PHYSICAL REVIEW A VOLUME 20, NUMBER 3 SEPTEMBER 1979

# Time-dependent harmonic oscillators

L. F. Landovitz
*Graduate Center of the City University of New York, New York, New York 10036*

A. M. Levine and W. M. Schreiber
*The College of Staten Island of the City University of New York, New York, New York 10301*
(Received 25 January 1979)

A formalism for a time-dependent harmonic oscillator is presented. The quantum-mechanical solution is developed and the Green's function is derived. Particular examples of runaway and dissipative behavior are considered.

## I. INTRODUCTION

The usual treatment of quantum-mechanical oscillators assumes that the Hamiltonian is time independent. It is unlikely that all physical oscillators are of this form. Camiz *et al.*[1] have considered a special case of the time-dependent oscillator where only the frequency is allowed to change with time. Kanai,[2] Kerner,[3] and Stevens[4] have considered another special case leading to a damped oscillator. In this paper, we consider the most general form of time-dependent linear oscillator subject to the normal commutation relations. The technique yields both the Green's function and a prescription for the explicit calculation of the important expectation values.

To illustrate the technique, we consider four interesting cases. The first is the damped oscillator originally proposed by Kanai.[2] The second is the corresponding runaway oscillator. The third is a new model where the frequency of the damped oscillator is also damped. The final case is the corresponding runaway solution to case three.

## II. FORMALISM

We represent the general time-dependent harmonic-oscillator Hamiltonian as

$$H=f(t)p^2/2m+g(t)\tfrac{1}{2}m\omega_0^2x^2\,, \tag{2.1}$$

where the time-independent harmonic-oscillator Hamiltonian is expressed by

$$H_0=p^2/2m+\tfrac{1}{2}m\omega_0^2x^2\,. \tag{2.2}$$

The condition that $H$ equal $H_0$ at $t=0$ implies

$$f(0)=g(0)=1\,. \tag{2.3}$$

The Hamiltonian equations

$$\dot{x}=\frac{\partial H}{\partial p}\,, \tag{2.4}$$

$$\dot{p}=-\frac{\partial H}{\partial x} \tag{2.5}$$

applied to the Hamiltonian (2.1) yield

$$\dot{x}=f(t)p/m\,, \tag{2.6}$$

$$\dot{p}=-g(t)m\omega_0^2x\,. \tag{2.7}$$

The equation of motion obtained is

$$\ddot{x}-\frac{d}{dt}[\ln f(t)]\dot{x}+f(t)g(t)\omega_0^2x=0\,. \tag{2.8}$$

The Lagrangian is expressed by

$$L=p\dot{x}-H\,. \tag{2.9}$$

From Eq. (2.6),

$$p=f(t)^{-1}m\dot{x}\,. \tag{2.10}$$

Hence the corresponding Lagrangian is

$$L=f(t)^{-1}\tfrac{1}{2}m\dot{x}^2-g(t)\tfrac{1}{2}m\omega_0^2x^2\,. \tag{2.11}$$

The equation of motion (2.8) can be rewritten in the form

$$\frac{d^2x}{d\theta^2}+Q(\theta)\omega_0^2x=0 \tag{2.12}$$

by a transformation

$$\theta=\theta(t)\,. \tag{2.13}$$

Equation (2.8) may be expressed as

$$\frac{d}{dt}[f(t)^{-1}\dot{x}]+g(t)\omega_0^2x=0\,. \tag{2.14}$$

Since

$$\frac{d}{dt}=\frac{d}{d\theta}\frac{d\theta}{dt}\,, \tag{2.15}$$

one obtains

$$f(t)^{-1}\dot{x}=\frac{dx}{d\theta} \tag{2.16}$$

for

$$\frac{d\theta}{dt}=f(t)\,. \tag{2.17}$$

Substituting Eqs. (2.15)–(2.17) into Eq. (2.14), one obtains Eq. (2.12) with $Q(\theta)$ expressed by

$$Q(\theta)=G(\theta)/F(\theta)\,, \tag{2.18}$$

where

$$F(\theta)=f(t)\,, \tag{2.19}$$

$$G(\theta)=g(t)\,. \tag{2.20}$$

Equation (2.17) defines the transformation (2.13). Integration of this equation yields

