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Spontaneous Emission by a System of Identical Atoms*

Demosthenes Dialetis
Duke University, Durham, North Carolina 27706
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The spontaneous emission of a set of N identical two-level atoms interacting with a quantized electromagnetic field is studied. The atoms are assumed to be close together compared to the mean wavelength of the emitted radiation, and their initial state is either a Dicke state or a superposition of the ground and excited state for each atom. Approximate expressions are obtained for transition probabilities, expectation values, and correlations as functions of time. The spontaneous emission for a very large separation of the atoms is also studied. In this case the directional properties of the emitted radiation, as well as the time within which the atoms decay to their ground state, depend on the position of each atom.

I. INTRODUCTION

In the following the evolution in time of a set of N identical atoms interacting with a quantized electromagnetic field is studied. It is assumed that the atoms lie far enough from each other so that their wave functions do not overlap. In addition it is assumed that only two levels of the energy spectrum of each atom are involved in the process of evolution, so that each individual atom is treated as a two-level system. The initial condition of the system is such that at $t = t_0$ there are no photons present in the field and the set of atoms is in an excited state which can be prepared experimentally.

When the electromagnetic field consists of a single mode it is possible to diagonalize the Hamiltonian[1] and the problem can be solved exactly. It may be shown then that the energy of the excited atoms is exchanged back and forth in time between the set of atoms and the one mode of the electromagnetic field. Hence no energy dissipation of the system of atoms occurs on a long time average. In the present work it will be assumed that the electromagnetic field consists of an infinite number of modes. In this case the atoms return to their ground state after a long time interval.

When the electromagnetic field has an infinite number of modes there are various methods of approach in solving the problem approximately. The method of approach that will be adopted here is to deal with the equations of motion of the matrix elements of the evolution operator. It turns out that there is only a finite number of such matrix elements, so that there is also a finite number of coupled equations of motion. In addition, these equations of motion are linear and hence easier to solve than, for example, the equations of motion of the operators themselves which are nonlinear. This same method of approach was used by Weisskopf and Wigner[2] for the first time to study the spontaneous emission of a single atom and rather recently it was also used by Ernst and Stehle[3] to study the spontaneous emission of N atoms. The latter investigators consider the particular case where all the atoms are excited initially, while the initial state of the set of atoms considered here is a superposition of the ground state and the excited state for each individual atom (Sec. IV).

Following a completely different approach, Dillard and Robl[4] have also treated the spontaneous emission of N two-level atoms which lie close together compared to the mean wavelength of the radiation. Some of the results obtained in their investigation are duplicated here in order to show explicitly that either method of approach gives identical results. In most of the present work the condition that the atoms lie close together com-

pared to the mean wavelength of the radiation is imposed, as it is done by Dillard and Robl, but the case where the atoms lie very far apart is also incorporated in a trivial manner (Sec. V).

II. EQUATIONS OF MOTION AND THEIR SOLUTION

Assuming that the atoms lie within a volume much smaller than λ_0^3 [$\lambda_0 = 2\pi c/\omega_0$, $\omega_0 = (1/\hbar)(E_e - E_g)$, where E_e, E_g are the energies of the excited and ground levels of each atom, respectively, and ω_0 is its resonance frequency], the following Hamiltonian is adopted[5]:

$$H = \hbar\omega_0(R_3 + \tfrac{1}{2}N) + \frac{(2\pi)^3}{V}\sum_{\vec{k}\lambda}\hbar\omega_{\vec{k}}a^{\dagger}_{\vec{k}\lambda}a_{\vec{k}\lambda} - \hbar\frac{(2\pi)^3}{V}\sum_{\vec{k}\lambda}[g_{\vec{k}\lambda}a^{\dagger}_{\vec{k}\lambda}R_- + g^{*}_{\vec{k}\lambda}a_{\vec{k}\lambda}R_+] \quad , \tag{1}$$

where

$$R_3 = \sum_{i=1}^{N} R_3^{(i)} \quad , \tag{2a}$$

$$R_+ = \sum_{i=1}^{N} R_+^{(i)} \quad , \tag{2b}$$

$$R_- = \sum_{i=1}^{N} R_-^{(i)} \quad . \tag{2c}$$

The notation of Dicke[6] is followed here. The commutation relations satisfied are

