stancy of $I(k)$, we suggest the soluble model of two-particle scattering where both the potential acting between the two particles and the metric potential are square wells with the same range R. Choosing η_f so that $R+\eta_f = (m+1/2)\pi$, $I(k)$ is the pole expansion of two cotangent functions $[(\cot z)/z]$. The constancy of $I(k)$ reflects the periodicity π of the cotangent function.

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Comments on the Faddeev Approach to Three-Particle Coulomb Systems

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Expressions correlating the transition operators to Faddeev's $T^{(i)}$'s are given for a three-body scattering problem. Using the results of Ball, Chen, and Wong, a revised partial-wave analysis for various transition processes in a three-particle Coulomb system is suggested. Some of the apparent contradictions between the off-shell two-particle partial-wave T matrix derived by Ball *et al.* and the full two-particle Coulomb T matrix derived by Nutt are resolved. In an Appendix, it is shown that $T^{(i)}$ is related to the transition operators. This establishes the equivalence between the limiting procedure of Ball *et al.* and the conventional procedure for obtaining the transition amplitudes.

I. INTRODUCTION

In recent years Faddeev's formulation[1] of the three-body problem has stimulated a great amount of interest in the study of three-particle Coulomb systems.[2–9] In principle, this approach gives a basically rigorous method for the solution of three-body problems like electron-hydrogen and proton-hydrogen scattering. However, some of the recent papers[3,4] on the partial-wave analysis of three-particle Coulomb systems do not seem to be consistent with the definition of transition amplitudes for various processes in a three-body problem. Secondly, there are some difficulties associated with the off-shell two-particle Coulomb partial-wave T matrix, which is, *apparently*, singular on the mass shell according to Refs. 3 and 4, in contrast with the Nutt's representation of the two-particle Coulomb T matrix, which is zero on the mass shell. In addition, in the Coulomb problem, the consistency of the T matrix with the asymptotic Coulomb distorted boundary conditions should always be retained.[7,8] In this paper, we resolve the above-mentioned difficulties and indicate a suitable revised partial-wave analysis of the three-particle Coulomb systems.

In order to appreciate the importance of this problem a brief survey of the present status of the three-particle Coulomb problem is desirable. The Faddeev trick is to express the full three-body T operator satisfying the usual Lippmann-Schwinger equation as a sum $\sum_{i=1}^{3} T^{(i)}$, $T^{(i)}$ in turn obeying a set of coupled integral equations, known as Faddeev equations. The kernel of this set is now compact. However, Lovelace[10] and Newton[11] have formulated a rigorous procedure to obtain the transition operators for various physical processes, for example, $(1,2)+3 \rightarrow (1,2)+3$; $(1,2)+3 \rightarrow (1,3)+2$; etc. Here (i,j) denotes the bound system of particles i and j. In order to accomplish this they have adopted the Faddeev trick to construct their coupled integral equations with a compact kernel. It is stressed in detail in these works[12] that the matrix element of $T = \sum_{i=1}^{3} T^{(i)}$ between a given set of initial and final states *does not* give the physical amplitude for that process.

In this connection, one notes that the Faddeev approach to three-particle Coulomb systems has resulted in two types of investigations: In the first type[2,5,6,8,9] methods are suggested to evaluate the leading terms in the Faddeev-type expansions for various transition operators. The elegant representation, due to Nutt, of the two-body off-shell Coulomb T matrix between plane-wave states makes the evaluation of terms up to second order feasible.[13] However, these are "Born-type" approaches and therefore, may not be reliable at lower ener-

gies. On the other hand, partial-wave analyses[3,4] of transition operators are useful at lower energies and also for the evaluation of bound-state energies of three-particle systems.[6c] Ball *et al.*[3] were the first to study the partial-wave analysis for the three-particle Coulomb systems. In attempting the numerical evaluation of the cross sections, they suggested a limiting procedure for specifying the initial and final states and identified the resulting $T^{(i)}$'s with the amplitudes. While their method of solution of the integral equations is novel, their expressions for transition amplitudes must be related to the works of Lovelace and Newton, because apparently their limiting procedure is not equivalent to taking matrix elements between the appropriate initial and final states[14] (see Appendix). However these problems do not arise in computing the three-body bound states of systems of three charged particles.