$$\theta(t)=\int f(t)\,dt+C\,. \tag{2.21}$$

One can choose the constant $C$ such that

$$\theta(0)=0\,. \tag{2.22}$$

If

$$g(t)=f(t)\,, \tag{2.23}$$

Eq. (2.12) becomes

$$\frac{d^2x}{d\theta^2}+\omega_0^2x=0\,. \tag{2.24}$$

## III. QUANTUM-MECHANICAL SOLUTION

For $H$ expressed by Eq. (2.1), $x$ and $p$ are taken as operators satisfying the commutation relation

$$[x,p]=i\hbar. \tag{3.1}$$

The corresponding Schrödinger equation for the wave function $\psi(x,t)$ is

$$i\hbar\frac{\partial\psi}{\partial t}=H\psi\,. \tag{3.2}$$

Setting

$$\psi(x,t)=U\psi(x,0)\,, \tag{3.3}$$

Eq. (3.2) implies

$$i\hbar\frac{\partial U}{\partial t}=HU\,, \tag{3.4}$$

where $U$ must be unitary to preserve norms.

For an operator $O$, the operators $O_\pm$ are defined by[5]

$$O_+=U^\dagger OU\,, \tag{3.5}$$

$$O_-=UOU^\dagger\,. \tag{3.6}$$

From Eq. (3.4),

$$\frac{dO_+}{dt}=\frac{1}{i\hbar}[O_+,H_+]+\left(\frac{\partial O}{\partial t}\right)_+\,. \tag{3.7}$$

Specifically,

$$\frac{dx_+}{dt}=\frac{1}{i\hbar}[x_+,H_+]\,, \tag{3.8}$$

$$\frac{dp_+}{dt}=\frac{1}{i\hbar}[p_+,H_+]\,, \tag{3.9}$$

$$\frac{d^2x_+}{dt^2}=\frac{1}{i\hbar}\left[\frac{dx_+}{dt},H_+\right]+\frac{\partial}{\partial t}\left(\frac{dx_+}{dt}\right). \tag{3.10}$$

Also, Eqs. (3.1) and (3.5) imply

$$[x_+,p_+]=i\hbar\,. \tag{3.11}$$

Application of Eqs. (3.8)–(3.11) to the Hamiltonian $H$ in Eq. (2.1) yields respective equations identical to Eqs. (2.6)–(2.8) with $x$ replaced by the operator $x_+$ and $p$ replaced by the operator $p_+$.

Since the Hamilton equations (2.6) and (2.7) are linear in $x$ and $p$, the operators $x_+$ and $p_+$ can be represented respectively by

$$x_+=a(t)x+b(t)p\,, \tag{3.12}$$

$$p_+=c(t)x+d(t)p\,, \tag{3.13}$$

where

$$a(0)=d(0)=1\,, \tag{3.14}$$

$$b(0)=c(0)=0\,. \tag{3.15}$$

Equations (2.6) and (2.7) imply

$$c=\frac{m}{f}\frac{da}{dt}\,, \tag{3.16}$$

$$d=\frac{m}{f}\frac{db}{dt}\,. \tag{3.17}$$

The condition

$$[x_+,p_+]=[x,p] \tag{3.18}$$

requires

$$ad-bc=1\,. \tag{3.19}$$

Assuming the oscillator to be in a state $|n\rangle$ at $t=0$, we compute the expectation values of $x_+^2$ and $\dot{x}_+^2$ at a later time $t$. These are denoted by $\langle x_+^2\rangle_n$ and $\langle\dot{x}_+^2\rangle_n$, respectively. The harmonic-oscillator operators $A$ and $A^\dagger$ are employed which have the following properties:

$$x=(\hbar/2m\omega_0)^{1/2}(A+A^\dagger)\,, \tag{3.20}$$

$$p=(1/i)(\tfrac{1}{2}m\hbar\omega_0)^{1/2}(A-A^\dagger)\,, \tag{3.21}$$

$$A|n\rangle=\sqrt{n}\,|n-1\rangle\,, \tag{3.22}$$

$$A^\dagger|n\rangle=(n+1)^{1/2}|n+1\rangle\,, \tag{3.23}$$

$$[A,A^\dagger]=1\,. \tag{3.24}$$

From Eqs. (3.20)–(3.24),

$$\langle n|x^2|n\rangle=(n+\tfrac{1}{2})\hbar/m\omega_0\,, \tag{3.25}$$

$$\langle n|p^2|n\rangle=(n+\tfrac{1}{2})m\hbar\omega_0\,, \tag{3.26}$$

$$\langle n|xp+px|n\rangle=0\,. \tag{3.27}$$

The quantities $\langle x_+^2\rangle_n$ and $\langle\dot{x}_+^2\rangle_n$ are defined in the following manner:

