PHYSICAL REVIEW A VOLUME 2, NUMBER 3 SEPTEMBER 1970

Polarized Maser Emission from Interstellar OH and H_2O†

M. M. Litvak

Lincoln Laboratory, Massachusetts Institute of Technology, Lexington, Massachusetts 02173
(Received 12 March 1970)

Polarization properties of interstellar OH and H_2O masers are related to the nonlinear properties of the medium, including that of the weak magnetoplasma. Because the Raman-type cross-saturation of oppositely polarized modes is negligible for this Gaussian-type broad-band signal, contrary to laboratory cases, the preference for circular polarization in the brightest OH sources is attributed to another mechanism: parametric down-conversion of one Zeeman-split microwave mode to another and to an electron cyclotron wave.

I. INTRODUCTION

Many regions of our galaxy contain clouds of OH and H_2O molecules that emit microwave signals so bright and unusually polarized that little doubt exists that they are maser sources.[1–3] Very long baseline interferometers have shown several of the emitters to have apparent sizes 10–100 A.U. (1 astronomical unit equals 1.5×10^{13} cm).[4] In the case of the OH-H_2O region near the radio continuum source W3, some seven emitting points, some being actually double or triple, are widely arranged in a rough circle having a diameter of about 1 arc sec [~ 3000 A.U. at a distance of ~ 3 kiloparsecs (kpc)]. Since these are regions of star formation, these masers are believed to be associated with protostars. However, OH-H_2O masers have also been found near infrared stars, many of which are probably highly evolved stars, M type, with circumstellar dust envelopes that are responsible for the infrared output. Compared to the masers near HII regions like W3, these infrared star masers show little polarization. Other masers have been found near the expanding shells of supernova remnants. These usually show even less polarization. General characteristics may be listed as follows: The OH emitters with the highest brightness, usually near HII regions, are almost always polarized, and usually highly circularly. However, particular Doppler-shifted features in a few such regions show elliptical, linear, or no polarization. This is true of all four of the hyperfine-split transitions of the OH ground state $\Pi_{3/2}, J=\frac{3}{2}$. However, only one of these transitions is dominant in a particular region, namely, the $\Delta F=0$ transitions at either 1665 or 1667 MHz near HII regions, often the $\Delta F=1$ transition at 1612 MHz near the infrared stars, and often the $\Delta F=-1$ transition at 1720 MHz near supernova remnants. No clear preference for linear polarization by the $|\Delta F| = 1$ transitions is to be found, but these are usually not strongly circularly polarized except when the $\Delta F=0$ transitions are also present and are circularly polarized.

The H_2O emission at the 22.2-GHz $6_{16}\rightarrow5_{23}$ transition is rarely polarized and then only linearly polarized.[5] The strongest emissions belong to $\Delta F=-1$ transitions, but time variations over weeks, as if the maser is unsaturated, have complicated the observations. In one region, W49, an unusual HII region, the time variations correspond to a length of about 0.1 arc sec or about 1400 A.U. at the probable distance of 14 kpc for the source. The brightness temperature corresponds to $\gtrsim 10^{13}$ °K for W49 since the emission points subtend less than 0.003 arc sec.[4] In Orion A the H_2O emission is highly polarized, linearly, but the emission points have yet to be resolved at about 0.003 arc sec, or 1.5 A.U. at 500 parsec (pc).[4] The brightness temperature then exceeds 10^{13} °K.

Maser emission has also been observed from excited rotational states of OH. The one, $\Pi_{1/2}$, $J=\frac{1}{2}, F=1\rightarrow0$ with very small Zeeman-splitting factors, shows no polarization and the others, $\Pi_{3/2}, J=\frac{5}{2}, F=3\rightarrow3$ and $2\rightarrow2$, show some circular polarization.[6] If interpreted as a distorted Zeeman splitting, the field would be about 10^{-2} G. Some features in the ground state might be interpreted as distorted Zeeman splittings at approximately 5×10^{-4} G.[7] Absorption by ground-state OH has shown no polarization whatsoever. The weak so-called normal OH emission from dark dust clouds has also shown little if any polarization.[7]

Maser amplification, under the proper conditions of saturation, can result in a high degree of polarized emission. Laboratory experiments on laser oscillators in weak or no magnetic field have verified to a fair degree the predictions of nonlinear optics that, for laser electric dipole transitions between states of the same angular momentum, the output has nearly pure circular polarization.[8] Cavity anisotropy will produce some ellipticity. Similarly, for laser transitions between states of angular momenta differing by one unit, excluding $1 \rightarrow 0$, the output has linear polarization.[9] Similar polarization characteristics are believed to occur in traveling-wave amplifiers of monochromatic signals. However, the OH (and H_2O) emission does not directly fit into these cases because first, the OH signal is broad band (~ 3 kHz), probably considerably larger than the radiative or collision linewidths of the OH states, and second, the Zeeman splitting due to the interstellar magnetic field also is likely to be larger than the linewidths. Both of these statements are probably also true for the H_2O molecules.

Recent analyses[10] of the OH problem merely assume that the case of monochromatic signal and weak magnetic field applies without qualification. For $\Delta J = 0$ or $\Delta F = 0$ transitions, it is supposed to be true that when both components of circular polarization are present there is less amplification than when only one component is present. This requires that the cross-saturation of one component by the other exceeds self-saturation. For this to occur, ordinary cross-saturation by induced changes of population must be supplemented by so-called Raman-type cross-saturation.[11] A critique[12] shows that the average power dissipation due to the Raman-type cross-saturation at a typical point is too small if the microwave signal being amplified has random phases within a bandwidth that exceeds the radiative or collision linewidths. Nevertheless, with the proper coupling of signals at different frequencies by other processes to be described, in addition to the nonlinear saturation effects, the OH and H_2O emission is likely to exhibit polarization properties similar to the previously mentioned lasers.

Two general regimes arise: the radiation-dominated one in which resonance radiation trapping and optical or weak collisional excitation determine the populations and maser properties, and the collision-dominated regime associated with high densities. For the collision rate to be faster than the microwave saturation rate, with the assumption of isotropic emission for the observed intensities, the density must exceed 10^{10}/cc. For a faster collision rate than the far-infrared resonance radiation spontaneous emission, with typical radiation trapping, the density must merely exceed 3×10^6/cc.

II. COLLISION-DOMINATED REGIME

The high-density regime affords two possible explanations for the circularly polarized emission, which we show are not likely. The first explanation is that the density is greater than 10^{13}/cc, so the pressure broadening exceeds the Doppler linewidth. The bandwidth of the amplified radiation is less than this linewidth. Therefore, the Raman cross-saturation is nearly as large as in the monochromatic case. The observed intensities are not large enough to produce saturation at such densities; but even if they were, the population cross-saturation which is usually present in the laboratory lasers does not exist for the ground state unless there is a faster rate of relaxing the Zeeman sublevel populations than for relaxing the population inversion. Contrary to the laboratory case, the pumping mechanism transfers population from one ground-state level to another, rather than drawing from a nearly unperturbed reservoir of population. The consequence is that the cross-saturation of gain for left-hand circularly polarized waves by population being transferred by the right-hand circularly stimulated emission might not even occur in the perturbation limit of weak saturation. The pumping acts to restore the population to the upper state by drawing from the lower state, a process which is not considered with a separate population reservoir.

The three-level analysis given below illustrates this point. Then, significant population cross-saturation occurs only when there is strong relaxation of population across the magnetic sublevels of the upper state or lower state. If this cross relaxation arises from strong resonance radiation trapping with the next-excited rotational states, then this cross-saturation of gain can nearly equal the self-saturation of the gain. Only a relatively small Raman cross-saturation or other effects, such as the parametric down-conversion to be described below, would be needed to make the system unstable to the presence of the oppositely polarized signals. If collisions are the cause of rapid relaxation of population among the Zeeman levels, the cross-saturation might still be small since the collisional rates are still nearly equal for hydrogen collisions for relaxing the effects of self- and cross-saturation. Electron, but perhaps not ion, collisions for cross-saturation by transferring Zeeman populations are too slow because of the magnetic dipole or electric quadrupole nature of the transitions among the Zeeman levels.