$$[a_{\vec{k}\lambda},\ a^{\dagger}_{\vec{k}'\lambda'}] = \frac{V}{(2\pi)^3}\delta_{\vec{k}\vec{k}'}\delta_{\lambda\lambda'} \quad , \tag{3}$$

$$[R_3^{(i)},\ R_+^{(j)}] = R_+^{(i)}\delta_{ij} \quad , \tag{4a}$$

$$[R_3^{(i)},\ R_-^{(j)}] = -R_-^{(i)}\delta_{ij} \quad , \tag{4b}$$

$$[R_+^{(i)},\ R_-^{(j)}] = 2R_3^{(i)}\delta_{ij} \quad . \tag{4c}$$

The rest of the commutation relations are zero. It follows that

$$[R_3,\ R_+] = R_+ \quad , \tag{5a}$$

$$[R_3,\ R_-] = -R_- \quad , \tag{5b}$$

$$[R_+,\ R_-] = 2R_3 \quad . \tag{5c}$$

The electromagnetic field has been quantized within a box of side $L(V = L^3)$ which is assumed to be so large that the transition

$$\frac{(2\pi)^3}{V}\sum_{\vec{k}} \rightarrow \int d^3k \tag{6}$$

will be allowed.

Now it is easy to show that the operators

$$R^2 \equiv \tfrac{1}{2}(R_+R_- + R_-R_+) + R_3^2 \quad , \tag{7}$$

$$S \equiv R_3 + \frac{(2\pi)^3}{V}\sum_{\vec{k}\lambda} a^{\dagger}_{\vec{k}\lambda}a_{\vec{k}\lambda} \quad , \tag{8}$$

commute with the Hamiltonian (1), so that they are conserved. It is most appropriate then to work with the Dicke states[6] $|r, m\rangle$ defined by the relations

$$R^2|r, m\rangle = r(r+1)|r, m\rangle \quad , \tag{9a}$$

$$R_3|r, m\rangle = m|r, m\rangle \quad , \tag{9b}$$

$$R_+|r, m\rangle = C_{r,m}|r, m+1\rangle \quad , \tag{9c}$$

$$R_-|r, m\rangle = C_{r,m-1}|r, m-1\rangle \quad , \tag{9d}$$

$$a_{\vec{k}\lambda}|r, m\rangle = 0 \quad , \tag{9e}$$

where

$$C_{r,m} = [r(r+1) - m(m+1)]^{1/2} \quad , \tag{10}$$

and $r = 0,\ \frac{1}{2},\ 2,\ \frac{3}{2},\ \ldots,\ \frac{1}{2}N$, while $|m| \leq r$. The ground state of the system is $|g\rangle = |r = \frac{1}{2}N, m = -\frac{1}{2}N\rangle$, for which $H|g\rangle = 0|g\rangle$. When the system is in the state $|r, m\rangle$, the number of excited atoms is equal to $r + m$.

Let $U(t, t_0)$ be the evolution operator, which satisfies the differential equation

$$i\hbar\frac{\partial U(t, t_0)}{\partial t} = HU(t, t_0) \quad , \tag{11}$$

and the initial condition $U(t_0, t_0) = 1$. Let $|r, m\rangle$ be the initial state of the system. The cooperation number r is conserved [Eq. (7)]. Also the total number of excited atoms and photons present in the field is conserved [Eq. (8)]. One concludes then that the only nonvanishing matrix elements of the evolution operator are

$$A(t - t_0) = \langle r, m|U(t, t_0)|r, m\rangle \quad , \tag{12a}$$

$$B_{\alpha_1\alpha_2\cdots\alpha_l}(t - t_0) = \frac{1}{\sqrt{l!}}\langle r, m - l|a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_l}U(t, t_0)|r, m\rangle \quad , \tag{12b}$$

for $l = 1,\ 2,\ \ldots,\ r + m$. For simplicity of notation $(\vec{k}, \lambda)$ has been replaced by the letter α. These matrix elements depend only on the time defference $t - t_0$, as the evolution operator $U(t, t_0)$ does. If one uses the relations (11), (1), (3), (5), and (9) together with the completeness relation of the photon states and the conservation relations (7) and (8), he can derive the equations of motion satisfied by the matrix elements