The plan of the paper is as follows: In Sec. II, we define the various transition operators for three-particle scattering and show their relation to Faddeev's $T^{(i)}$'s. We shall also clarify the interconnections between various T operators used by Faddeev, Lovelace, and Newton. In Sec. III, we study the off-shell two-body Coulomb T matrix and show that the representations of Nutt and Ball *et al.* are equivalent. In Sec. IV, we indicate the partial-wave analysis and point out that in order to obtain the physical amplitudes one has to carry out the usual overlap integral between the initial and final states. In the Appendix, it is shown that specifying the initial and final states using a limiting procedure as prescribed in Refs. 3 and 4 is equivalent to the evaluation of the overlap integral. Section V summarizes the main conclusions obtained.

II. TRANSITION OPERATORS FOR THREE-PARTICLE SYSTEMS

In this section, a brief outline of the transition operators given by Newton in his book[11] and the relation of these with the Faddeev operators are given. We begin with a system of three interacting particles 1, 2, and 3 with masses m_1, m_2, and m_3, respectively. Let V_{ij} denote the potential between particles i and j and E the total energy of the system. We consider the following scattering processes:

$$1+(2,3)\rightarrow 1+(2,3)\ , \tag{2.1a}$$

$$1+(2,3)\rightarrow 2+(3,1)\ , \tag{2.1b}$$

$$1+(2,3)\rightarrow 3+(1,2)\ , \tag{2.1c}$$

$$1+(2,3)\rightarrow 1+2+3\ . \tag{2.1d}$$

Equation (2.1d) denotes the breakup process; other processes are self-explanatory. By permuting 1, 2, 3 we obtain the rest of the processes. We denote our three-particle free Hamiltonian by H_0 and define the following operators:

$$G_i^+=(E-H_0-V_{jk}+i\epsilon)^{-1}\ , \tag{2.2}$$

$$G_0^+=(E-H_0+i\epsilon)^{-1}\ , \tag{2.3}$$

$$T_i=V_{jk}+V_{jk}G_i^+V\ ,\quad i\neq j\neq k,\ i,j,k=1,2,3. \tag{2.4}$$

T_i is the two-particle T operator for (j,k) system with i as spectator.

Following Newton,[11] we denote by T_{11}, T_{21}, and T_{31} the transition operators for processes (2.1a)–(2.1c), respectively. The following relations hold[11]:

$$T_{j1}=\sum_{k\neq j}V_{jk}+\sum_{k\neq j}V_{jk}G_1^+T_{11}\ . \tag{2.5}$$

The following alternative relations are also given by Newton:

$$T_{j1}=V_{23}(\delta_{2j}+\delta_{3j})+\sum_{k\neq j}\mathcal{T}_{k1}\ . \tag{2.6}$$

$\mathcal{T}_{k1}$, $k=1,2,3$ satisfy the following Faddeev-type coupled integral equations with compact kernel:

$$\mathcal{T}_{k1}=T_k(\delta_{k2}+\delta_{k3})+\sum_{j\neq k}T_kG_0^+\mathcal{T}_{j1} \tag{2.7}$$

It is to be noted that Eq. (2.7) facilitates, at least in principle, evaluating $\mathcal{T}_{k1}$ correctly, and it can be used in turn to obtain transition operators T_{j1} through Eq. (2.6).

Now we relate T_{j1} to the solutions of conventional Faddeev equations. The usual Faddeev T operators $T^{(j)}$ are the solutions of the following set of coupled integral equations:

$$T^{(j)}=T_j+T_jG_0^+(T^{(k)}+T^{(i)}),\quad i\neq j\neq k \tag{2.8}$$

and the sum $T=\sum_{j=1}^{3}T^{(j)}$ satisfies the Lippmann-Schwinger-type equation for the three-particle system:

$$T=\sum_{\substack{i,j\\ i\neq j}}V_{ij}(1+G_0^+T). \tag{2.9}$$

Now we use a set of auxiliary operators $T_i^{(j)}$ satisfying

$$T_i^{(j)}=T_j\delta_{ij}+T_jG_0^+\sum_{k\neq j}T_i^{(k)} \tag{2.10}$$

and therefore

$$T=\sum_j T^{(j)}=\sum_{i,j}T_i^j\ . \tag{2.11}$$

Using Eqs. (2.7), (2.8), and (2.10) we obtain

$$\mathcal{T}_{j1}=T_2^{(j)}+T_3^{(j)}, \tag{2.12}$$

$$T_{j1}=\sum_{i\neq j}(T^{(i)}-T_1^{(i)})+(\delta_{2j}+\delta_{3j})V_{23}\ . \tag{2.13}$$