$$\langle x_+^2\rangle_n=\langle n|x_+^2|n\rangle\,, \tag{3.28}$$

$$\langle \dot{x}_+^2\rangle_n = \langle n|\dot{x}_+^2|n\rangle \,. \tag{3.29}$$

The expressions for $x_+$ and $p_+$ are given in Eq. (3.12) and (3.13), respectively. The operator $\dot{x}_+$ is related to $p_+$ through Eq. (2.6), i.e.,

$$\dot{x}_+ = f p_+/m \,. \tag{3.30}$$

Substitution in Eqs. (3.28) and (3.29), respectively, and using the relationships (3.25)–(3.27) yields

$$\langle x_+^2\rangle_n = (a^2 + m^2\omega_0^2 b^2)(n+\tfrac{1}{2})\hbar/m\omega_0 \,, \tag{3.31}$$

$$\langle \dot{x}_+^2\rangle_n = f^2[(1/m^2\omega_0^2)c^2 + d^2](n+\tfrac{1}{2})\hbar\omega_0/m \,. \tag{3.32}$$

## IV. GREEN'S FUNCTION

The Green's function $G(x,x';t)$ is defined by

$$\langle x|U|x'\rangle = G(x,x';t)\,. \tag{4.1}$$

The wave function $\psi(x,t)$ is obtainable from $\psi(x,0)$ by the formula

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x,x';t)\psi(x',0)\,dx' \,. \tag{4.2}$$

The condition

$$\int_{-\infty}^{\infty} \psi^\dagger(x,t)\psi(x,t)\,dx = \int_{-\infty}^{\infty}\psi^\dagger(x,0)\psi(x,0)\,dx \tag{4.3}$$

implies

$$\int_{-\infty}^{\infty} G^\dagger(x,x'';t)G(x,x';t)\,dx = \delta(x''-x')\,. \tag{4.4}$$

The boundary condition

$$\lim_{t\to 0}\psi(x,t) = \psi(x,0) \tag{4.5}$$

implies

$$\lim_{t\to 0} G(x,x';t) = \delta(x'-x)\,. \tag{4.6}$$

Equations (3.5) and (3.6) applied to the operator $x$ yield

$$x_+ = U^\dagger x U \,, \tag{4.7}$$

$$x_- = U x U^\dagger \,. \tag{4.8}$$

The expression for $x_+$ is given in Eq. (3.12). Since

$$U x_+ U^\dagger = x \,, \tag{4.9}$$

$$U p_+ U^\dagger = p \,, \tag{4.10}$$

one obtains from the respective Eqs. (3.12) and (3.13):

$$x = a x_- + b p_- \,, \tag{4.11}$$

$$p = c x_- + d p_- \,. \tag{4.12}$$

Thus

$$x_- = dx - bp \,. \tag{4.13}$$

From Eq. (4.1),

$$\langle x|xU|x'\rangle = xG(x,x';t)\,. \tag{4.14}$$

Also,

$$\langle x|xU|x'\rangle = \langle x|Ux_+|x'\rangle \,. \tag{4.15}$$

The expression for $x_+$ is given in Eq. (3.12) and

$$\langle x|Up|x'\rangle = i\hbar\frac{\partial}{\partial x'}G(x,x';t)\,. \tag{4.16}$$