To illustrate the situation of two Λ-doublet maser states under the influence of pumping, broad-band microwave, and resonance radiation trapping or collisions to the next rotational states, we consider a simple three-sublevel system. These three sublevels could be the $M_F = \pm 1$ of an $F = 1$ state and $M_F = 0$ of either an $F = 1$ or 0 state. With propagation along the magnetic field these are the only states which are involved in the microwave transitions, the right-hand-polarized wave being absorbed in the transition from $M_F=0$, $F=0, 1$ to $M_F=+1, F=1$ if the latter state is chosen as the upper one. Note that in the case for which both are $F=1$ states and propagation is along the magnetic field, the microwave transitions involve a pair of three-sublevel systems. We consider that only a right-hand circularly polarized signal is present and ask for its effect on its own and the opposite polarization's gain coefficient.

If we call A the net rate of pumping from either the $M_F=+1$ or -1 upper states to the lower state $M_F=0$, B the net rate of pumping from the $M_F=0$ lower state to either upper sublevel, C the population transfer rate between $M_F=+1$ and -1 due to resonance radiation trapping or collisions, and W the microwave rate between the $M_F=+1$ and 0 states, then we find that the fractional population inversion $\Delta n_{10}/n$ between $M_F=+1$ and $M_F=0$ is

$$\frac{\Delta n_{10}}{n}=\frac{B-A}{A+2B+W(2A+B+3C)/(A+2C)}=\frac{\Delta n^0/n}{1+W(2A+B+3C)/(A+2C)\,(A+2B)}\ , \qquad (1)$$

while the fractional population inversion $\Delta n_{-10}/n$ between $M_F=-1$ and $M_F=0$ is

$$\frac{\Delta n_{-10}}{n}=\frac{B-A}{A+2B+W(A-B+3C)/(A+2C+W)}=\frac{\Delta n^0/n}{1+W(A-B+3C)/[\,(A+2C+W)\,(A+2B)\,]}\ , \qquad (2)$$

where

$$\Delta n^0/n=(B-A)/(A+2B)$$

is the unsaturated fractional population inversion. n is the total population in all three levels. For small W, that is, a weak microwave signal, we see by an expansion of Eq. (1) in powers of W that the self-saturation contribution to the fractional population inversion is

$$-W[\,(2A+B+3C)/(A+2C)\,(A+2B)\,]\,(\Delta n^0/n)\ , \qquad (3)$$

while the cross-saturation contribution, obtained from the expansion of Eq. (2) in powers of W, is

$$-W[\,(A-B+3C)/(A+2C)\,(A+2B)\,]\,(\Delta n^0/n)\ . \qquad (4)$$

Since $B>A$ for there to be any inversion, we find that unless $3C>B-A$, there is no cross-saturation, but there is, in fact, an enhancement of gain for the opposite polarization. Therefore, the fast transfer of population across the sublevels is essential for suppression of the opposite polarization. This is a characteristic of cases in which the population reservoir for pumping consists of the maser states themselves, as is the case for the ground state. Excited Λ-doublet states would not be of this type and would not require fast sublevel transfer for some, but probably not enough, cross-saturation. The cross-saturation contribution is less negative than the self-saturation contribution by an amount $[\,W/(A+2C)\,]\,(\Delta n^0/n)$. Then, additional cross-saturation due to some other mechanism such as the parametric down-conversion must overcome this deficit in order to suppress the opposite polarization. The faster the transfer rate C across the sublevels is, the smaller is the deficit to be overcome. However, the Raman-type cross-saturation decreases with C. Note that the criterion for self-saturation to occur, namely, that

$$W>(A+2B)\,(A+2C)/(2A+B+3C)\,,$$

is not very sensitive to this rate C. This calculation has neglected the influence of off-diagonal density-matrix elements and their interconnection with the diagonal elements, the populations, on the basis that the microwave signal is broad band and that the trapped resonance radiation is everywhere locally isotropic.[13]

If the Zeeman splitting is less than collision linewidths, the Heer nonlinear (monochromatic) analysis[10] applies. However, the magnetic splittings are likely to be larger than the collision linewidths in any reasonable range of densities n_H. The magnetic field B (tangential component) will be compressed with the gas until the magnetic pressure $\sim B^2/8\pi$ is not much less than the gas pressure $\sim n_H\kappa T$. If the compression is one dimensional, as in a shock front, then B is proportional to n_H to conserve flux. Similarly, if the compression is spherical, then B is proportional to $n_H^{2/3}$. In a protostar, the azimuthal field might be proportional to $n_H^{1/2}$. In fact, the Zeeman splitting may then be much greater than the Doppler widths. Then the second explanation offered is that the amplification for each Zeeman component can be much different because of velocity gradients and accidents of clouds in the line of sight with mean velocities and magnetic field directions just right for providing gain for one sense of circular polarization and not the other.[14] A similar explanation has been proposed for a protostar.[14a] However, this applies to propagation mostly parallel to the magnetic field. For other directions the polarizations are elliptical, contrary to the rarity of observed linear polarization components for OH. In both these explanations, microwave saturation cannot be large if the present estimates of brightness temperatures are correct. Yet, the similarity of intensities, despite strong polarization differences from various points in a given region, would not be expected unless saturation were dominant.

With these reservations about the collision-dominated regime, we have made the major effort here to explain the reasons that conditions in the radiation-dominated regime best explain the observed polarized emission. General nonlinear properties are derived for broad-band signals and large magnetic splittings.

III. STOKES PARAMETERS OF THE MODES

The propagation of intense polarized microwaves can be influenced by various nonlinear mechanisms in the case of a medium consisting of population-inverted OH molecules and plasma in a magnetic field. We consider the case for which the Zeeman splittings of the OH energy levels are larger than the inverse lifetimes of these levels. This is probably also true for H_2O, even though the g factor is 10^{-3} times smaller. Because of arguments given above, the dielectric susceptibility that is contributed by the OH is then determined mainly by the sublevel populations, the diagonal density-matrix elements, and not by various off-diagonal density-matrix elements that may be generated by the microwave signal. The Raman-type cross-saturation, important in Heer's description of polarization effects in laboratory lasers, involves such off-diagonal elements, which we now neglect for this broad-band case.

The dielectric susceptibility $\overleftrightarrow{\chi}$ including the magneto-plasma contribution has the form

$$\overleftrightarrow{\chi} = \begin{pmatrix} \chi_{11} & \chi_{12} & 0 \\ -\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix}$$

in a coordinate system with the dc magnetic field $\vec{B}_0$ along the x_3 axis.[15] It is convenient to define a gyrotopy parameter $\eta = 2\chi_{12}(\cos\theta)/\chi_{11}$ and an alignment parameter $\zeta = (\chi_{33} - \chi_{11})(\sin^2\theta)/\chi_{11}$, where θ is the angle between $\vec{B}_0$ and the wave vector $\vec{k}$. Then two modes at the same frequency are obtained from the usual Maxwell equations such that $\vec{k}\cdot(\overleftrightarrow{I} + 4\pi\overleftrightarrow{\chi})\cdot\vec{E}(k) = 0$ and $|\vec{k}| = (\omega/c) + \Delta k$, where

$$\Delta k \simeq [2 + \zeta \pm (\zeta^2 - \eta^2)^{1/2}]\,(\pi\omega/c)\,\chi_{11} \quad . \tag{5}$$

The gain coefficient is $\alpha = 2\,\mathrm{Im}\Delta k$, and the ratio of the electric field components transverse to $\vec{k}$ is

$$E_x/E_y \simeq -[\zeta \pm (\zeta^2 - \eta^2)^{1/2}]/\eta = -e^{\pm\psi} \tag{6}$$

in the coordinate system with $\vec{e}_y = \vec{e}_{B_0} \times \vec{e}_k$ and $\vec{e}_x = \vec{e}_y \times \vec{e}_k$. The $\vec{e}_i$ vectors are the unit vectors in the direction of $\vec{B}_0$, $\vec{x}$, $\vec{y}$, or $\vec{k}$.