The corresponding operators $T_{jk}(k=2,3)$ for the rest of the processes can be obtained by a proper permutation of the indices 1, 2, 3. Equation (2.13) establishes the connection between transition operators T_{j1} and Faddeev's $T^{(j)}$ and $T_i^{(j)}$ operators.[15] This clearly demonstrates that evaluation of matrix elements of $T^{(j)}$'s between given initial and final states does not correspond to the amplitude for that process. Secondly, one also notes that in the evaluation of amplitudes for rearrangement collisions, the bare potential V_{23} is also to be taken into account in addition to T operators. This feature is apparently in contrast with the imlications to the contrary in Refs. 3 and 4. It is also clear that partial-wave analysis of transition amplitudes T_{j1} can be carried out once we derive a partial-wave expansion for the matrix elements of $T_j^{(i)}$'s. This is indicated in Sec. IV.

The S-matrix element for any three-particle process except ionization is given by[10,16]

$$S_{in,jm}(\vec{q}_i,\vec{q}_j') = \delta_{ij}\delta_{nm}\delta_3(\vec{q}_i-\vec{q}_i') - 2\pi i\delta(q_i^2-E_{in}-q_j'^2+E_{jm})\int d^3p_i\int d^3p_j' \times\psi_{in}^*(\vec{p}_i)\,T_{ij}(\vec{p}_i,\vec{q}_i,\vec{p}_j',\vec{q}_j',q_i^2-E_{in}+i\epsilon)\psi_{jm}(\vec{p}_j'). \quad (2.14)$$

Here ψ_{in} and ψ_{jm} are the bound-state wave functions.

The quantities $\vec{q}_i,\vec{q}_j',\vec{p}_i,\vec{p}_j'$ are defined in Ref. 10. Equations (2.13) and (2.14) establish the relations between Faddeev's $T^{(j)}$'s, $T_i^{(j)}$'s, transition operators T_{ij}, and the S-matrix elements and thereby demonstrate the correct method for the evaluation of amplitudes in three-particle scattering.

III. OFF-SHELL COULOMB T MATRIX

In order to evaluate the transition operators for three-particle Coulomb systems using the procedure outlined in Sec. II, an unambiguous definition of two-particle off-shell T matrix is needed. Using Schwinger's representation[7] for Coulomb Green's function Nutt,[8] constructed a representation for off-shell two-body Coulomb T matrix. Solving the Lippmann-Schwinger equation, Ball *et al.*[3] obtained an eigenfunction expansion for the off-shell two-particle partial-wave T matrix $t_l(p,p',E)$. The difficulties in this representation and its apparent contradictions with the representation of Nutt can be seen as follows: Consider the representation used in Refs. 3 and 4 for $t_l(p,p',E)$:

$$t_l(p,p',E)=\sum_{\lambda=1}^{\infty}(\gamma_{\lambda l}-1)^{-1}\phi_{\lambda l}(p,E)\phi_{\lambda l}(p',E). \quad (3.1)$$

Here $\phi_{\lambda l}$ are the two-particle Coulomb wave functions in the momentum space as defined in Refs. 3 and 4 and can be expressed in terms of Gegenbauer polynomials. $\gamma_{\lambda l}$ are given by $i\gamma_{\lambda l}\nu=\lambda$ and $\nu=me^2/k$, $E=k^2/2m$. Moreover, $t_l(p,p',E)$ satisfy the partial-wave Lippmann-Schwinger equation. In Refs. 3 and 4, the physical partial-wave amplitude is related to $t_l(p,p',E)$ through the following equation:

$$t_l(p,p,p^2/2m)=-[e^{i\delta_l(p)}\sin\delta_l(p)]/p. \quad (3.2)$$

This contradicts the subsequent discussion of the analytical properties of $t_l(p,p',E)$ in one of their papers,[3] for, on the mass shell each term in the expansion (3.1) is singular and therefore one will not obtain the physical partial-wave amplitude as implied in Eq. (3.2).[17] This statement is further clarified by noting that $t_l(p,p',E)$ is the solution of the Lippmann-Schwinger equation which has plane-wave boundary conditions built in it, whereas the physical Coulomb partial-wave amplitude is defined with respect to Coulomb-distorted asymptotic states. Careful attention is given to this point in the papers of Schwinger and Nutt. Nutt's representation for the off-shell two-particle Coulomb T-matrix has the following form in the case of an attractive potential:

$$T(\vec{p},\vec{p}',E)=\frac{-e^2}{2\pi^2}\frac{1}{|\vec{p}-\vec{p}'|^2}\times\left(1-\frac{4i\nu}{e^{2\pi\nu}-1}\int_C\frac{dt\,t^{-i\nu}}{\epsilon_0(1-t^2)-4t}\right), \quad (3.3)$$

with

$$\epsilon_0=(k^2-p^2)(k^2-p'^2)/[k^2|\vec{p}-\vec{p}'|^2].$$

The contour C starts at $t=1$ slightly above the real axis, moves to and around the origin and then moves to $t=1$ slightly below the real axis.

In the works of Schwinger[7] and of Nutt[8] it is clearly demonstrated that the physical amplitudes can be obtained from Eq. (3.3) *only* after taking a correct overlap between the Coulomb-distorted asymptotic states. As it is, $T(\vec{p},\vec{p}',E)$ is defined only with respect to plane-wave boundary conditions. Essentially due to this aspect, it was found that $T(\vec{p},\vec{p}',E)$ is zero on the mass shell. This contradicts the statements of Ball *et al.*[3] However, since both $T(\vec{p},\vec{p}',E)$ and $t_l(p,p',E)$ are defined with respect to plane-wave boundary conditions, and are exact solutions, such contradictions can only be apparent. In fact, in the remaining part of this section we show that one can obtain $t_l(p,p',E)$ given by Eq. (3.1) from Nutt's $T(p,p',E)$ which proves that $t_l(p,p',E)$ is zero on the mass shell in spite of the singularity of each term in the expansion (3.1) at that point. This fact indicates the importance of the overlap integral between the correct asymptotic states in order to get the physical scattering amplitude.

In order to prove the equivalence of Eqs. (3.3) and (3.1) we proceed as follows: Using expres-

sions[18]

$$[1-2hZ+h^2]^{-\upsilon}=\sum_{n=0}^{\infty} C_n^{\nu}(Z)h^n \ , \tag{3.4}$$

and

$$i(2\sin\pi\alpha)^{-1}e^{i\pi\alpha}\int_C d\rho\rho^{-\alpha}\cdots=\int_0^1 d\rho\rho^{-\alpha}\cdots, \ \mathrm{Re}\alpha<-1 \tag{3.5}$$

we obtain

$$\frac{-4i\nu e^2}{(e^{2\pi\nu}-1)}\frac{1}{\epsilon_0|\vec{p}-\vec{p}'|^2}\int_C dt\, t^{-i\nu}\,[1-2t(1+2/\epsilon_0)+t^2]^{-1}$$
$$=\frac{4e^2}{|\vec{p}-\vec{p}'|^2}\frac{1}{\epsilon_0}\sum_{\lambda=1}^{\infty} C^1_{\lambda-1}(1+2/\epsilon_0)/(1-\gamma_\lambda)^{-1} \ . \tag{3.6}$$

Here $\gamma_\lambda=\gamma_{\lambda l}=-i\lambda/\nu$. The condition $\mathrm{Re}(i\nu)>-1$ can be relaxed now, and Eq. (3.6) can be defined for all values of ν (except the points of singularity) through analytic continuation. Equation (3.5) when substituted in Eq. (3.3) gives another new representation for $T(\vec{p},\vec{p}',E)$:

$$T(\vec{p},\vec{p}',E)=\frac{1}{2\pi^2}\times\left(\frac{e^2}{|\vec{p}-\vec{p}'|^2}-\frac{4e^2}{|\vec{p}-\vec{p}'|^2\,\epsilon_0}\sum_{\lambda=1}^{\infty}\frac{C^1_{\lambda-1}(1+2/\epsilon_0)}{1-\gamma_\lambda}\right) \tag{3.7}$$