Substitution for $x_+$ into Eq. (4.15) and then equating to Eq. (4.14), one obtains

$$\frac{\partial G}{\partial x'} = (i/\hbar b)(ax'-x)G \,. \tag{4.17}$$

Hence

$$G = g(x,t)\exp[(i/\hbar b)(a\tfrac{1}{2}x'^2 - xx')]\,. \tag{4.18}$$

Substitution of Eq. (4.13) for $x_-$ into the equation

$$\langle x|Ux|x'\rangle = \langle x|x_- U|x'\rangle \tag{4.19}$$

and noting that

$$\langle x|Ux|x'\rangle = x'G(x,x';t)\,, \tag{4.20}$$

yields the expression

$$\frac{\partial G}{\partial x} = \frac{i}{\hbar b}(dx - x')G \,. \tag{4.21}$$

From Eq. (4.18), one obtains

$$\frac{\partial g}{\partial x} = \frac{id}{\hbar b}xg \,. \tag{4.22}$$

Thus

$$g = F(t)\exp[(id/2\hbar b)x^2]\,. \tag{4.23}$$

Substituting this expression for $g$ into Eq. (4.18) yields

$$G = F(t)\exp[(i/2\hbar b)(dx^2 + ax'^2 - 2xx')]\,. \tag{4.24}$$

The expression for $F(t)$ can be obtained from the requirements (4.4) and (4.6). The result is

$$F(t) = 1/(2\pi i\hbar b)^{1/2}\,. \tag{4.25}$$

Hence

$$G(x,x';t) = (\beta/i\pi)^{1/2}\exp[i\beta(dx^2 + ax'^2 - 2xx')]\,, \tag{4.26}$$

where

$$\beta = 1/2\hbar b \,. \tag{4.27}$$

An alternative method of deriving the Green's function is through the path integral formalism.[6] The Green's function is expressible through the relationship

$$G(x,x';t) = F(t)\exp[(i/\hbar)S(x,x';t)]\,, \tag{4.28}$$

where

$$S=\int_0^t L\,dt\,. \tag{4.29}$$

The expression for $F(t)$ is obtained from Eqs. (4.4) and (4.6).

For an operator 0, the expression for the Green's function can be used to determine $\langle O\rangle_t$, where

$$\langle O\rangle_t=\int_{-\infty}^{\infty}\psi^\dagger(x,t)O\psi(x,t)\,dx\,. \tag{4.30}$$

Thus

$$\langle O\rangle_t=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dx''\,\psi^\dagger(x'',0)G^\dagger(x,x'';t)OG(x,x';t)\psi(x',0)\,. \tag{4.31}$$

$\langle O\rangle_t$ is also expressible as

$$\langle O\rangle_t=\int_{-\infty}^{\infty}\psi^\dagger(x,0)O_+\psi(x,0)\,dx\,, \tag{4.32}$$

where $O_+$ is defined in Eq. (3.5).

Consider, for instance, the Gaussian distribution

$$\psi(x,0)=(\alpha/\pi)^{1/4}\exp[-\tfrac{1}{2}\alpha(x-x_0)^2]\,. \tag{4.33}$$

This function has the property

$$\int_{-\infty}^{\infty}\psi^\dagger(x,0)x\psi(x,0)\,dx=x_0\,. \tag{4.34}$$

Also, since from Eqs. (2.3) and (2.6),

$$\dot{x}(0)=p(0)/m\,, \tag{4.35}$$

this implies that

$$\int_{-\infty}^{\infty}\psi^\dagger(x,0)\dot{x}\psi(x,0)\,dx$$

$$=\frac{\hbar}{im}\int_{-\infty}^{\infty}\psi^\dagger(x,0)\frac{\partial}{\partial x}\psi(x,0)\,dx=0\,. \tag{4.36}$$

We evaluate $\langle x\rangle_t$ for the Gaussian expression given in Eq. (4.33). Utilizing either the Green's function (4.26) or Eq. (3.12) for $x_+$, one obtains

$$\langle x\rangle_t=a(t)x_0\,, \tag{4.37}$$

which is the same as the classical solution for $x$ with the initial conditions $x=x_0$ and $\dot{x}=0$.

The wave function $\psi(x,t)$ for the Gaussian $\psi(x,0)$ defined in Eq. (4.33) is obtainable from Eq. (4.2) using the expression (4.26) for the Green's function. The result is

$$\psi(x,t)=(\beta/i)^{1/2}(\alpha/\pi B)^{1/4}e^{\gamma}e^{i\varphi}\,, \tag{4.38}$$

where

$$B=\tfrac{1}{4}\alpha^2+\beta^2a^2\,, \tag{4.39}$$

$$\gamma=-(\alpha/2A)\beta^2(x-ax_0)^2\,, \tag{4.40}$$

$$\varphi=\tfrac{1}{2}\arctan(2\beta a/\alpha)+\beta[d-(1/B)\beta^2a]x^2$$
$$-(\alpha^2/2B)\beta x_0x+(\alpha^2/4B)\beta a x_0^2\,. \tag{4.41}$$

