1}(\epsilon-1)(I_{1102}-I_{1112})\,,\\&G_1^{12}=-\tfrac{1}{2}\pi\beta^2\eta_1\{\epsilon^{-1}(\epsilon-1)\eta^2\exp[2i\bar{\sigma}(\eta)](I_{2102}-I_{2112})-\tfrac{1}{2}i\eta_1(1+2i\eta_1)\exp[-2i\bar{\sigma}(\eta_1)](I_{2103}-I_{2113})\}\,,\\&\bar{\sigma}(\eta)=\sigma(\eta)-\sigma'(\eta)\,,\end{aligned} \qquad (20)$$

in terms of integrals

$$\begin{aligned}I_{KLMN}(\zeta)&=-4\pi^{-1}i\eta^{-1}(ak)^{-4-2i\Delta\eta}\exp\{2i[\sigma(\eta_1)-\sigma(\eta)]\}[C(\eta_1)]^{-1}\zeta^{-i\eta}\\&\quad\times\frac{\partial}{\partial\zeta}\int_0^\infty v^{3-K+2i\Delta\eta}J_L(xv)\,dv\int_0^\infty u^{1+M}K_0(\zeta^{1/2}xvu)\,du\int_0^{2\pi}d\phi\cos(M\phi)y^{-N-2i\eta_1}\\\Delta\eta&=\eta-\eta_1,\quad y=(1+u^2-2u\cos\phi)^{1/2},\quad x=q/k\,.\end{aligned} \qquad (21)$$

By using methods similar to those employed in Ref. 4 we find that

$$\begin{aligned}I_{KLMN}&=(ak)^{-K}i(M!\eta)^{-1}[\Gamma(1-\tfrac{1}{2}N-i\eta_1)/\Gamma(1-i\eta_1)]^2[\Gamma(M+\tfrac{1}{2}N+i\eta_1)/\Gamma(\tfrac{1}{2}N+i\eta_1)][\Gamma(1-\nu+i\eta)/\Gamma(\mu-i\eta)]\\&\quad\times\exp[-2i\sigma(\eta)]\zeta^{(1/2)(N-K)}F_{KLMN}(\zeta)\,,\\F_{KLMN}&=(1-\tfrac{1}{2}N-i\eta_1)\,{}_3F_2(M+\tfrac{1}{2}N+i\eta_1,1-\mu+i\eta,1-\nu+i\eta;\tfrac{1}{2}N+i\eta_1,M+1;-\zeta)+[(M+1)(\tfrac{1}{2}N+i\eta_1)]^{-1}(M+\tfrac{1}{2}N+i\eta_1)\\&\quad\times(1-\mu+i\eta)(1-\nu+i\eta)\zeta\,{}_3F_2(M+\tfrac{1}{2}N+1+i\eta_1,2-\mu+i\eta,2-\nu+i\eta;\tfrac{1}{2}N+1+i\eta_1,M+2;-\zeta)\,,\\\mu&=\tfrac{1}{2}(K+L-N),\quad\nu=\tfrac{1}{2}(K-L-N)\,,\end{aligned} \qquad (22)$$

where ${}_3F_2$ is a generalized hypergeometric function. Since in our case M is an integer, ${}_3F_2$ can be expressed in terms of ordinary hypergeometric functions ${}_2F_1$, the cases of interest to us being

$$F_{KL0N}=(1-\tfrac{1}{2}N-i\eta_1)_2F_1(1-\mu+i\eta,1-\nu+i\eta;1;-\zeta)+(1-\mu+i\eta)(1-\nu+i\eta)\zeta_2F_1(2-\mu+i\eta,2-\nu+i\eta;2;-\zeta)\,,$$
$$F_{KL1N}=(1-\tfrac{1}{2}N-i\eta_1)_2F_1(1-\mu+i\eta,1-\nu+i\eta;2;-\zeta)+(1-\mu+i\eta)(1-\nu+i\eta)\zeta_2F_1(2-\mu+i\eta,2-\nu+i\eta;3;-\zeta)$$
$$-\tfrac{1}{6}(\tfrac{1}{2}N+i\eta_1)^{-1}(1-\mu+i\eta)(2-\mu+i\eta)(1-\nu+i\eta)(2-\nu+i\eta)\zeta^2{}_2F_1(3-\mu+i\eta,3-\nu+i\eta;4;-\zeta)\,. \qquad (23)$$