Now we make use of the following expressions:

$$\frac{e^2}{|\vec{p}-\vec{p}'|^2}=\sum_{l=0}^{\infty}(2l+1)V_l(p,p')P_l(\mu)$$
$$=\sum_{l=0}^{\infty}(2l+1)P_l(\mu)\sum_{\lambda}\gamma_\lambda^{-1}\phi_{\lambda l}\times(p,E)\phi_{\lambda l}(p',E) \tag{3.8}$$

and the expansion[18]

$$[\Gamma(\nu)]^2 C_n^{\nu}(ZZ_1-(Z^2-1)^{1/2}(Z_1^2-1)^{1/2}\cos\phi)$$
$$=\Gamma(2\nu-1)\left(\sum_{l=0}^{n}(-1)^l 4^l\Gamma(n-l+1)\times[\Gamma(\nu+l)]^2(2\nu+2l-1)\times[\Gamma(N+2\nu+l)]^{-1}(Z^2-1)^{l/2}(Z_1^2-1)^{l/2}\times C_{n-l}^{\nu+l}(Z)\,C_{n-l}^{\nu+l}(Z_1)C_l^{\nu-1/2}(\cos\phi)\right) \tag{3.9}$$

in Eq. (3.7) and after some simplification arrive at the result

$$T(\vec{p},\vec{p}',E)=-\frac{1}{2\pi^2}\sum_{l=0}^{\infty}(2l+1)P_l(\mu)\times\sum_{\lambda}(1-\gamma_\lambda)^{-1}\phi_{\lambda l}(p,E)\phi_{\lambda l}(p',E). \tag{3.10}$$

From (3.10) we get the expansion (3.1) for the partial-wave off-shell T matrix $t_l(p,p',E)$. But $T(\vec{p},\vec{p}',E)$ is zero on the mass shell[8] for all $1>\mu>-1$, and therefore it follows that quite generally $t_l(p,p',E)$ is also zero on the mass shell. The physical scattering amplitude can only be obtained after the evaluation of the Coulomb T matrix between the Coulomb-distorted asymptotic states.

IV. PARTIAL-WAVE ANALYSIS

One of the important aspects of the partial-wave analysis carried out in Refs. 3 and 4 is the reduction of the partial-wave elements of $T^{(i)}$'s to the *single-variable integral equations* which are suitable for numerical computation. We indicate a similar partial-wave analysis for $T_i^{(j)}$'s which can be used to evaluate the partial-wave matrix elements of T_{11}, T_{21}, and T_{31}, through Eq. (2.13). However, it should be noted that we differ from the procedure of Ball *et al.* in two important aspects: First, we are aiming at the partial-wave analysis of T_{ij}'s and not just $T^{(i)}$'s. Secondly, we believe that evaluation of the overlap integral given by Eq. (2.14) between the initial and final states must be explicitly carried out and that the limiting procedure suggested in Refs. 3 and 4 also accomplishes this. Details of the mathematical manipulations are described in Refs. 3 and 4 and therefore we give only the equations to be solved, for the partial-wave reduction of $T_i^{(j)}$'s.

From Eq. (2.13), it is clear that partial-wave amplitudes for any given transition operator T_{j1} can be obtained in terms of the corresponding partial-wave projections of $T^{(i)}$, $T_1^{(i)}$, and V_{23} operators. The partial-wave projections of $T^{(i)}$ between any two three-particle states are studied by Ball *et al.* Since $T^{(i)}=\sum_j^{(i)}$, the partial-wave analysis for $T^{(i)}$ can be easily adopted for $T_j^{(i)}$'s. In analogy with Ref. 3 and 4 we define

$$\Psi^{(i)}_{\alpha\mu}=i\langle pq\alpha|\,T^{(i)}_{\mu}(s)\,|\vec{k}_1\vec{k}_2\vec{k}_3\rangle \ . \tag{4.1}$$

All the symbols are as defined in Ref. 3 and 4. Then from Eq. (2.10) one obtains the following coupled integral equations for $\Psi^{(i)}_{\alpha\mu}$:

$$\Psi^{(i)}_{\alpha\mu}(p,q,s)=\Phi^i_\alpha(pqs)\,\delta_{i\mu}-\tfrac{1}{4}\sum_{\alpha_j}\sum_{j\neq i}\int dp_j^2\int dq_j^2\,K_j^{(i)}(pq\alpha|p_jq_j\alpha_j)\times\frac{p_jq_j}{(p_j^2+q_j^2-s)}\,\Psi^{(j)}_{\alpha_j\mu}(p_jq_js),$$
$$i=1,2,3,\quad j=1,2,3,\quad \mu=1,2,3. \tag{4.2}$$