The Zeeman splittings are determined by the values of the Landé factors g_F: $g_1 = 1.24$ and $g_2 = 0.74$ for the hyperfine states $F = 1$ and 2, respectively, for either upper or lower Λ-doublet OH ground states. The sublevels are denoted by $(F, M_F, +)$, where M_F is the component of $\vec{F}$ along $\vec{B}_0$ and + is the parity of the upper Λ-doublet state, and by $(F', M_{F'}, -)$ for the lower Λ-doublet state. The Zeeman splittings for the microwave fields of opposite circular polarization, $\vec{e}_\pm = (\vec{e}_x \pm i\vec{e}_y)/\sqrt{2}$, are $g_1\omega_H$ and $g_2\omega_H$ for the $F = 1 \rightarrow 1$ and $F = 2 \rightarrow 2$ transitions, respectively, where ω_H is the cyclotron frequency $eB_0/(mc)$. For OH, the Zeeman splitting is less than the Doppler width when $B_0 < 10^{-3}$ G. For H_2O, the Zeeman splitting is a factor of 10^{-3} less for the same field. The parameter ζ is due both to the difference of sublevel populations because of molecular alignment with respect to $\vec{B}_0$ and to the Zeeman splitting of the resonance frequencies for the different senses of polarization. The contribution to ζ from the magnetoplasma is usually negligible. Because of resonance radiation trapping between the ground OH state and the next rotational states, the ground sublevels are rapidly equilibrated, provided that this far-infrared radiation is nearly isotropic, which is very likely for the clouds of large optical depth that we are considering. Then, for small $\vec{B}_0$, ζ is not large compared to η, where η may be dominated by the contribution from the magnetoplasma once the OH contribution is heavily saturated by the microwave signal. In general, we have an elliptically polarized pair of modes which are not orthogonal because of the amplification properties of the medium, i.e., there is an anti-Hermitean component to $\overleftrightarrow{\chi}$.

Normalized Stokes parameters have been defined from Eq. (6) for the two separate modes:

$$\begin{aligned} s_0 &\equiv 1 \;, \\ s_1 &= \pm\tanh\psi' \simeq \mp\mathrm{Im}(\zeta/\eta), && \text{for small } \zeta/\eta \\ s_2 &= \cos\psi''\,\mathrm{sech}\psi' \simeq \mathrm{Re}(\zeta/\eta), && \text{for small } \zeta/\eta \\ s_3 &= \pm\sin\psi''\,\mathrm{sech}\psi' \simeq \pm 1\,, && \text{for small } \zeta/\eta \end{aligned} \tag{7}$$

where $\psi = \psi' + i\psi''$ is the complex angle defined in Eq. (6). At a given frequency ω, s_1 is the difference between the intensity for linear polarization along $\phi = 0$ and that along $\phi = \frac{1}{2}\pi$. s_2 is the difference between the intensity for linear polarization along $\phi = \frac{1}{4}\pi$ and that along $\phi = \frac{3}{4}\pi$, where ϕ is the azimuthal angle with respect to the x axis in the x-y plane. $\vec{k}$ is the polar axis. The magnetic field lies in the x-k plane. s_3 is the difference between the right-handed circularly polarized[16] intensity and the left-handed one, assuming $\cos\theta_k > 0$. For negative $\cos\theta_k$, right and left handedness should be interchanged. Only the sense of rotation about the magnetic field is important to the parametric process.

Without serious error, the quantity ζ/η may be represented by

$$\frac{\zeta}{\eta} \simeq \frac{i(\frac{1}{2}g_F\,\omega_H)\sin^2\theta/(2\cos\theta)}{\omega - \omega_F + i\Delta\omega} \quad \text{for } F \to F \quad , \tag{8}$$

when only *molecular* Zeeman splitting contributes to ζ and η. Here, ω_F is the resonant frequency for the transition $F \to F$ with no magnetic field and $\Delta\omega$ is the Doppler half-width at e^{-1} of maximum. More precisely, $\Delta\omega$ in the denominator of (8) should be multiplied by $(2i)^{-1}d\log I(z)/dz$, where

$$z = (\omega - \omega_F + i\Gamma)/\Delta\omega \ ,$$

$$I(z) = \int z_H^2 dz' z'^{-1}(z_H^2 - z'^2)^{-1}\exp[-(z-z')^2] \ .$$

Γ is the radiative or collisional damping rate and $z_H = g_F\,\omega_H/2\Delta\omega$.

If Φ and Ψ are the standard ellipse position and eccentricity angles for elliptical polarization, then

$$\tan 2\Phi = s_2/s_1 \quad , \quad \tan 2\Psi = s_3/(s_1^2 + s_2^2)^{1/2} \ , \tag{9}$$

where Φ is measured counterclockwise from the x axis toward the y axis. Equations (7) and (8) show that at line center the polarization ellipses lie at $\pm\frac{1}{4}\pi$ with respect to the x axis when $|\zeta/\eta| < 1$. When $|\zeta/\eta| > 1$, the modes are more linearly polarized. Such may be the case for the H_2O emission, for which molecular alignment (ζ) might be large and gyrotropy (η) due to the magnetic field very small, even for the magnetoplasma. Then the Stokes parameters are

$$s_1 \simeq \pm 1, s_2 \simeq [\mathrm{Re}(\zeta/\eta)]^{-1}, s_3 \simeq \pm[\mathrm{Im}(\zeta/\eta)][\mathrm{Re}(\zeta/\eta)]^{-2} \quad \text{for} \quad |\zeta/\eta| \gg 1 \ . \tag{10}$$

The difference of maser gain coefficients for the two modes is approximately $[4\pi\omega(\vec{k})/c]\,\mathrm{Im}(\zeta\chi_{11})$, as can be seen from Eq. (5).

IV. PARAMETRIC DOWN-CONVERSION

We discuss the propagation of polarized microwaves that interact with one another by various nonlinear mechanisms. The medium consists of population-inverted OH molecules and plasma in the presence of a magnetic field. The Zeeman splitting of the sublevels of the OH hyperfine states is assumed to be comparable to the Doppler linewidth. Microwaves may then be amplified in separate Zeeman modes. Each corresponds to an electric dipole transition between a particular pair of sublevels of opposite parity across the Λ-doublet. Also present is a low degree of ionization, which nevertheless contributes a large plasma term to the dielectric susceptibility. Two microwaves of different elliptical polarization in Zeeman modes differ in frequency by a Zeeman splitting or less. There is an electric field proportional to the amplitudes of both waves, which drives a current at the difference frequency. This current source causes the growth of an electron cyclotron wave whose wave vector is the difference between the two microwave vectors. Because of frequency dispersion such that the vectors cannot be collinear and, because of its short length, this difference vector is nearly perpendicular to the microwave vectors. An electron cyclotron wave consists of both electron gyromotion, which is transverse to the wave vector, and space-charge effects, which are parallel to the wave vector.[17] The latter becomes more important the closer the frequency of the wave approaches the resonance at $\omega_H\cos\theta$, where ω_H is the gyrofrequency and θ is the angle of propagation with respect to the magnetic field. The group velocity is perpendicular to the electric field, which will be mostly parallel to the wave vector for space-charge effects. Thus, the energy of this wave propagates nearly perpendicular to its wave vector or parallel to the energy flux of the microwaves. This is necessary for appreciable volume of interaction of the three waves.[18]

As is usual in calculating parametric coupling of waves, the Fourier transforms of the fields are most convenient. The Maxwell wave equation has the form

$$\overleftrightarrow{R}(k)\cdot\vec{E}(k) = -4\pi\vec{P}^{nl}(k) \ , \tag{11}$$

where

$$\vec{E}(k) = \int d^4x\,\vec{E}(x)e^{-ik\cdot x} \ , \tag{12}$$

$$\overleftrightarrow{R}(k) = \overleftrightarrow{\epsilon}(k) + k_0^{-2}\vec{k}\times(\vec{k}\times\overleftrightarrow{I}) \ , \tag{13}$$

$$\vec{P}^{nl}(k) = \int d^4p\,d^4q\,\delta^{(4)}(p+q-k)\,(2\pi)^{-4} \times \delta\underline{\chi}(k,p,q) : \vec{E}(p)\vec{E}(q) \ . \tag{14}$$

The quantity $\vec{E}(x)$ is the electric field, as a function of the four-vector $(\vec{x}, ict)$, while $\vec{P}^{nl}(k)$ is the Fourier transform of the polarization vector due to nonlinear effects, in this case, the coupling of two electric field components by the nonlinear susceptibility tensor $\delta\underline{\chi}(k,p,q)$. The quantities k, p, and q are four-vectors such that, for example, $k = (\vec{k}, ik_0)$, where $k_0 c$ is a frequency variable of integration, not necessarily equal to a specific frequency $\omega(\vec{k})$ identified with a mode of the linear medium with wave vector $\vec{k}$ and dielectric tensor $\overleftrightarrow{\epsilon}(k)$. The inner product is

$$k\cdot x = \vec{k}\cdot\vec{x} - k_0 ct \ .$$

The usual linear dielectric susceptibility is

$$\overleftrightarrow{\chi}(k) = (\overleftrightarrow{\epsilon}(k) - \overleftrightarrow{I})/4\pi \ ,$$

where $\overleftrightarrow{I}$ is the unit tensor.