(12). They are as follows:

$$i\frac{\partial A(t-t_0)}{\partial t} = (\tfrac{1}{2}N+m)\omega_0 A(t-t_0) - C_{r,m-1}\frac{(2\pi)^3}{V}\sum_{\alpha_1} g^*_{\alpha_1} B_{\alpha_1}(t-t_0) \quad , \tag{13a}$$

$$i\frac{\partial B_{\alpha_1}(t-t_0)}{\partial t} = [(\tfrac{1}{2}N+m-1)\omega_0+\omega_{\alpha_1}]B_{\alpha_1}(t-t_0) - C_{r,m-1}g_{\alpha_1}A(t-t_0) - \sqrt{2}\,C_{r,m-2}\frac{(2\pi)^3}{V}\sum_{\alpha_2} g^*_{\alpha_2}B_{\alpha_1\alpha_2}(t-t_0) \quad , \tag{13b}$$

$$\begin{aligned} i\frac{\partial B_{\alpha_1\ldots\alpha_l}(t-t_0)}{\partial t} &= [(\tfrac{1}{2}N+m-l)\omega_0+\omega_{\alpha_1}+\cdots+\omega_{\alpha_l}]B_{\alpha_1\cdots\alpha_l}(t-t_0) \\ &\quad -\frac{1}{\sqrt{l}}\,C_{r,m-l}[g_{\alpha_1}B_{\alpha_2\alpha_3\cdots\alpha_l}(t-t_0)+g_{\alpha_2}B_{\alpha_1\alpha_3\ldots\alpha_l}(t-t_0)+\cdots+g_{\alpha_l}B_{\alpha_1\alpha_2\cdots\alpha_{l-1}}(t-t_0)] \\ &\quad -(l+1)^{1/2}C_{r,m-l-1}\frac{(2\pi)^3}{V}\sum_{\alpha_{l+1}} g^*_{\alpha_{l+1}}B_{\alpha_1\alpha_2\cdots\alpha_l\alpha_{l+1}}(t-t_0) \quad , \end{aligned} \tag{13c}$$

for $l=2,3,\ldots,r+m$, and ω_α stands for $\omega_{\vec{k}}=c|\vec{k}|$. When $l=r+m$, the last term in Eq. (13c) is absent, since $C_{r,-r-1}=0$. The initial conditions of the matrix elements (12) are

$$A(0)=1 \quad , \tag{14a}$$

$$B_{\alpha_1\alpha_2\cdots\alpha_l}(0)=0 \quad , \tag{14b}$$

for $l=1,2,\ldots,r+m$. Relations (13) and (14) form the set of linear integrodifferential equations and initial conditions that a set of identical two-level atoms lying close together compared to $\lambda_0=2\pi c/\omega_0$ and interacting with a quantized electromagnetic field satisfies, if initially there are no photons present in the field and the atoms are in the state $|r,m\rangle$.

The next task is to solve the set of linear equations (13) with the initial conditions (14). It is convenient to introduce the functions $B^{(l)}(\omega_{\alpha_1},\omega_{\alpha_2},\ldots,\omega_{\alpha_l};t-t_0)$ and $f(\omega_{\vec{k}})$ with the help of the relations

$$B_{\alpha_1\alpha_2\cdots\alpha_l}(t-t_0)=g_{\alpha_1}g_{\alpha_2}\cdots g_{\alpha_l}B^{(l)}(\omega_{\alpha_1},\ldots,\omega_{\alpha_l};t-t_0) \quad , \tag{15}$$

for $l=1,2,\ldots,r+m$, and

$$f(\omega_{\vec{k}})=\frac{\omega^2_{\vec{k}}}{c^3}\int\sum_{\lambda}|g_{\vec{k}\lambda}|^2 d\Omega_{\vec{k}} \quad , \tag{16}$$

where the integration in Eq. (16) is over the solid angle spanned by the wave vector $\vec{k}$ in the momentum space. In relation (15) it is assumed that the dependence of $B_{\alpha_1\alpha_2\cdots\alpha_l}(t-t_0)$ on the polarization of the phoons and the direction of the $\vec{k}$ vectors is expressed only through the proportionality factors g_{α_1}, g_{α_2}, $\ldots$, g_{α_l}. Substitution of Eq. (15) into the relations (13) and (14) leads to the new set of linear equations:

$$i\frac{\partial A(t-t_0)}{\partial t} = (\tfrac{1}{2}N+m)\omega_0 A(t-t_0) - C_{r,m-1}\int_0^\infty f(\omega_1)B^{(1)}(\omega_1;t-t_0)d\omega_1 \quad , \tag{17a}$$

$$\begin{aligned} i\frac{\partial B^{(1)}(\omega_1;t-t_0)}{\partial t} &= [(\tfrac{1}{2}N+m-1)\omega_0+\omega_1]B^{(1)}(\omega_1;t-t_0) - C_{r,m-1}A(t-t_0) \\ &\quad -\sqrt{2}\,C_{r,m-2}\int_0^\infty f(\omega_2)B^{(2)}(\omega_1,\omega_2;t-t_0)d\omega_2 \quad , \end{aligned} \tag{17a}$$

$$\begin{aligned} i\frac{\partial B^{(l)}(\omega_1,\ldots,\omega_l;t-t_0)}{\partial t} &= [(\tfrac{1}{2}N+m-l)\omega_0+\omega_1+\cdots+\omega_l]B^{(l)}(\omega_1,\ldots,\omega_l;t-t_0) \\ &\quad -\frac{1}{\sqrt{l}}\,C_{r,m-l}[B^{(l-1)}(\omega_2,\omega_3,\ldots,\omega_l;t-t_0)+B^{(l-1)}(\omega_1,\omega_3,\ldots,\omega_l;t-t_0) \\ &\quad +\cdots+B^{(l-1)}(\omega_1,\omega_2,\ldots,\omega_{l-1};t-t_0)] \\ &\quad -(l+1)^{1/2}C_{r,m-l-1}\int_0^\infty f(\omega_{l+1})B^{(l+1)}(\omega_1,\omega_2,\ldots,\omega_l,\omega_{l+1};t-t_0)d\omega_{l+1} \quad , \end{aligned} \tag{17c}$$

for $l=2,3,\ldots,r+m$, and the initial conditions

$$A(0)=1 \quad , \tag{18a}$$

$$B^{(l)}(\omega_1,\omega_2,\ldots,\omega_l;0)=0 \quad , \tag{18b}$$

for $l=1,2,\ldots,r+m$. Relations (16) and (6) were taken into account in deriving Eqs. (17) and ω_l stands for

ω_{α_l} (or $\omega_{\vec{k}_l} = c|\vec{k}_l|$) for simplicity of notation. The set of equations (17) and (18) do not depend either on the polarization of the photons or the direction of the $\vec{k}$ vectors and the same is true for the solutions of these equations. Therefore, the assumption implied in Eq. (15) is justified.

The Laplace transform method will be employed to solve the set of linear equations (17) with the initial conditions (18). For this the Laplace transforms

$$a(E) = \int_{t_0}^{\infty} A(t-t_0)e^{-E(t-t_0)}dt \quad , \tag{19a}$$

$$a(E)b^{(l)}(\omega_1, \ldots, \omega_l; E) = \int_{t_0}^{\infty} B^{(l)}(\omega_1, \ldots \omega_l; t-t_0)e^{-E(t-t_0)}dt \quad , \tag{19b}$$

for $l = 1, 2, \ldots, r+m$ are introduced together with their inverse

$$A(t-t_0) = (1/2\pi)\int_{-\infty}^{\infty} a[E = -i(z+i\epsilon)]e^{-iz(t-t_0)}dz \quad , \tag{20a}$$

$$B^{(l)}(\omega_1, \ldots \omega_l; t-t_0) = (1/2\pi)\int_{-\infty}^{\infty} a[E = -i(z+i\epsilon)]b^{(l)}[\omega_1, \ldots, \omega_l; E = -i(z+i\epsilon)]e^{-iz(t-t_0)}dz \quad , \tag{20b}$$