In this notation the amplitudes (20) then become

$$F_0^{00}=F_{0000},\quad F_0^{02}=-\tfrac{1}{4}\epsilon^{-1}(\epsilon-1)x^2(F_{2000}-\zeta F_{2002}),\quad F_0^{11}=-\tfrac{1}{4}\pi\beta^2\eta x[F_{1000}-i(\eta_1/\eta)tgh(\pi\eta_1)\zeta^{1/2}F_{1001}]\,,$$
$$F_0^{12}=\tfrac{1}{8}\pi\beta^2\eta x^2[F_{2000}+(\eta_1/\eta)\zeta F_{2002}],\quad F_1^{02}=-\tfrac{1}{4}(\eta/\eta_1)[\epsilon^{-1}(\epsilon-1)]^2\zeta x^2[F_{2002}-(1+i\eta_1)F_{2012}]\,,$$
$$G_0^{01}=\tfrac{1}{2}\epsilon^{-1}(\epsilon-1)xF_{1100},\quad G_0^{12}=-\tfrac{1}{8}\pi\beta^2\eta x^2[F_{2100}-i(\eta_1/\eta)\epsilon^{-1}(\epsilon-1)tgh(\pi\eta_1)\zeta^{1/2}F_{2101}]\,, \qquad (24)$$
$$G_1^{01}=-\tfrac{1}{2}i\epsilon^{-1}(\epsilon-1)\eta_1^{-1}(1+i\eta)\zeta x[F_{1102}-(1+i\eta_1)F_{1112}]\,,$$
$$G_1^{12}=\tfrac{1}{16}i\pi\beta^2(\eta/\eta_1)\zeta x^2\{\epsilon^{-1}(\epsilon-1)(1+2i\eta)[F_{2102}-(1+i\eta_1)F_{2112}]$$
$$+4(\eta_1^2/\eta)(1+2i\eta_1)^{-1}(1+i\eta)tgh(\pi\eta_1)\zeta^{1/2}[F_{2103}-(\tfrac{3}{2}+i\eta_1)F_{2113}]\}\,.$$

We see that the lowest-order term

$$c(\eta,\zeta)\exp[i\Phi(\eta)]C(\eta)F_0^{00}$$

is in agreement with the result of Thomas and Gerjuoy[4] [their Eq. (28a)] for scattering on the hydrogen atom, i.e., for $\eta_1=\eta$. It should be noted that our η and theirs are defined with opposite signs, ours being positive for positron scattering. Another limit to be checked is the case where no target electron is present, which means letting $\eta_1\to 0$, $\zeta\to 0$ and $(\zeta/\eta_1)\to 0$; the amplitudes (4) and (18) do then indeed coincide.

For finite η_1 and large momentum transfers ($\zeta\ll 1$) the role of the target electron is simply to multiply the scattering amplitude from the bare nucleus by a factor

$$\exp[i\Phi(\eta_1)]C(\eta_1)(1-i\eta_1)\,,$$

i.e., the projectile-nucleus cross section is multiplied by a q-independent factor

$$D(\eta_1)=C^2(\eta_1)(1+\eta_1^2)\,. \qquad (25)$$

IV. NUMERICAL RESULTS AND DISCUSSION

So far, the only experimental differential cross sections available for comparison with the present theoretical ones are those concerning e^- scattering on hydrogen.[11-13] Since the highest projectile energy here is 680 eV, relativistic effects play a very small role in these cases, and for practical purposes the comparison between theory and experiment comes out exactly as shown in Ref. 12, except for the fact that the newer results of van Wingerden *et al.* at 100 and 200 eV should now be included.

It is clear that the present version of the Glauber approximation does not do too well in explaining the experimental results at 200 eV and below. From Ref. 12 one is, however, left with the impression that this can be stated with certainty also at 400 and 680 eV. We believe that Figs. 1–2, where we have refrained from using the often instructive but at times misleading semilog plots familiar to these areas of physics, depict the actual situation rather clearly. At these energies the large-angle differential cross section is essentially that of e^-p scattering, whereas the target electron makes itself felt at small scattering angles. However, these experiments do not justify any choice between different theories based on small differences in the calculated cross section.