In the analysis which follows, ion motions can be neglected, and the electron velocity in a mode is then given by

$$\vec{V}(q) = i\omega(\vec{q})\overleftrightarrow{\chi}(q)\cdot\vec{E}(q)/n_e e, \text{ with } q_0 = \omega(\vec{q})/c \quad , \quad (15)$$

where n_e is the electron density and e is the magnitude of electron charge. From the Maxwell induction equation, the magnetic field in a mode is given by

$$\vec{B}(q) = c\vec{q}\times\vec{E}(q)/\omega(\vec{q}), \text{ with } q_0 = \omega(\vec{q})/c \quad . \quad (16)$$

Specifically for a microwave mode, because of its high frequency, we have

$$\vec{E}(k)^*\cdot\overleftrightarrow{\chi}(k) \simeq -\vec{E}(k)^*\omega_0^2[4\pi\omega(\vec{k})^2]^{-1} \quad , \quad (17)$$

where ω_0 is the plasma frequency $(4\pi n_e e^2/m)^{1/2}$ and the asterisk denotes the complex-conjugate operation.

As will become clear below, the damping of the cyclotron wave occurs over a distance somewhat smaller than that for significant parametric interaction. Then, it is convenient to reserve the four-vector q for this wave and to eliminate the explicit appearance of the "driven" quantity $\vec{E}(q)$ in Eq. (14) by solving for $\vec{E}(q)$ from Eq. (11), after k is replaced by q, q is replaced by $-p'$, and p by k'. Because the fields $\vec{E}(x)$ are real valued, the relationship $\vec{E}(-p) = \vec{E}(p)^*$ holds. Then, Eq. (11) becomes

$$\begin{aligned}\overleftrightarrow{R}(k)\cdot\vec{E}(k) \simeq{}& 4(2\pi)^{-6}\int d^4p\, d^4q\, \delta^{(4)}(p+q-k) \\ &\times\int d^4k'\, d^4p'\, \delta^{(4)}(k'-p'-q)\,\delta\underline{\chi}(k,p,q) \\ &:\vec{E}(p)\overleftrightarrow{R}(q)^{-1}\cdot\delta\underline{\chi}(q,k',-p'):\vec{E}(k')\vec{E}(p')^* \quad . \end{aligned} \quad (18)$$

Only *microwave* electric fields are now exhibited, but the distinctive properties of the cyclotron wave are contained in the inverse tensor $\overleftrightarrow{R}(q)^{-1}$ and the explicit form of $\delta\underline{\chi}(q,k',-p')$ to be given below.

We will estimate the parametric gain coefficient when the microwave signals have Gaussian statistics over a bandwidth $\delta\omega$ which may be $\sim 10^4$ to $10^5\,\mathrm{sec}^{-1}$ for the OH and H_2O signals, respectively. We average pairs of electric field factors over random initial phases for traveling waves:

$$\begin{aligned}\langle E(p)_\beta E(p')^*_{\beta'}\rangle/8\pi ={}& \kappa T_B\,\pi^{1/2}\,\delta\omega\, g(\omega(\vec{p})-\omega)_{\beta\beta'} \\ &\times(2\pi)^5\,\delta^{(4)}(p-p')\,\delta(p_0-\omega(\vec{p})/c), \end{aligned} \quad (19)$$

where $g_{\beta\beta'}$ is the normalized spectral distribution function, usually Gaussian in shape because of maser amplification, with central frequency ω and width $\delta\omega = \pi^{-1/2}g(0)^{-1}$.
The tensor properties of $g_{\beta\beta'}$ describe the polarization of the mode, having components principally transverse to $\vec{p}$. The line-center brightness temperature T_B, taken independent of angle, is usually much greater than $\hbar\omega/\kappa$ where κ is the Boltzmann constant, so the Rayleigh-Jeans approximation to the blackbody formula has been used. In contrast, the bandwidth $2\delta\omega$ might correspond to the reciprocal of the time duration of individual coherent pulses in a steady train, for which the above averaging is inapplicable. Examination of data seems to exclude this case, however.

By using the various δ functions in Eqs. (18) and (19), we obtain

$$\begin{aligned}R(k)_{\alpha\rho}E(k)_\rho \simeq{}& 16\kappa T_B\,\pi^{1/2}\,\delta\omega\int d^3p\,\delta\chi(k,p,k-p)_{\alpha\beta\gamma} \\ &\times R(k-p)^{-1}_{\gamma\gamma'}g_{\beta\beta'}\,\delta\chi(k-p,k,-p)_{\gamma'\rho'\beta'}E(k)_{\rho'} \quad , \end{aligned} \quad (20)$$

where $p_0 = \omega(\vec{p})/c$, and $|\vec{p}| \simeq \omega(\vec{p})/c$ for microwaves since the index of refraction is close to unity. The summation convention on repeated subscripts is being used.

It can be shown that the response due to the cyclotron wave, as contained in the term $\overleftrightarrow{R}(k-p)^{-1}$, has a resonance in frequency that is narrow compared to $\delta\omega$. If the unit polarization vector $\vec{e}(\vec{q})$ for the cyclotron wave is introduced, the cyclotron wave component of $\overleftrightarrow{R}^{-1}$ is equivalent to

$$\begin{aligned}\vec{e}(\vec{q})^*\cdot\overleftrightarrow{R}(q)^{-1}\cdot\vec{e}(\vec{q}) \sim{}& [\omega(\vec{q})/2i(\cos^2\alpha_q)n_q^2] \\ &\times 2\pi\delta(\omega(\vec{q})-\omega_H\cos\theta_q) \quad , \end{aligned} \quad (21)$$

where $\vec{q} = \vec{k}-\vec{p}$ and $\omega(\vec{q}) = \omega(\vec{k})-\omega(\vec{p})$. The integration in Eq. (10) is over $\vec{p}$ for fixed $\vec{k}$. The angle α_q is that between $\vec{q}$ and the group velocity $\partial\omega(\vec{q})/\partial\vec{q}$. n_q is the index of refraction for the cyclotron wave, $n_q = |\vec{q}|c/\omega(\vec{q})$. We will use

$$\begin{aligned}n_q\cos\alpha_q \simeq{}& |\vec{q}|c[\cos^2\theta_q + (m/M)\,(1+|\vec{q}|^{-2}r_e^{-2})\,]^{1/2} \\ &\times(\omega_H\sin\theta_q\cos\theta_q)^{-1}\{2(1+|\vec{q}|^2r_e^2)^{-1} - m(M|\vec{q}|^2r_e^2)^{-1} \\ &\times[\cos^2\theta_q + (m/M)\,(1+|\vec{q}|^{-2}r_e^{-2})\,]^{-1}\} \quad , \end{aligned} \quad (22)$$

when $|\vec{q}|c_T \ll \omega_H\cos\theta_q$, where θ_q is the angle between $\vec{q}$ and $\vec{B}_0$, and c_T is the ion-gas sound velocity. m/M is the ratio of electron to the average ion mass, perhaps that of Na^+ in dense HI regions, and $r_e = c/\omega_0$ is the electron plasma length $(mc^2/4\pi n_e e^2)^{1/2}$. Equation (22a) is obtained from the dispersion relation given by Stringer that

$$\omega(\vec{q})^2 \simeq \omega_H^2\left[\cos^2\theta_q + \frac{m}{M}\left(1+\frac{1}{|\vec{q}|^2r_e^2}\right)\right] \times \left(1-\frac{1}{1+|\vec{q}|^2r_e^2}\right)^2 . \quad (23)$$

We will use

$$n_q\cos\alpha_q \simeq \frac{|\vec{q}|c\,\omega_H\sin\theta_q\,\omega_0^{-2}\,(|\vec{q}|\,\lambda_{\mathrm{De}})^2(1+|\vec{q}|^2\lambda_{\mathrm{De}}^2)^{-2}}{1-\omega_H^2\sin^2\theta_q\,\omega_0^{-2}(1+|\vec{q}|^2\lambda_{\mathrm{De}}^2)^{-1}} \quad , \quad (22')$$

when $|\vec{q}| \gg$ either $\omega_H\cos\theta_q/c_T$ or $\lambda_{\mathrm{De}}^{-1}$, where the electron Debye length λ_{De} equals $(kT_e/4\pi n_e e^2)^{1/2}$. Equation (22′) is obtained from the dispersion relation given by Stringer that

$$\omega(\vec{q})^2 \simeq \omega_H^2\cos^2\theta_q\,[1-\omega_H^2\sin^2\theta_q\omega_0^{-2}(1+|\vec{q}|^2\lambda_{De}^2)^{-1}]\,. \tag{23′}$$

Collisional, Landau, and cyclotron damping may be important and will be dealt with below.