$l = 1, 2, \ldots, r+m$. Here ϵ is a positive number which tends to zero. Next the Laplace transform of Eqs. (17) is taken, as defined by Eqs. (19), and the initial conditions (18) are taken into consideration. The functions $b^{(l)}(\omega_1, \ldots \omega_l; E)$, $l = 1, 2, \ldots, r+m$ satisfy the following set of coupled integral equations:

$$\{iE - [(\tfrac{1}{2}N + m - 1)\omega_0 + \omega_1]\}b^{(1)}(\omega_1; E) = -C_{r,m-1} - \sqrt{2}\,C_{r,m-2}\int_0^{\infty} f(\omega_2)b^{(2)}(\omega_1, \omega_2; E)d\omega_2 \quad ,$$

$$\{iE - [(\tfrac{1}{2}N + m - l)\omega_0 + \omega_1 + \cdots + \omega_l]\}b^{(l)}(\omega_1, \ldots \omega_l; E) \tag{21a}$$

$$= -(1/\sqrt{l})\,C_{r,m-l}[b^{(l-1)}(\omega_1, \omega_2, \ldots, \omega_{l-1}; E) + b^{(l-1)}(\omega_1, \ldots, \omega_{l-2}, \omega_l; E) + \cdots$$

$$+ b^{(l-1)}(\omega_2, \omega_3, \ldots, \omega_l; E)] - (l+1)^{1/2}C_{r,m-l-1}\int_0^{\infty} f(\omega_{l+1})b^{(l+1)}(\omega_1, \omega_2, \ldots, \omega_l, \omega_{l+1}; E)d\omega_{l+1} \quad , \tag{21b}$$

for $l = 2, 3, \ldots, r+m$, while the function $a(E)$ is equal to

$$a(E) = \frac{i}{iE - (\frac{1}{2}N + m)\omega_0 + C_{r,m-1}\int_0^{\infty} f(\omega_1)b^{(1)}(\omega_1; E)d\omega_1} \quad . \tag{22}$$

When $l = r+m$, the last term in Eq. (21b) is absent, since $C_{r,-r-1} = 0$.

The problem of solving the set of equations (13) with the initial conditions (14) has been reduced to that of solving the set of coupled integral equations (21). This set can be written as follows:

$$\{iE - [(\tfrac{1}{2}N + m - 1)\omega_0 + \omega_1] - D_{r,m-1}(\omega_1; E)\}b^{(1)}(\omega_1; E)$$

$$= -C_{r,m-1} - C_{r,m-2}\int_0^{\infty} f(\omega_2)\left[\sqrt{2}\,b^{(2)}(\omega_1, \omega_2; E) + \frac{C_{r,m-2}b^{(1)}(\omega_1; E)}{iE - [(\tfrac{1}{2}N + m - 2)\omega_0 + \omega_1 + \omega_2] - D_{r,m-2}(\omega_1, \omega_2; E)}\right]d\omega_2 \quad , \tag{23a}$$

$$\{iE - [(\tfrac{1}{2}N + m - l)\omega_0 + \omega_1 + \cdots + \omega_l] - D_{r,m-l}(\omega_1, \ldots, \omega_l; E)\}b^{(l)}(\omega_1, \ldots, \omega_l; E)$$

$$= -\frac{1}{\sqrt{l}}\,C_{r,m-l}[b^{(l-1)}(\omega_1, \omega_2, \ldots, \omega_{l-1}; E) + b^{(l-1)}(\omega_1, \ldots, \omega_{l-2}, \omega_l; E)$$

$$+ \cdots + b^{(l-1)}(\omega_2, \omega_3, \ldots, \omega_l; E)] - C_{r,m-l-1}\int_0^{\infty} f(\omega_{l+1})\left[(l+1)^{1/2}b^{(l+1)}(\omega_1, \ldots \omega_l, \omega_{l+1}; E)\right.$$

$$\left. + \frac{C_{r,m-l-1}b^{(l)}(\omega_1, \omega_2, \ldots, \omega_l; E)}{iE - [(\tfrac{1}{2}N + m - l - 1)\omega_0 + \omega_1 + \cdots + \omega_{l+1}] - D_{r,m-l-1}(\omega_1, \ldots, \omega_{l+1}; E)}\right]d\omega_{l+1} \quad , \tag{23b}$$

for $l = 2, 3, \ldots, r+m$. The function $D_{r,m-l}(\omega_1, \ldots, \omega_l; E)$ is defined by the recurrence relation

$$D_{r,m-l}(\omega_1, \omega_2, \ldots, \omega_l; E) = C_{r,m-l-1}^2\int_0^{\infty} \frac{f(\omega_{l+1})d\omega_{l+1}}{iE - [(\tfrac{1}{2}N + m - l - 1)\omega_0 + \omega_1 + \cdots + \omega_{l+1}] - D_{r,m-l-1}(\omega_1, \ldots, \omega_{l+1}; E)} \tag{24a}$$