## V. EXPONENTIALLY TIME-DEPENDENT OSCILLATORS

For a Hamiltonian of the form (2.1) with

$$f(t)=e^{t/\alpha_1}\,, \tag{5.1}$$

$$g(t)=e^{t/\alpha_2}\,, \tag{5.2}$$

where $\alpha_1$ and $\alpha_2$ are constants, Eqs. (2.6)–(2.8) and (2.11), respectively, yield

$$\dot{x}=e^{t/\alpha_1}p/m\,, \tag{5.3}$$

$$\dot{p}=-e^{t/\alpha_2}m\omega_0^2x\,, \tag{5.4}$$

$$\ddot{x}-\frac{1}{\alpha_1}\dot{x}+\exp[(1/\alpha_1+1/\alpha_2)t]\omega_0^2x=0\,, \tag{5.5}$$

$$L=e^{-t/\alpha_1}\tfrac{1}{2}m\dot{x}^2-e^{t/\alpha_2}\tfrac{1}{2}m\omega_0^2x^2\,. \tag{5.6}$$

We consider the following particular Hamiltonians $H_1-H_4$ of the above type:

$$H_1=e^{-t/\tau}p^2/2m+e^{t/\tau}\tfrac{1}{2}m\omega_0^2x^2\,, \tag{5.7}$$

$$H_2=e^{t/\tau}p^2/2m+e^{-t/\tau}\tfrac{1}{2}m\omega_0^2x^2\,, \tag{5.8}$$

$$H_3=e^{-t/\tau}p^2/2m+e^{-t/\tau}\tfrac{1}{2}m\omega_0^2x^2\,, \tag{5.9}$$

$$H_4=e^{t/\tau}p^2/2m+e^{t/\tau}\tfrac{1}{2}m\omega_0^2x^2\,. \tag{5.10}$$

Thus, in the notation of Eq. (2.1),

$$f_1=e^{-t/\tau}=g_1^{-1} \tag{5.11}$$

$$f_2=e^{t/\tau}=g_2^{-1} \tag{5.12}$$

$$f_3=e^{-t/\tau}=g_3 \tag{5.13}$$

$$f_4=e^{t/\tau}=g_4\,. \tag{5.14}$$

Equation (5.3)–(5.5) yield the following equations for the Hamiltonians $H_1$–$H_4$, respectively:

$$\dot{x}_1=e^{-t/\tau}p_1/m\,, \tag{5.15}$$

$$\dot{p}_1=-e^{t/\tau}m\omega_0^2x_1\,, \tag{5.16}$$

$$\ddot{x}_1+(1/\tau)\dot{x}_1+\omega_0^2x_1=0; \tag{5.17}$$

$$\dot{x}_2=e^{t/\tau}p_2/m\,, \tag{5.18}$$

$$\dot{p}_2=-e^{-t/\tau}m\omega_0^2x_2\,, \tag{5.19}$$

$$\ddot{x}_2-(1/\tau)\dot{x}_2+\omega_0^2x_2=0; \tag{5.20}$$

$$\dot{x}_3=e^{-t/\tau}p_3/m\,, \tag{5.21}$$

$$\dot{p}_3=-e^{-t/\tau}m\omega_0^2x_3\,, \tag{5.22}$$

$$\ddot{x}_3+(1/\tau)\dot{x}_3+e^{-2t/\tau}\omega_0^2x_3=0; \tag{5.23}$$

$$\dot{x}_4=e^{t/\tau}p_4/m\,, \tag{5.24}$$

$$\dot{p}_4=-e^{t/\tau}m\omega_0^2x_4\,, \tag{5.25}$$

$$\ddot{x}_4-(1/\tau)\dot{x}_4+e^{2t/\tau}\omega_0^2x_4=0\,. \tag{5.26}$$

Equation (5.17) is the equation of motion for a dissipative harmonic oscillator, and Eq. (5.20) is the corresponding equation of motion for a runaway harmonic oscillator. Equations (5.23) and (5.26) can, respectively, be written as

$$\ddot{x}_3 + (1/\tau)\dot{x}_3 + \omega_3^2(t)x_3 = 0\,, \tag{5.27}$$

$$\omega_3(t) = e^{-t/\tau}\omega_0; \tag{5.28}$$

$$\ddot{x}_4 - (1/\tau)\dot{x}_4 + \omega_4^2(t)x_4 = 0\,, \tag{5.29}$$

$$\omega_4(t) = e^{t/\tau}\omega_0\,. \tag{5.30}$$

Thus Eq. (5.23) corresponds to a system whose $\omega$ is decaying with time and Eq. (5.26) to a system whose $\omega$ is growing with time.