If $\delta\omega$ were narrower than the damping rate for the cyclotron wave or if these were coherent microwave pulses, then we would use

$$\vec{e}(\vec{q})^*\cdot\overleftrightarrow{R}(q)^{-1}\cdot\vec{e}(\vec{q})\sim\omega(\vec{q})^2/(ic^2K_q|\vec{q}|\cos\alpha_q)\ , \tag{21′}$$

where $K_q=\gamma_q n_q\cos\alpha_q/c$ is the attenuation coefficient (cm^{-1}) for the cyclotron wave and the damping rate (sec^{-1}) γ_q is nearly equal to the electron collision frequency, when collisional damping dominates.

As discussed earlier, the source of nonlinear polarization with four-vector k is driven by the $\vec{V}\times\vec{B}$ force generated by the waves with p and q:

$$\delta\underline{\chi}(k,p,q):\vec{E}(p)\vec{E}(q) = \overleftrightarrow{\chi}(k)\cdot[\vec{V}(p)\times\vec{B}(q)+\vec{V}(q)\times\vec{B}(p)]c^{-1}\,. \tag{24}$$

The term involving $\vec{V}(p)\times\vec{B}(q)$ actually can be neglected when q denotes the cyclotron wave. Then, from Eq. (24), we may write

$$\delta\chi(k,p,q)_{\alpha\beta\gamma}=[i\omega(\vec{q})\,e/m\omega(\vec{k})^2\omega(\vec{p})] \times[p_\alpha\chi(q)_{\beta\gamma}-\delta_{\alpha\beta}p_{\beta'}\chi(q)_{\beta'\gamma}]\ . \tag{25}$$

Then, with the help of Eqs. (15) and (16), the energy component becomes

$$\vec{E}(k)^*\cdot\delta\underline{\chi}(k,p,q):\vec{E}(p)\vec{E}(q) \simeq[i\omega(\vec{q})\,e/m\omega(\vec{k})^2\omega(\vec{p})][\vec{E}(k)^*\cdot\vec{p}\,\vec{E}(p) \cdot\overleftrightarrow{\chi}(q)\cdot\vec{E}(q)-\vec{E}(k)^*\cdot\vec{E}(p)\vec{p}\cdot\overleftrightarrow{\chi}(q)\cdot\vec{E}(q)]\ . \tag{26}$$

The last term is the dominant one. Similarly,

$$\delta\chi(q,k,-p)_{\gamma\rho\beta}\simeq[(ie/m)/\omega(\vec{k})\omega(\vec{p})][\chi(q)_{\gamma\sigma}q_\sigma\delta_{\rho\beta} -\chi(q)_{\gamma\rho}q_\beta-\chi(q)_{\gamma\beta}q_\rho]\ . \tag{27}$$

The last two terms will be the dominant ones.

The tensor $R(k)_{\alpha\beta}$, having to do with microwave signals, has a value for a particular mode, denoted by subscript α:

$$R(k)_{\alpha\alpha}\simeq -i|\vec{k}|Kk_0^{-2}\ , \tag{28}$$

when evaluated at $k_0=\omega(\vec{k})/c$, where K is the microwave *gain* coefficient (cm^{-1}) for the parametric process and for maser amplification, should both processes be occurring in the same region, and where $\omega(\vec{k})$ is the frequency associated with this mode and wave vector $\vec{k}$. The OH molecules contribute to the mocrowave complex dielectric susceptibility. Modes polarized almost entirely transverse to $\vec{k}$ can be found, even in the presence of microwave saturation. However, the polarization is generally elliptical and a complicated function of frequency within the Doppler width of the OH transition. We have previously allowed for this by introducing the tensor $g(\omega(\vec{p})-\omega)_{\beta\beta'}$ in Eqs. (19) and (20). However, to numerically estimate the parametric gain we will have to be more specific. We consider $\vec{E}(k)$ to belong to a particular mode having a unit polarization vector $\vec{e}(\vec{k})$. Then, from Eqs. (20), (21), (25), (27), and (28) the parametric gain coefficient is given by

$$K\simeq\frac{16\pi\kappa T_B\pi^{1/2}\,\delta\omega}{\omega(\vec{k})|\vec{k}|}\left(\frac{e}{mc}\right)^2\int d^3p\,\delta(\omega(\vec{k})-\omega(\vec{p})-\omega_H\cos\theta_q) \times\frac{\omega(\vec{q})^2A(k,p,k\text{-}p)g(\omega(\vec{p})-\omega)}{\omega(\vec{p})^2n_q^2\cos^2\alpha_q}\,, \tag{29}$$

where

$$A(k,p,k\text{-}p)=\mathrm{Re}\,e(\vec{k})^*_\alpha[p_\alpha\chi_{\beta\gamma}-\delta_{\alpha\beta}p_{\sigma'}\chi_{\sigma'\gamma}] \times g(\omega(\vec{p})-\omega)_{\beta\beta'}e(\vec{q})_\gamma e(\vec{q})_{\gamma'}^*[\chi_{\gamma'\sigma}q_\sigma\delta_{\rho'\beta'} -\chi_{\gamma'\rho'}q_{\beta'}-\chi_{\gamma'\beta'}q_{\rho'}]\,e(\vec{k})_{\rho'}g(\omega(\vec{p})-\omega)^{-1} \simeq|\chi_{R,\parallel}|^2|\vec{p}|\,|\vec{q}|\,s(\vec{k},\vec{p})\cos\tfrac{1}{2}\theta_{kp}\ ,$$

where

$$s(\vec{k},\vec{p})=[s_1(\vec{p})s_3(\vec{k})-s_1(\vec{k})s_3(\vec{p})]\cos2\phi_q +[s_2(\vec{p})s_3(\vec{k})-s_2(\vec{k})s_3(\vec{p})]\sin2\phi_q\ .$$

The quantities s_1, s_2, and s_3 are the normalized Stokes parameters for linear and circular polarization in the modes. $g(\omega(\vec{p})-\omega)$ is the normalized Gaussian that represents the intensity profile in frequency. The angle ϕ_q is that between the plane formed by $\vec{k}$ and $\vec{q}$ and the plane formed by $\vec{k}$ and $\vec{B}_0$. The angle θ_{kp} is between $\vec{k}$ and $\vec{p}$. The χ's appearing in the expression for A refer to the cyclotron wave with wave vector $\vec{q}$. We have taken only the largest element of the tensor, $\chi_{R,\parallel}$, one component parallel to $\vec{q}$ and the other, right circularly polarized around $\vec{q}$, as discussed by Stringer, where

$$\chi_{R,\parallel}\simeq(-\omega_0^2/\sqrt{2}\,\pi\omega_H^2\sin2\theta_q)f(|\vec{q}|\lambda_{De})\ , \tag{30}$$

where $f(|\vec{q}|\lambda_{De})\simeq1$ when

$$r_e^{-1}<|\vec{q}|\leq\omega_H\cos\theta_q/c_T \text{ or } \lambda_{De}^{-1}\,,$$
$$f(|\vec{q}|\lambda_{De})\simeq2+(|\vec{q}|\lambda_{De})^2\ ,$$

when $|\vec{q}|\gg$ either $\omega_H\cos\theta_q/c_T$ or λ_{De}^{-1}. The element $\chi_{\parallel,\parallel}$ is smaller by a factor of $\frac{1}{2}(\omega_H/\omega_0)^2\sin2\theta_q$ in the first case and $\frac{1}{2}(\omega_H/\omega_0)^2\sin2\theta_q(1+|\vec{q}|^2\lambda_{De}^2)^{-1}$ in the second case. ω_H/ω_0 is greater than unity only when $B_0n_e^{-1/2}>3\times10^{-3}$ G- cm$^{3/2}$ in special regions of high field and mass density but low fractional ionization. We see from Eq. (29) that $A(k, p, k-p)$ is a factor $|\vec{q}|/|\vec{p}|$ smaller when $\chi_{\parallel,\parallel}$ is used. We will neglect the $\chi_{\parallel,\parallel}$ element henceforth and assume $\omega_H/\omega_0 \ll 1$ when $|\vec{q}| < \lambda_{De}^{-1}$.