The function $K_j^{(i)}$ is given by

$$K_j^{(i)}(pq\alpha|p_jq_j\alpha_j)=\int_{-1}^{1}d\cos\theta_{p_jq_j}A_{\alpha\alpha_j}(\theta_{p_ip_j},\theta_{q_ip_j},\theta_{q_jp_j})\times\delta(q^2-q_i^2)\,t_l^{(i)}(p,p_i,s-q^2). \tag{4.3}$$

$A_{\alpha\alpha_j}$ is defined in Refs. 3 and 4. Following Ball *et al.* for $J=M=0$ case, we define

$$\Psi_{i\mu}^{(i)}(pqs)=\Phi_l^{(i)}(pqs)\delta_{i\mu} + \sum_{\lambda}[\gamma_{\lambda l}^{(i)}(s-q^2)-1]^{-1} \times \phi_{\lambda l}^{(i)}(p, s-q^2)\chi_{\lambda l\mu}^{(i)}(qs) \tag{4.4}$$

and obtain the following integral equation for $\chi_{\lambda l\mu}^{(i)}$:

$$\chi_{\lambda l\mu}^{(i)}=\eta_{\lambda l\mu}^{(i)}(qs)+\sum_{\lambda' l'}\sum_{j\neq i}\int_0^\infty dq_j^2\,\mathcal{K}_{\lambda l\lambda' l'}^{(ij)}(qq_j) \times \chi_{\lambda' l'\mu}^{(j)}(q_j s), \tag{4.5}$$

with

$$\eta_{\lambda l\mu}^{(i)}=\tfrac{1}{4}\sum_{l'}\sum_{j\neq i}\int_0^\infty dp_j^2\int_0^\infty dq_j^2\int_{-1}^{1}d\,\cos\theta_{p_jq_j} \times A_{ll'}(\theta_{p_ip_j},\ \theta_{q_ip_j},\ \theta_{q_jp_j})\,\delta(q^2-q_i^2) \times \phi_{\lambda l}^{(i)}(p_i, s-q^2)\,\Phi_l^{(j)}(p_jq_js)\delta_{i\mu} \tag{4.6}$$

and $\mathcal{K}^{(ij)}$ is the same as that given in Ref. 4.

Now, knowing $\langle pq\alpha|T^{(i)}|\vec{k}_1\vec{k}_2\vec{k}_3\rangle$ and $\langle pq\alpha|T_\mu^{(i)}|\vec{k}_1\vec{k}_2\vec{k}_3\rangle$ we can evaluate T_{jl} for any j. However, we stress here once again that these evaluations give only the T_{jl} in the integrand of Eq. (2.14) given by Lovelace, and to obtain the amplitude from these an additional integration is to be carried out between the correct initial and final states. This procedure may prove to be more amenable to practical calculations involving iterative series for T^i and the like.

V. CONCLUSIONS

We draw the following conclusions from the present analysis: (i) The physical transition operators are related to Faddeev's $T^{(i)}$'s through Eq. (2.13) and the prescription of Ball *et al.* is equivalent to the calculation of the physical amplitudes. (ii) Representations given by Ball *et al.* for the off-shell Coulomb two-particle partial-wave T matrix are equivalent to Nutt's representation for the off-shell Coulomb T matrix. (iii) In order to evaluate the amplitudes for various processes an additional overlap integral between initial and final states given by Eq. (2.14) is necessary, as shown by Lovelace[10] and illustrated in Refs. 8 and 9. The evaluation of this overlap integral is not apparently equivalent to the limiting procedure suggested by Ball *et al.* For, consider T_{11} to the leading order:

$$T_{11}=T_2+T_3+\cdots.$$

Neither T_2 nor T_3 has the bound system (2, 3). Therefore, the limiting procedure of Ball *et al.* does not seem to specify the initial and final states. The presence of V_{23} in the expansion for T_{21} and T_{31} makes this point still more explicit. To realize the equivalence explicitly, see the Appendix.

APPENDIX

In Ref. 4, a limiting procedure for finding the appropriate transition matrix element is given. However, the procedures of Lovelace[10] and Newton[11] commonly adopted here and elsewhere must be shown to be equivalent to the limiting procedure. Since such a proof is not given in Ref. 4 and since it seems to be a source of confusion, we construct the following arguments for it here (see also the Appendix in Ref. 10).