In general, we find that for hydrogen and the light hydrogenlike ions it is never necessary to consider the Glauber deviation from the bare-nucleus cross section and the relativistic effects [i.e., $f/c\neq 1$, $g\neq 0$ in Eq. (4)] at the same time. As an example we show in Figs. 3–5 the differential cross section for $e^\pm$ scattering on He^+ at the energies 100 eV, 10 keV, and 1 MeV. In the range of angles that is considered in each case the relativistic effects, and therefore also the $e^\pm$ difference, are too small to be shown in the figure. At 100 eV the deviation from the bare-nucleus cross section is large and the relativistic effects small at all angles; in the two other cases the Glauber cross section becomes identical to the bare-nucleus cross section [except for the small difference introduced at these energies through the factor (25)] long before the relativistic effects become noticeable.

The situation is different for higher Z_2, as exemplified in Fig. 6 for scattering of 0.5-MeV positrons at $Z_2=20$. In this case we see that in the angular region from 10° to 30°, say, the relativistic Glauber cross section differs appreciably both from the relativistic bare-nucleus cross section and the nonrelativistic Glauber cross section.

Our conclusion is, then, that the type of calculations presented here will be of value when one considers high-energy $e^\pm$ scattering in complex atoms, whereas scattering on actual hydrogenlike

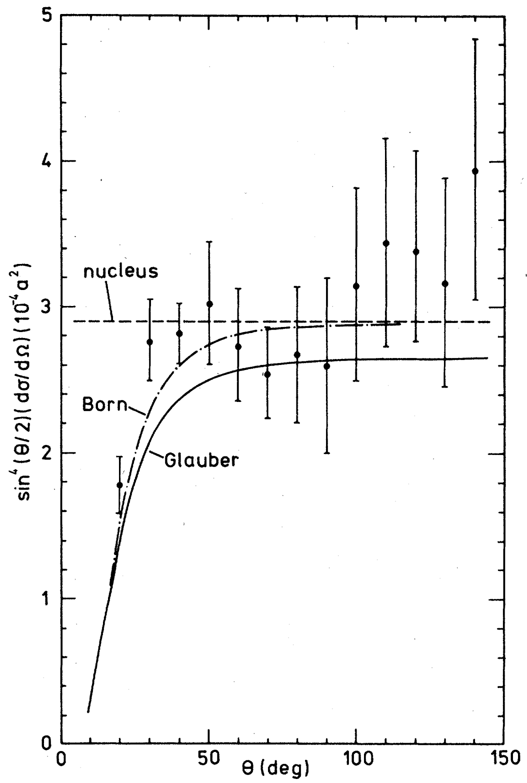

FIG. 1. Electron scattering on hydrogen at 400 eV in the Born and Glauber approximations. The straight line corresponds to the Rutherford cross section for scattering on the proton. The experimental points are from Ref. 12.

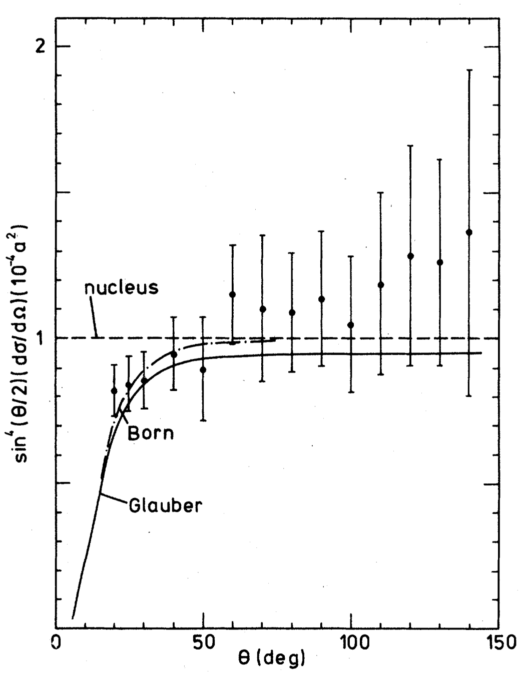

FIG. 2. As Fig. 1, but at 680 eV.