Finally, we obtain for the parametric gain

$$K \simeq \frac{\kappa T_B (e/mc)^2 \omega_0^4}{2\pi\omega(\vec{k})^3 \omega_H^2} \int d^3p \, \delta(\omega(\vec{k}) - \omega(\vec{p}) - \omega_H \cos\theta_q) \times \frac{|\vec{q}|\pi^{1/2}\delta\omega g(\omega(\vec{p}) - \omega) s(\vec{k},\vec{p}) \cos\frac{1}{2}\theta_{kp}}{(n_q \cos\alpha_q)^2}, \qquad (31)$$

where

$$d^3p = \sin\theta_{kp} d\theta_{kp} d\phi_q \omega(\vec{p})^2 d\omega(\vec{p})/c^3.$$

We have used

$$|\vec{p}|/|\vec{k}| \simeq 1, \; \omega(\vec{p})^2/\omega(\vec{k})^2 \simeq 1, \; \omega(\vec{q})^2/\omega_H^2 \cos^2\theta_o \simeq 1$$

to obtain Eq. (31) from (29). For the magnitude $|\vec{q}|$ we have

$$|\vec{q}| \simeq 2|\vec{k}| \sin\tfrac{1}{2}\theta_{kp} \;.$$

The integral over ϕ_q is complicated by the dependence of s and g on ϕ_q through

$$\omega(\vec{p}) = \omega(\vec{k}) - \omega_H \cos\theta_q \;.$$

Also, for the small $|\vec{q}|$ that is being considered, we have

$$\cos\theta_q \simeq -\sin\theta_k \cos\phi_q \;.$$

The polarization factor becomes approximately

$$s(\vec{k},\vec{p}) \simeq -\operatorname{sgn}(\cos\phi_q)[(s_1(\vec{k}) + s_1(\vec{p})) \times \cos 2\phi_q + (s_2(\vec{k}) + s_2(\vec{p}))\sin 2\phi_q]$$

because s_3 is approximately $+1$ for the mode associated with the higher Zeeman frequency and -1 for the lower-frequency one, when the complex-valued ratio,

$$\zeta/\eta = (\chi_{33} - \chi_{11})(\sin^2\theta)/(2\chi_{12}|\cos\theta|) \;,$$

is small, where θ equals θ_k or θ_p, as is appropriate. The quantity $\operatorname{sgn}(\cos\phi_q)$ denotes the algebraic sign of $\cos\phi_q$. One can show for small ζ/η that $s_1 \simeq -\operatorname{Im}(\zeta/\eta)$ and $s_2 \simeq \operatorname{Re}(\zeta/\eta)$. If ζ/η is determined mainly by Zeeman splitting that is comparable to the molecular Doppler width $\Delta\omega$ (half-width at the e^{-1} point), then

$$s(\vec{k},\vec{p}) \simeq \operatorname{sgn}(\cos\phi_q) g_F \omega_H \operatorname{Re}\left[e^{i2\phi_q}\left([\omega(\vec{k}) - \omega_F + i\Delta\omega]^{-1} \times \frac{\sin^2\theta_k}{2|\cos\theta_k|} + (\omega(\vec{p}) - \omega_F + i\Delta\omega)^{-1} \frac{\sin^2\theta_p}{2|\cos\theta_p|}\right)\right].$$

We will simplify $s(\vec{k},\vec{p})$:

$$s(\vec{k},\vec{p}) \simeq \operatorname{sgn}(\cos\phi_q) g_F \omega_H \sin^2\theta_k \sin 2\phi_q (\Delta\omega|\cos\theta_k|)^{-1}$$

by assuming that $\theta_p \simeq \theta_k$ and that the reasonant frequency factors are merely $(i\Delta\omega)^{-1}$. This is consistent with small Zeeman splittings and a small frequency separation of the peaks in the maser gain for the two modes.

In order to avoid excessive Landau damping of the cyclotron wave, we have $|\vec{q}|\lambda_{De} < \omega_H/\omega_0$.[19] Cyclotron resonance damping and departures from "cold-plasma" dispersion relations will be entirely negligible only when

$$|\cos\theta_q| < (1 + |\vec{q}| v_T/\omega_H)^{-1},$$

where v_T is the electron thermal velocity $(2kT_e/m)^{1/2}$.[19] For the present application to broad-band microwave signals, as long as the microwave bandwidth $\delta\omega$ exceeds the damping rate γ_q, the cyclotron damping is not very important. However, when

$$|\vec{q}\cos\theta_q| v_T/\omega_H > 1,$$

dispersive effects (omitted by Stringer) due to the electron Doppler width will cause $\omega(\vec{q})$ to deviate from $\omega_H \cos\theta_q$ and will reduce $|\chi_{R,\parallel}|$ and n_q^2 each by a factor

$$\sim \pi^{1/2}\omega_H(\sin^2\theta_q)/(2|\vec{q}\cos\theta_q| v_T) \;.$$

Since this factor is usually of order unity, we will not cause further, possibly unnecessary, complications by including it.

The integration in Eq. (31) over ϕ_q then involves

$$\int d\phi_q \sin 2\phi_q g(\omega(\vec{p}) - \omega)\sqrt{\pi}\, \delta\omega \;,$$

such that

$$|\sin\theta_k| \geq |\sin\theta_k \cos\phi_q| \geq |\vec{q}| c_T/\omega_H \;,$$

when $|\vec{q}| c_T/\omega_H \leq 1$. However, when $|\vec{q}|\lambda_{De}$ or $|\vec{q}| c_T/\omega_H > 1$, the integrand contains a factor $(\sin\theta_q)^{-2}$ in addition, and only the constraint $|\cos\theta_q| < |\sin\theta_k|$ applies. The frequency splitting of the peaks in the maser gains of the oppositely polarized modes is $2\epsilon\Delta\omega$, where

$$\epsilon \simeq \operatorname{sgn}(\cos\phi_q) g_F \omega_H \cos\theta_k / 2\Delta\omega,$$

if χ_{12} is determined by molecular Zeeman splitting; but, if it is determined by the magnetoplasma, we have

$$\epsilon \simeq \operatorname{sgn}(\cos\phi_q)|\zeta^2/\eta(2+\zeta)|,$$

with[15]

$$\zeta = (\chi_{33} - \chi_{11})\sin^2\theta_k/\chi_{11}.$$

Then, we obtain $\omega = \omega_F + \epsilon\Delta\omega$. For small ϵ the dependence of g on ϕ_q will be neglected. The above integral in the first case gives $1 - |\vec{q}|^2 c_T^2/(\omega_H^2 \times \sin^2\theta_k)$, and in the second case gives $\sim \ln(\cos\theta_k)^{-2}$.