The Hamiltonians $H_3$ and $H_4$ satisfy the condition (2.23). Hence these Hamiltonians satisfy the equation of motion (2.24), which is the equation of motion for $H_0$ with $t$ replaced by $\theta$. The respective expressions $\theta_3$ and $\theta_4$ for the Hamiltonians $H_3$ and $H_4$ are obtainable from Eqs. (2.21)–(2.22). Thus

$$\theta_3 = \tau(1 - e^{-t/\tau}) \tag{5.31}$$

$$\theta_4 = \tau(e^{t/\tau} - 1)\,. \tag{5.32}$$

As noted in Sec. III, the corresponding quantum-mechanical expressions are identical to Eqs. (5.15)–(5.26) with $x$ replaced by $x_+$ and $p$ replaced by $p_+$.

For the Hamiltonian $H_1$, the solution obtained is

$$x_+ = e^{-t/2\tau}(O_1 \cos\omega t + O_2 \sin\omega t)\,, \tag{5.33}$$

$$\omega = (\omega_0^2 - 1/4\tau^2)^{1/2}\,, \tag{5.34}$$

where $O_1$ and $O_2$ are operators to be determined. The corresponding expression for $p_+$ is obtained from the relationship

$$p_+ = e^{t/\tau} m \frac{dx_+}{dt}\,. \tag{5.35}$$

At $t = 0$,

$$x_+ = x\,, \tag{5.36}$$

$$\frac{dx_+}{dt} = \frac{p}{m}\,. \tag{5.37}$$

Thus

$$O_1 = x\,, \tag{5.38}$$

$$O_2 = (1/\omega)[p/m + (1/2\tau)x]\,. \tag{5.39}$$

Substituting Eqs. (5.38)–(5.39) into Eqs. (5.33)–(5.35), one obtains

$$x_+ = e^{-t/2\tau}\{x[\cos\omega t + (1/2\omega\tau)\sin\omega t] + (p/m\omega)\sin\omega t\}\,, \tag{5.40}$$

$$p_+ = e^{t/2\tau}\{p[\cos\omega t - (1/2\omega\tau)\sin\omega t] - xm(\omega + 1/4\omega\tau^2)\sin\omega t\}\,. \tag{5.41}$$

The corresponding expressions for $H_2$ are obtained by the substitution $\tau \to -\tau$ in the above expressions.

The solution to the Hamiltonian $H_3$ may be obtained from the above-mentioned property that the equation of motion for $x_+$ is of the time-independent harmonic-oscillator type in the coordinate $\theta(t)$. Thus

$$x_+ = O_1 \cos\omega_0\theta + O_2 \sin\omega_0\theta\,, \tag{5.42}$$

where the expression for $\theta$ is given in Eq. (5.31). The operator $p_+$ is given by

$$p_+ = e^{t/\tau} m \frac{dx_+}{dt}\,. \tag{5.43}$$

The $t = 0$ conditions (5.36) and (5.37) imply

$$O_1 = x\,, \tag{5.44}$$

$$O_2 = p/m\omega_0\,. \tag{5.45}$$

Hence

$$x_+ = x\cos[\omega_0\tau(1 - e^{-t/\tau})] + (p/m\omega_0)\sin[\omega_0\tau(1 - e^{-t/\tau})]\,, \tag{5.46}$$

$$p_+ = p\cos[\omega_0\tau(1 - e^{-t/\tau})] - xm\omega_0\sin[\omega_0\tau(1 - e^{-t/\tau})]\,. \tag{5.47}$$

The substitution $\tau \to -\tau$ in the above expressions yields the corresponding expressions for the Hamiltonian $H_4$.