We set

$$T^i=T_i+T_iG_0^+\mathcal{T}_i \tag{A1}$$

in the Faddeev equations (2.8) of the text. This substitution is made only to correlate with the Eq. (4.4) of the text (and Ref. 4). The $\mathcal{T}_i$'s then satisfy the equations

$$\mathcal{T}_i=T_j+T_k+T_jG_0^+\mathcal{T}_j+T_kG_0^+\mathcal{T}_k \tag{A2}$$

provided $T_iG_0^+$'s are not null. It must be shown that the suitable matrix elements of the transition operators T_{11}, T_{21} for the typical processes $1(23)\to1(23)$, $1(23)\to2(31)$, respectively, are obtained when the limiting procedure of Ref. 4 is employed on T^1 and T^2, respectively. It suffices for our purposes to show this equivalence for the usual series development of these operators. The series development corresponds to an iterative solution of the Faddeev equations or its equivalent [Eq. (2.7)]. Since our initial state is 1(23), only the terms with T_1 in the rightmost of each term of the series for $\mathcal{T}_1$ and $\mathcal{T}_2$ and hence in $T^1$and T^2 survive (T_1 has in it this bound state).[19] Thus,

$$T^1=T_1+T_1G_0^+[T_2+T_3+T_2G_0^+T_3 + T_3G_0^+T_2+\cdots]\,G_0^+T_1, \tag{A3}$$

$$T^2=T_2G_0^+[(G_0^+)^{-1}+T_3+T_1G_0^+T_2 + T_1G_0^+T_3+T_3G_0^+T_2+\cdots]\,G_0^+T_1. \tag{A4}$$

The terms in the square brackets in (A3) and (A4) *indeed correspond* to the appropriate series for the transition operators T_{11} and T_{21}, respectively. The initial- and the final-state specification then gives directly the matrix elements of these operators with respect to the appropriate initial and final states because $\lim_{\text{final state}} T_2G_0^+$ and $\lim_{\text{initial state}} G_0^+T_1$ give $\langle 2(13)|$ and $|1(23)\rangle$, respectively. lim here is as given in Ref. 4. In T^1, the term T_1 vanishes when the final-state limit is taken; the term $(G_0^+)^{-1}$ in T^2 simply becomes the potential V_1 or equivalently V_{23}.

We have presented this "proof" because all the arguments given in the text and Ref. 14 give one

a strong suspicion as to the validity of the limiting procedure. While a single T operator does not exist which gives the appropriate transition matrix element for a given process,[12] the limiting procedure shows that from the total T $(=\sum_i T^i)$ operator, one could obtain *all* the transition amplitudes.[19]

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[10]C. Lovelace, Phys. Rev. 135, B1225 (1964).

[11]R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966), p. 557.

[12]See Ref. 11, pp. 484, 485.

[13]L. Kumar, C. S. Shastry, and J. Callaway (unpublished).

[14]The transition operator for any rearrangement process begins with a bare potential as the leading term; see, for example, T_{21} of Newton (Ref. 11) or Shastry *et al.* (Ref. 9). This is *not* apparently obtained when the prescription of Ball *et al.* (Ref. 4) is used. The seeming nonequivalence of the usual procedure of taking matrix elements of appropriate T operator between the initial and final states and the limiting procedure of Ball *et al.* (Refs. 3 and 4) may be seen by a glance at T_{11} of Newton, the calculation of the leading term in T_{11} by Nutt (Ref. 8), and the corresponding expression of Ball *et al.* (Ref. 3).

[15]Lovelace's $U_{\alpha\beta}^{(\pm)}$ are the same as T_{jk} used in this paper (see Ref. 11). Faddeev's book denotes $T_i^{(j)}$ of the present paper by $M_{\alpha\beta}$. All these developments are equivalent.

[16]The expression for S matrix for ionization process may be found for example in Ref. 10 and 11.

[17]Apparently because of the logarithmic singularity in Born term $V_l(p, p' E)$, in Ref. 3 it is also stated that t_l $(p,p' E)$ is singular for $p=p'$ for all E; which does not seem to be true.

[18]A. Erdélyi, *Bateman Manuscript Project* (McGraw-Hill, New York, 1953), Vol. 1, p. 176.

[19]It may be shown quite generally that

$$T^i = T_i + T_i G_0^+ [T_{ii} G_0^+ T_i + T_{ij} G_0^+ T_j + T_{ik} G_0^+ T_k] ,$$
$$i \neq j \neq k ,$$

where T_{ij} are the transition operators defined in the text.