For very large $|\vec{k}|\lambda_{De}$ or $|\vec{k}| c_T/\omega_H$, the integration in Eq. (31) over θ_{kp}, then involves

$$\int_0^{\theta_m} d\theta_{kp} \sin\theta_{kp} \cos\tfrac{1}{2}\theta_{kp} |\vec{q}|^{-5} \lambda_{De}^{-4} (1+|\vec{q}|^2\lambda_{De}^2)^4 \times (2+|\vec{q}|^2\lambda_{De}^2)^2 \ln(\cos\theta_k)^{-2} .$$

If the parametric gain is to be calculated at some distance r from a maser source whose apparent diameter is l_*, then $\theta_m \simeq l_*/2r$, which we will assume is $\lesssim 1$. If $|\vec{k}| l_* \lambda_{De}/r$ is still $\gg 1$, then we can calculate the integral in the large $|\vec{q}|$ limit and obtain $\sim(\frac{1}{9}\theta_m)(\theta_m |\vec{k}|\lambda_{De})^8/|\vec{k}|$. If $|\vec{k}| l_* \lambda_{De}/r \ll 1$, then we calculate the integral in the small - $|\vec{q}|$ limit, provided that

$$|\vec{q}| > \omega_H |\cos\theta_q| / c_T \quad \text{or} \quad \lambda_{De}^{-1},$$

as mentioned earlier. If $|\vec{q}|$ is too small, the dispersion relation [Eq. (23′)] and the various quantities based on it no longer apply, although (22) and (23) do. Dispersive effects when

$$|\vec{q}\cos\theta_q| v_T/\omega_H > 1,$$

partially omitted by Stringer, may be particularly severe when $|\vec{q}|\lambda_{De} > 1$. However, the parametric gain coefficient, an overestimate, given below for $|\vec{q}|\lambda_{De} > 1$, is much smaller than that for $|\vec{q}|\lambda_{De} \ll 1$. Thus, for our present purposes there is no need to further complicate the analysis.

When

$$2|\vec{k}| \sin\tfrac{1}{2}\theta_m < \omega_H (\sin\theta_k)/c_T,$$

we use Eq. (22) so that the integral over θ_{kp} in Eq. (21) then involves

$$\int_0^{\theta_m} d\theta_{kp} \sin\theta_{kp} \cos\tfrac{1}{2}\theta_{kp} \int d\phi_q \sin 2\phi_q \cos^2\theta_q \times \left(\cos^2\theta_q + \frac{m}{M}\right)^{-1} \left[2 - \frac{m}{M}\left(\cos^2\theta_q + \frac{m}{M}\right)^{-1}\right]^{-2} \frac{(|\vec{q}| r_e)^4}{|\vec{q}|}.$$

The integration over ϕ_q is such that

$$|\vec{q}| c_T/\omega_H < |\cos\theta_q| \lesssim |\sin\theta_k| \quad .$$

Upon neglecting m/M, we obtain for the above integral the approximate value $\theta_m (r_e |\vec{k}| \theta_m)^4 (10|\vec{k}|)^{-1}$, assuming that $\omega_H \sin\theta_k / |\vec{k}| c_T$ is larger than $l_*/2r$.

We now evaluate the parametric gain coefficient to be

$$K \simeq \kappa T_B \left(\frac{e}{mc}\right)^2 \theta_m^5 |\vec{k}|^2 (5\pi c^2)^{-1} \frac{g_F \omega_H \sin^2\theta_k}{\Delta\omega |\cos\theta_k|} \quad ,$$

$$\text{for} \quad \theta_m |\vec{k}| c_T/\omega_H \sin\theta_k < 1 \tag{32a}$$

$$K \simeq \kappa T_B (e/mc)^2 (v_T/c)^8 (|\vec{k}| c/\omega_H)^4 (18\pi c^2)^{-1} \theta_m^9 \times |\vec{k}|^2 (g_F \omega_H \ln(\cos\theta_k)^{-2}/\Delta\omega |\cos\theta_k|) \quad ,$$

$$\text{for} \quad \theta_m |\vec{k}| \lambda_{De} \text{ or } \theta_m |\vec{k}| c_T/\omega_H > 1. \tag{32b}$$

For the following values of the parameters ,

$$g_F \omega_H \sin^2\theta_k / \Delta\omega |\cos\theta_k| \simeq 1,$$

corresponding to

$$\omega_H \simeq 3\times 10^3 \text{ sec}^{-1}, \text{ for the OH case},$$

$$\omega(\vec{k}) \simeq 10^{10} \text{ sec}^{-1},\ T_B = 10^{11}\ ^\circ\text{K},\ T = 10\ ^\circ\text{K},$$

we obtain from Eq. (32a), for $\theta_m \simeq 1$,

$$K_{OH} \simeq 3\times 10^{-14} \text{ cm}^{-1} \quad .$$

Similarly, for

$$g_F \omega_H \ln(\cos\theta_k)^{-2}/\Delta\omega |\cos\theta_k| \simeq 3\times 10^{-4},$$

corresponding to

$$\omega_H \simeq 3\times 10^4 \text{ sec}^{-1},$$

and nearly equal g factors for upper and lower levels, for the H_2O case

$$\omega(\vec{k}) \simeq 1.4\times 10^{11} \text{ sec}^{-1},\ T_B = 10^{13}\ ^\circ\text{K},\ T = 1000\ ^\circ\text{K},$$

we obtain from Eq. (23b), for

$$|\vec{k}| c_T/\omega_H \simeq 10 \text{ and } \theta_m \simeq 1,$$

$$K_{H_2O} \simeq 3\times 10^{-15} \text{ cm}^{-1} \propto T^{7/2} B^{-3} \quad ,$$

while from Eq. (32a), we obtain a more favorable value for $\omega_H \simeq 10^5 \text{ sec}^{-1}$, $T = 100\ ^\circ\text{K}$, and $\theta_m \simeq 1$, namely,

$$K_{H_2O} \simeq 3\times 10^{-12} \text{ cm}^{-1} \quad .$$

We note the importance of small values of $\cos\theta_k$ and no explicit dependence on n_e for both cases. However, to apply Eq. (32a) we must have

$$\omega_H < \omega_0 \quad \text{or} \quad n_e > 10(B_0/10^{-2}\text{ G})^2 \text{ cm}^{-3}.$$

Because of the observed maser properties, sizes about 10^{16} cm and densities of 10^8–10^9 cm^{-3} for the H_2O region are expected. Fields of 10^{-3}–10^{-2} G would correspond to fluxes per unit mass, a conserved quantity during collapse of a protostar, that lie between the typical values 10^{-4}–10^{-3} G cm^2/g.[20] The H_2O emission is distinguished from the OH emission by the importance of linear polarization in the H_2O case when polarization is present, and circular polarization for OH, at least for most of the sources associated with HII regions. As already suggested for infrared stars with OH and H_2O maser emission, a complicated flow with one or more shockfronts may be present.[21] Because the component of the field that is parallel to the front is increased as the density is increased, the denser OH and H_2O maser medium might involve propagation perpendicular to both the shockfront and the largest components of the magnetic field, although the general flow might be nearly along the undisturbed field, in filaments. The shockwave might have the further role of allowing the formation of H_2O from OH and H_2 in atom-exchange reactions. The added effect of heavy saturation for OH emission is conducive to circular polarization, because of small alignment but large gyrotopy properties due to the (unsaturated) magnetoplasma.[15] With less saturation for H_2O emission, molecular alignment could easily override in importance the weak Zeeman-splitting effects and perhaps even the magnetoplasma gyrotopy. At line center, in order to have linearly polarized modes, we must have a sufficiently low electron density,[15] i.e.,

$$n_e < \frac{|\zeta|\Delta n A m c^3}{16\delta\nu\omega_H e^2 \cos\theta_k} \simeq \left(\frac{|\zeta|\Delta n}{\cos\theta_k}\right) 10^3 \text{ cm}^{-3} \quad ,$$

with $\delta\nu \simeq 10^5$ Hz, $A \simeq 2\times10^{-9}$ sec^{-1}, and $\omega_H \simeq 10^5$ sec^{-1}. Since the population inversion density, $\Delta n \simeq 0.1$–1 cm^{-3}, according to estimates of the amplification and the amplifier length, and since perhaps $|\zeta| \sim 0.01$, the bound on the electron density seems easily satisfied if $n_e < 1$ cm^{-3}. For this case, using Eq. (10) and the definition of $s(\vec{k},\vec{p})$ following Eq. (19), and using