Note that the expressions (5.40) and (5.41), (5.46) and (5.47), and the corresponding expressions for $H_2$ and $H_4$ are of the general forms (3.12) and (3.13). They satisfy the conditions (3.16), (3.17), and (3.19). Thus for the Hamiltonian $H_1$,

$$a = e^{-t/2\tau}[\cos\omega t + (1/2\omega\tau)\sin\omega t]\,, \tag{5.48}$$

$$b = e^{-t/2\tau}(1/m\omega)\sin\omega t\,, \tag{5.49}$$

$$c = -e^{t/2\tau} m[\omega + (1/4\omega\tau^2)]\sin\omega t\,, \tag{5.50}$$

$$d = e^{t/2\tau}[\cos\omega t - (1/2\omega\tau)\sin\omega t]\,. \tag{5.51}$$

For the Hamiltonian $H_3$,

$$a = \cos[\omega_0\tau(1 - e^{-t/\tau})]\,, \tag{5.52}$$

$$b = (1/m\omega_0)\sin[\omega_0\tau(1 - e^{-t/\tau})]\,, \tag{5.53}$$

$$c = -m\omega_0\sin[\omega_0\tau(1 - e^{-t/\tau})]\,, \tag{5.54}$$

$$d = \cos[\omega_0\tau(1 - e^{-t/\tau})]\,. \tag{5.55}$$

The corresponding expressions for the Hamiltonians $H_2$ and $H_4$ are obtainable by the substitution $\tau \to -\tau$ in Eqs. (5.48)–(5.51) and (5.52)–(5.55), respectively.

For the Hamiltonian $H_1$, Eqs. (3.31) and (3.32) yield, respectively,

$$\langle x_+^2\rangle_n = e^{-t/\tau}\{[\cos\omega t + (1/2\omega\tau)\sin\omega t]^2 + (\omega_0^2/\omega^2)\sin^2\omega t\}(n+\tfrac{1}{2})\hbar/m\omega_0\,, \tag{5.56}$$

$$\langle \dot{x}_+^2\rangle_n = e^{-t/\tau}\{[\cos\omega t - (1/2\omega\tau)\sin\omega t]^2 + (\omega/\omega_0 + 1/4\omega_0\omega\tau^2)^2\sin^2\omega t\}(n+\tfrac{1}{2})\frac{\hbar\omega_0}{m}\,. \tag{5.57}$$

The energy operator $E_+$ is defined by

$$E_+ = \tfrac{1}{2}m\dot{x}_+^2 + \tfrac{1}{2}m\omega_0^2 x_+^2\,. \tag{5.58}$$

From Eqs. (5.56) and (5.57), one obtains

$$\langle E_+\rangle_n = e^{-t/\tau}[1 + (1/2\omega^2\tau^2)\sin^2\omega t](n+\tfrac{1}{2})\hbar\omega_0\,. \tag{5.59}$$

The corresponding expressions for the Hamiltonian $H_2$ are obtained by the substitution $\tau \to -\tau$.

For the Hamiltonian $H_3$, Eqs. (3.31) and (3.32) yield, respectively,

$$\langle x_+^2\rangle_n = (n+\tfrac{1}{2})\hbar/m\omega_0\,, \tag{5.60}$$

$$\langle \dot{x}_+^2\rangle_n = e^{-2t/\tau}(n+\tfrac{1}{2})\hbar\omega_0/m\,. \tag{5.61}$$

The energy operator $E_+$ is given by

$$E_+ = \tfrac{1}{2}m\dot{x}_+^2 + \tfrac{1}{2}m\omega^2 x_+^2\,. \tag{5.62}$$

From Eq. (5.28),

$$\omega = e^{-t/\tau}\omega_0\,. \tag{5.63}$$

Substitution of Eqs. (5.60) and (5.61) yields

$$\langle E_+\rangle_n = e^{-2t/\tau}(n+\tfrac{1}{2})\hbar\omega_0\,. \tag{5.64}$$

The substitution $\tau \to -\tau$ in the above equations yields the corresponding expressions for the Hamiltonian $H_4$.