for $l = 1, 2, \ldots, r+m-1$, and for $l = r+m$

$$D_{r,-r}(\omega_1, \ldots, \omega_{r+m}; E) = 0 \quad . \tag{24b}$$

It is seen then that the systems (21) and (23) are identical. The reason for writing the system (21) in the form (23) is the following: From Eq. (21b), when one sets $l \to l+1$, it follows that $b^{(l+1)}(\omega_1, \ldots, \omega_{l+1}; E)$ is

a sum of terms one of which is proportional to $b^{(l)}(\omega_1, \omega_2, \ldots \omega_l; E)$. Hence, the integral $\int_0^\infty f(\omega_{l+1})$ $b^{(l+1)}(\omega_1, \ldots, \omega_{l+1}; E)\, d\omega_{l+1}$ will also be a sum of terms one of which is proportional to $b^{(1)}(\omega_1, \omega_2, \ldots, \omega_l; E)$ (this latter function is independent of the variable ω_{l+1}, so that integration over this variable does not affect it). One concludes then that, due to the presence of the integral in Eq. (21b), there is a term on the right-hand side of this equation which is proportional to $b^{(l)}(\omega_1, \omega_2, \ldots, \omega_l; E)$ [a similar statement applies for Eq. (21a)]. If this term is subtracted from the right-hand side of Eqs. (21) and added to the left-hand side, the system (23) is obtained. In Eq. (23b) then there is no term in the right-hand side of this equation proportional to $b^{(l)}(\omega_1, \omega_2, \ldots, \omega_l; E)$ [a similar statement applies for Eq. (23a)].

Up to now no approximations were made beyond the postulation of the Hamiltonian given by Eq. (1). But at this point one has to assume that the coupling constant is small in order to solve approximately the system of coupled integral equations (23). This system can be solved by an iteration method in powers of the function $f(\omega)$, which is proportional to the coupling constant squared [Eq. (16)]. To lowest order then one has from Eqs. (23)

$$b^{(1)}(\omega_1; E) \simeq - \frac{C_{r,m-1}}{iE - [(\tfrac{1}{2}N + m - 1)\omega_0 + \omega_1] - D^{(0)}_{r,m-1}(\omega_1; E)} \ , \tag{25a}$$

$$b^{(l)}(\omega_1, \omega_2, \ldots, \omega_l; E) \simeq - \frac{1}{\sqrt{l}} \frac{C_{r,m-l}}{iE - [(\tfrac{1}{2}N + m - l)\omega_0 + \omega_1 + \cdots + \omega_l] - D^{(0)}_{r,m-l}(\omega_1, \ldots, \omega_l; E)}$$

$$\times [b^{(l-1)}(\omega_1, \omega_2, \ldots, \omega_{l-1}; E) + b^{(l-1)}(\omega_1, \omega_2, \ldots, \omega_{l-2}, \omega_l; E) + \cdots + b^{(l-1)}(\omega_2, \omega_3, \ldots, \omega_l; E)] \ , \tag{25b}$$

for $l = 2, 3, \ldots, r+m$, where

$$D^{(0)}_{r,m-l}(\omega_1, \omega_2, \ldots, \omega_l; E) = C^2_{r,m-l-1} \int_0^\infty \frac{f(\omega_{l+1})\, d\omega_{l+1}}{iE - [(\tfrac{1}{2}N + m - l - 1)\omega_0 + \omega_1 + \cdots + \omega_{l+1}]} \ . \tag{26}$$

Here $D^{(0)}_{r,m-l}(\omega_1, \ldots, \omega_l; E)$ is the lowest-order expression for $D_{r,m-l}(\omega_1, \ldots, \omega_l; E)$ as defined by Eq. (24). From Eqs. (22) and (25a) one also has

$$\alpha(E) \simeq i \Big/ \left(iE - (\tfrac{1}{2}N + m)\omega_0 - C^2_{r,m-1} \int_0^\infty \frac{f(\omega_1)\, d\omega_1}{iE - [(\tfrac{1}{2}N + m - 1)\omega_0 + \omega_1] - D^{(0)}_{r,m-1}(\omega_1; E)} \right) . \tag{27}$$