$$\chi_{12} = i\omega_0^2\omega_H/(4\pi\omega^3)$$

due to the magnetoplasma, we obtain

$$s(\vec{k},\vec{p}) \simeq -\operatorname{sgn}(\operatorname{Im}\zeta\chi_{11})[s_3(\vec{k}) \pm s_3(\vec{p})]\cos 2\phi_q$$
$$\simeq \frac{\omega_0^2\omega_H^2\cos\theta_k\sin\theta_k\cos\phi_q\cos 2\phi_q}{|\operatorname{Im}(\zeta\chi_{11})|2\pi\omega(\vec{k})^3\Delta\omega} \quad ,$$

where the upper sign applies when $\vec{k}$ and $\vec{p}$ belong to opposite modes and the lower sign when they belong to the same modes. We have also used

$$\omega(\vec{k}) - \omega(\vec{p}) = \omega_H\cos\theta_q \ , \ \cos\theta_q \simeq -\sin\theta_k\cos\phi_q \ ,$$

and $\theta_k \simeq \theta_p$

to obtain the above results. The factor multiplying the trigonometric functions is $\sim 10^{-4} n_e (B_0/10^{-2}\text{ G})^2 (\Delta\alpha/\alpha)^{-1}$, where $\Delta\alpha/\alpha$ is the fractional difference of gain for the opposite linearly polarized modes. There is parametric gain for down-converting either mode into the same or the other mode. This will mainly cause a shifting to lower frequency of the more intense linearly polarized component by some fraction of the Doppler width, perhaps causing some of the observed asymmetry of the H_2O line shapes. This applies to any one of the hyperfine-split H_2O transitions in particular. Since the hyperfine splitting and the Doppler width are often comparable, there may be confusion about whether the signal actually belongs to a lower-frequency hyperfine transition or is a down-shifted component.

V. SUMMARY

Under suitable conditions of low temperature and high magnetic field, the parametric down-conversion of one microwave mode to the other mode and to an electron cyclotron wave seems capable of qualitatively accounting for circular polarization, or more generally, elliptical polarization in the OH emission. Because the signal may be propagating outward from an apparent source whose dimensions are much smaller than those of the maser amplifier, the parametric gain coefficient drops sharply with distance. In order for this down-conversion to compete with the tendency toward equal intensities of right- and left-handed circular polarization due to maser saturation effects for broad-band signals, the maser amplification would have to be well saturated over most of the distance from the apparent source. This implies that the lengths for unsaturated growth are less than the apparent size. That this is violated for H_2O emission may be indicated by the result that under similar conditions, but of high field ($>10^{-2}$ G) and electron density (>10 cm^{-3}), the calculated parametric down-conversion would be strong enough to produce similar tendencies toward elliptical polarization for H_2O, but this is inconsistent with observations of very little polarization. Since the Zeeman splitting is probably very small compared to the Doppler widths for H_2O, perhaps molecular alignment due to directional pumping produces the observed small amounts of linear polarization. Parametric down-conversion weakly shifts the emission to lower frequencies by an amount comparable to the Doppler width of the molecules, but the preference of maser amplification for one sense of linear polarization would be derived mainly from the molecular alignment.

ACKNOWLEDGMENTS

Numerous fruitful discussions with Dr. H. J. Zeiger, Professor A. L. McWhorter, and Professor A. Javan are gratefully acknowledged.

[†]Work sponsored by the Dept. of the Air Force.

[1]A. H. Barrett, Science 157, 881 (1967); Sci. Am. 219, 36 (1968).

[2]B. J. Robinson and R. X. McGee, *Annual Reviews of Astronomy and Astrophysics* (Annual Reviews, Palo Alto, Calif., 1967), Vol. 5, p. 183.

[3]M. M. Litvak, Science 165, 855 (1969).

[4]J. M. Moran, B. F. Burke, A. H. Barrett, A. E. E. Rogers, J. A. Ball, J. C. Carter, and D. D. Cudaback, Astrophys. J. (Letters) 152, L97 (1968); B. F. Burke, D. C. Papa, G. D. Papadopoulos, P. R. Schwartz, S. H. Knowles, W. T. Sullivan, M. L. Meeks, and J. M. Moran, *ibid.* 160, L63(1970). More recently they have been found to be ten or more times smaller still [J. M. Moran, private communication].

[5]A. C. Cheung, D. M. Rank, C. H. Townes, D. D. Thornton, and W. J. Welch, Nature 221, 626 (1969); S. H. Knowles, C. H. Mayer, A. C. Cheung, D. M.

Rank, and C. H. Townes, Science 163, 1055 (1969); M. L. Meeks, J. C. Carter, A. H. Barrett, P. R. Schwartz, J. W. Waters, and W. E. Brown III, *ibid.* 165, 180 (1969); D. Buhl, L. E. Snyder, P. R. Schwartz, and A. H. Barrett, Astrophys. J. (Letters) 158, L97 (1969); B. E. Turner, D. Buhl, E. B. Churchwell, P. G. Mezger, and L. E. Snyder, Astron. Astrophys. 4, 165 (1970).

[6]B. Zuckerman, P. Palmer, H. Penfield, and A. E. Lilley, Astrophys. J. (Letters) 153, L69 (1968); J. L. Yen, B. Zuckerman, P. Palmer, and H. Penfield, *ibid.* 156, L27 (1968); P. R. Schwartz and A. H. Barrett, *ibid.* 157, L109 (1969).

[7]P. Palmer and B. Zuckerman, Astrophys. J. 148, 727 (1967); J. A. Ball and M. L. Meeks, *ibid.* 153, 577 (1968); W. A. Coles, V. H. Rumsey, W. J. Welch, Astrophys. J. (Letters) 154, L61 (1968); E. Raimond and B. Eliasson, Astrophys. J. 155, 817 (1969); W. A. Coles and V. H. Rumsey, *ibid.* 159, 247 (1970).

[8]W. Culshaw and J. Kannelaud, Phys. Rev. 145, 257 (1966); W. J. Tomlinson and R. L. Fork, *ibid.* 164, 466 (1967); 180, 628 (1969); M. Sargent III, W. E. Lamb, Jr., and R. L. Fork, *ibid.* 164, 436 (1967); 164, 450 (1967); C. H. Wang and W. J. Tomlinson, *ibid.* 181, 115 (1969); C. H. Wang, W. J. Tomlinson, and R. T. George, Jr., *ibid.* 181, 125 (1969); H. W. DeLang and G. Bouwhuis, Phys. Letters 20, 383 (1966).

[9]W. M. Doyle and M. B. White, Phys. Rev. 147, 359 (1966).

[10]C. V. Heer and R. D. Graft, Phys. Rev. 140, A1088 (1965); C. V. Heer and R. A. Settles, J. Mol. Spectry. 23, 448 (1967); C. V. Heer and J. R. Bupp, Phys. Letters 26A, 354 (1968).

[11]H. R. Schlossberg and A. Javen, Phys. Rev. 150, 267 (1966).

[12]P. L. Bender, Phys. Rev. Letters 18, 562 (1967).

[13]If the state which is excited by the resonance radiation has a decay coefficient for spontaneous emission which is greater than the Zeeman-splitting frequencies, then quantum-coherent effects among the sublevel wave functions can be carried from the ground state back to the ground state despite the intervening process of spontaneous emission. We have for the moment neglected such processes even though the near-infrared and ultraviolet excited states will fall into this category for interstellar magnetic field strengths.

[14]A. H. Cook, Nature 211, 503 (1966).

[14a]I. S. Shklovskii, Astron. Zh. 46, 3 (1969) [Soviet Astron AJ 13, 1 (1969)].

[15]M. M. Litvak, A. L. McWhorter, M. L. Meeks, and H. J. Zeiger, Phys. Rev. Letters 17, 821 (1966).

[16]Right circular polarization involves rotation of the electric vector in a clockwise sense when viewed along the direction of propagation, according to the standard radio definition.

[17]T. E. Stringer, J. Nucl. Energy Pt. C 5, 89 (1963).

[18]N. Bloembergen, *Nonlinear Optics* (Benjamin, New York, 1965), pp. 96–100.

[19]T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill, New York, 1962), pp. 193–196; V. L. Ginzburg, *The Propagation of Electromagnetic Waves in Plasmas* (Addison-Wesley, Reading, Mass., 1964), pp. 125–148.

[20]G. L. Verschuur, Phys. Rev. Letters 21, 775 (1968); Astrophys. J. (Letters) 155, L155 (1969).

[21]M. M. Litvak, Bull. Am. Astron. Soc. 2, 206 (1970).