Equations (5.57) and (5.61) imply that both the Hamiltonians $H_1$ and $H_3$ yield $\langle \dot{x}_+^2\rangle_n \to 0$ for $t\to\infty$. Equation (5.56) implies that the Hamiltonian $H_1$ yields $\langle x_+^2\rangle_n \to 0$ for $t\to\infty$. Equation (5.60) for the corresponding expression for the Hamiltonian $H_3$ implies that $\langle x_+^2\rangle_n$ is time independent and equal to its value at $t=0$. This is because the value of $\omega$ is decaying with time in accordance with Eq. (5.63). Equations (5.59) and (5.64) imply that for both $H_1$ and $H_3$, $\langle E_+\rangle_n \to 0$ for $t\to\infty$.

For $t=0$, let the system have the wave function

$$\psi(x,0) = \sum_n a_n\psi_n(x)\,, \tag{5.65}$$

where $\psi_n(x)$ are eigenfunctions of the Hamiltonian $H_0$. The expressions for $\psi(x,t)$ for the Hamiltonians $H_3$ and $H_4$ will be of the corresponding form for $H_0$ with $t$ replaced by the respective $\theta$. Specifically,

$$\psi(x,t) = \sum_n a_n\psi_n(x) \exp[\mp i(n+\tfrac{1}{2})\omega_0\tau(1-e^{\mp t/\tau})]\,, \tag{5.66}$$

where the upper signs refer to $H_3$ and the lower signs refer to $H_4$.

The respective Green's functions are obtainable by the substitution of the appropriate expressions for $a$, $b$, and $d$ in Eq. (4.26). Similarly, one can obtain the Gaussian expressions for $\langle x\rangle_t$ and $\psi(x,t)$ by substitution in Eqs. (4.37) and (4.38), respectively.

## VI. DISCUSSION

This paper has presented a method for obtaining the expression values for all observables of a time-dependent harmonic oscillator. Physically, such oscillators can occur in at least two different ways. In each case, the quantum-mechanical system under consideration must contain smaller subsystems. For example, a diatomic molecule is a quantum-mechanical system with smaller systems within both the electron cloud and the nuclei.

The first class of time-dependent oscillators occurs when oscillatory behavior of the subsystem produces a periodic variation in either the effective mass or the effective spring constant of the larger system. This will be examined in detail in a future paper.

In the second class of problems, the subsystem possesses a large number of degrees of freedom with a quasicontinuous spectrum of energy eigenvalues. The subsystem acts to dissipate the energy of the large scale mode. One way of describing this case is to use the time-dependent Hamiltonian denoted by $H_1$ in Sec. V. This approach, originally suggested by Kanai,[2] corresponds to the classical damped harmonic oscillator as described by Eq. (5.17).

The above technique for treating dissipative systems has been questioned by Senitzky,[7] who suggested using a Hamiltonian describing the coupling between a lossless oscillator and the loss mechanism. With approximations, he obtained an equation of motion that included both a damping term and a fluctuating driving term. Senitzky claimed that this driving term is needed to obtain both the correct commutation relation and the correct thermal fluctuation level. We disagree with the first point, since, as explained in Sec. III, it is the commutation relation for the generalized position and momentum that must be, and, in our treatment, is conserved. We agree with the second point. The correct classical equation of motion should have a Langevinlike fluctuating driving term. This term may be neglected whenever the oscillator's energy is much greater than the thermal fluctuation level. Within this approximation, the energy of the oscillator goes to zero [see Eq. (5.59)]. The treatment by Kanai,[2] and our treatment of $H_1$, corresponds to the classical limit with the fluctuating driving term neglected. A better treatment, and one that will be published in a future paper, is to add terms to the Hamiltonian that will produce the classical equation of motion with the fluctuating driving term.

[1]P. Camiz, A. Gerardi, C. Marchioro, E. Presutti, and E. Scacciatelli, J. Math. Phys. 12, 2040 (1971), and references therein; see also D. C. Khandekar and S. V. Lawande, *ibid.* 16, 384 (1975).
[2]E. Kanai, Prog. Theor. Phys. (Kyoto) 3, 440 (1948).
[3]E. H. Kerner, Can. J. Phys. 36, 371 (1958).
[4]W. K. H. Stevens, Proc. Phys. Soc. London 72, 1027 (1958).
[5]L. F. Landovitz, Phys. Rev. A 11, 67 (1975). Also, for a very thorough discussion of quantum mechanics in terms of Green's functions cf. J. Schwinger, *Quantum Kinematics and Dynamics* (Benjamin, New York, 1970).
[6]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
[7]I. R. Senitzky, Phys. Rev. 119, 670 (1960).