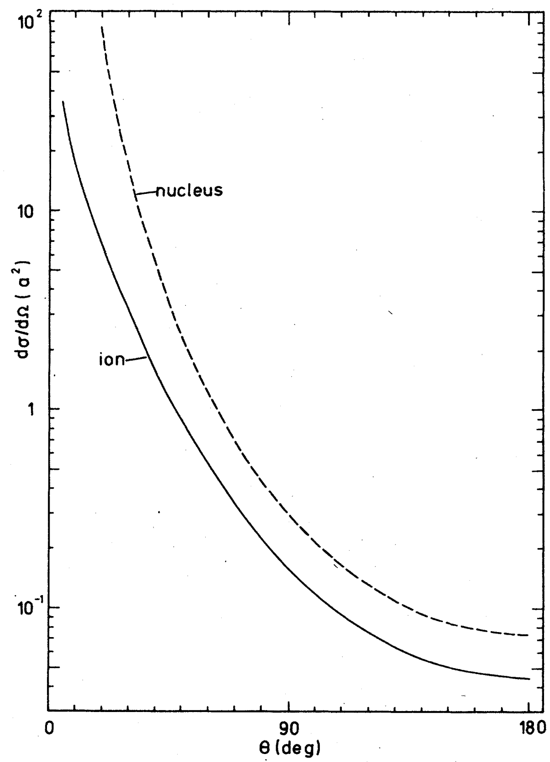

FIG. 3. $e^{\pm}$ scattering on He^{+} at 100 eV in the Glauber approximation, compared with the cross section for scattering on the bare nucleus.

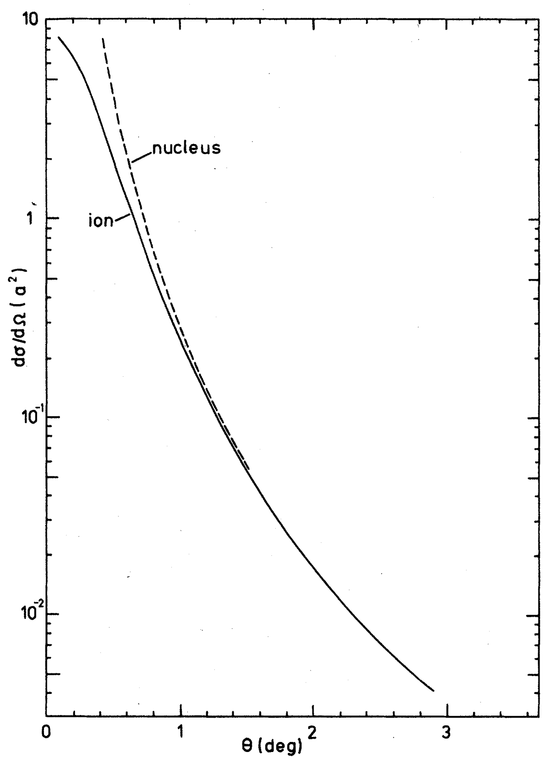

FIG. 4. As Fig. 3, but at 10 keV.

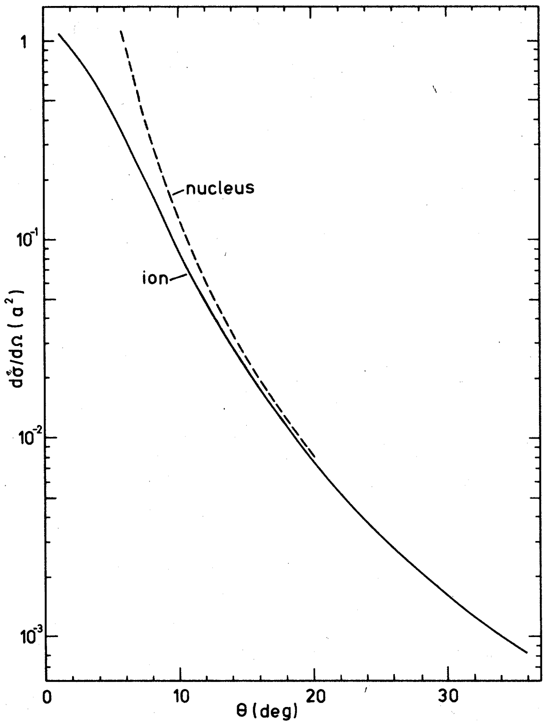

FIG. 5. As Fig. 3, but at 1 MeV.

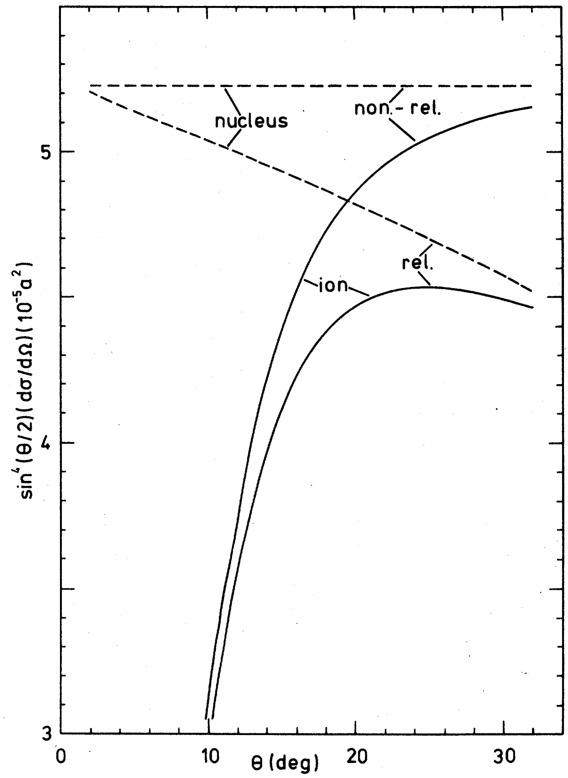

FIG. 6. Positron scattering on a hydrogenlike ion with $Z=20$ at 0.5 MeV in the relativistic and nonrelativistic Glauber approximations, compared with the corresponding cross sections for scattering on the bare nucleus.

ions (low Z_2) may be suitably described either by a nonrelativistic theory or as relativistic scattering from the bare nucleus, depending on the energy and the scattering angle in question.

As far as the scattering at very small angles is concerned, it is clear that this process is inadequately described by the present methods. We mention, however, that the direct term in the amplitude now behaves as $F(\zeta) \propto i(1+\frac{1}{2}i\Delta\eta)\ln q$ as $q \to 0$; for a hydrogenlike ion the divergence therefore appears in the real as well as in the imaginary part of the amplitude. The spin-flip term $G(\zeta)$ has an even stronger q^{-1} divergence as $q \to 0$. As suggested already by Franco[14] these difficulties could be remedied by relaxing the frozen-target assumption which is implicit in the method used here.

In conclusion, we should like to mention very briefly some of the attempts that have been made to improve upon the present "restricted" version of the Glauber theory, as applied to atomic scattering problems. These include the "unrestricted" Glauber approximation considered, e.g., in Ref. 15, where the path of integration is chosen along the direction of the incident beam ($q_z \neq 0$). It is, however, our opinion that this method does not remove a restriction as much as violate a basic principle, viz., that of time-reversal invariance, due to the asymmetric treatment of the initial and final directions. In the "Glauber angle" method of Chen *et al.*,[16] one uses two intersecting straight lines as the path of integration, and we also mention the "eikonal-Born series" of Byron and Joachain[17] and its close relative the "modified" Glauber approximation,[18,19] where the scattering amplitude is obtained partly from the Glauber and partly from the Born approximation. The virtues and disadvantages of the different approaches depend to a great extent on the projectile energy in question. If we limit the discussion to energies of some 400 eV and above (for $e^{\pm}$H scattering) it is clear that over a large range of scattering angles the differential cross section is largely determined by the scattering against the nucleus, the influence of the target electron being important only at fairly small angles. In these circumstances it seems natural to use a theory which has built into it the property that it reproduces at least the dominant part of the amplitude, i.e., the amplitude for scattering against the bare nucleus, well. In this respect the "restricted, wide-angle" Glauber approximation, in spite of its several shortcomings, has the advantage that it reproduces the Rutherford amplitude exactly.[9,20] We should like to stress that it is not our purpose to advocate the indiscriminate use of the Glauber approximation in atomic scattering problems, but only its usefulness in situations where the target electrons have a fairly small influence on the scattering cross section. In other cases the problem should be tackled by different methods. We mention, for instance, that in the case of electron scattering on He, for which very accurate data exist, the eikonal-Born series method has made it possible to reproduce the measured differential cross sections in the energy range 100–700 eV very well.[17]

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