The relations (25) and (27) form an approximate solution of the system (23) and Eq. (22). One must set $iE = z + i\epsilon$ in these relations in order to evaluate the functions $A(t - t_0)$ and $B^{(l)}(\omega_1, \ldots, \omega_l; t - t_0)$ from Eqs. (20). In this case a typical denominator, for example in Eq. (25b), is of the form

$$\left(z + i\epsilon - \Omega_l - C^2_{r,m-l-1} \int_0^\infty \frac{f(\omega_{l+1})\, d\omega_{l+1}}{z + i\epsilon - (\Omega_l - \omega_0 + \omega_{l+1})} \right)^{-1}$$

$$= \left(z - \Omega_l + C^2_{r,m-l-1} P \int_0^\infty \frac{f(\omega)\, d\omega}{\omega - (z - \Omega_l + \omega_0)} + i[\epsilon + \pi C^2_{r,m-l-1} f(z - \omega_l + \omega_0)\Theta(z - \Omega_l + \omega_0)] \right)^{-1} \ , \tag{28}$$

as $\epsilon \to 0$. Here $\Omega_l = (\tfrac{1}{2}N + m - l)\omega_0 + \omega_1 + \cdots + \omega_l$ and

$$\Theta(x) = 1 \quad \text{if} \quad x > 0, \qquad \Theta(x) = 0 \quad \text{if} \quad x < 0 \ . \tag{29}$$

In Eq. (28) use has been made of the identity

$$1/(x + i\epsilon) = \pi i \delta(x) + P(1/x) \ , \tag{30}$$

where P denotes the Cauchy principal value. Now it will be assumed in the following that there is no $z = z_0$ value which satisfies both the inequality $z_0 - \Omega_l + \omega_0 < 0$ and the equation

$$z_0 - \Omega_l + C^2_{r,m-l-1} P \int_0^\infty \frac{f(\omega)\, d\omega}{\omega - (z_0 - \Omega_l + \omega_0)} = 0 \ , \tag{31}$$

for $l = 0, 1, 2, \ldots, r+m$. When this is true, the system of atoms is said to form an *unstable* system, i.e., if initially the atoms are in an excited state, they will decay in the ground state as time evolves. Assuming then that the system of atoms is unstable and also that the coupling constant is small, it is seen from Eq. (28) that the pole in this equation lies very close to $z = \Omega_l$. Hence Eq. (28) may be approximated as follows:

$$\left(z+i\epsilon-\Omega_l-C^2_{r,m-l-1}\int_0^\infty \frac{f(\omega_{l+1})d\omega_{l+1}}{z+i\epsilon-(\Omega_l-\omega_0+\omega_{l+1})}\right)^{-1} \simeq \left(z+i\epsilon-\Omega_l+C^2_{r,m-l-1}\int_0^\infty \frac{f(\omega)d\omega}{\omega-\omega_0-i\epsilon}\right)^{-1} , \tag{32}$$

for $l=0,1,2,\ldots,r+m$. Substitution of the last relation into Eqs. (25) gives as an approximate solution of the system (23) the following expressions, as $\epsilon \to 0$:

$$b^{(1)}[\omega_1; E=-i(z+i\epsilon)] \simeq \frac{C_{r,m-1}}{z-[(\frac{1}{2}N+m-1)\omega_0+\omega_1]+C^2_{r,m-2}(\frac{1}{2}i\gamma-\Delta\omega_0)} , \tag{33a}$$

$$b^{(l)}[\omega_1,\omega_2,\ldots,\omega_l; E=-i(z+i\epsilon)] \simeq \frac{1}{\sqrt{l}} \frac{C_{r,m-l}}{z-[(\frac{1}{2}N+m-l)\omega_0+\omega_1+\cdots+\omega_l]+C^2_{r,m-l-1}(\frac{1}{2}i\gamma-\Delta\omega_0)}$$
$$\times\{b^{(l-1)}[\omega_1,\omega_2,\circ\circ\circ,\omega_{l-1}; E=-i(z+i\epsilon)]+b^{(l-1)}[\omega_1,\omega_2,\ldots,\omega_{l-2},\omega_l; E=-i(z+i\epsilon)]+\cdots$$
$$+b^{(l-1)}[\omega_2,\omega_3,\ldots,\omega_l; E=-i(z+i\epsilon)]\} , \tag{33b}$$

for $l=2,3,\ldots,r+m$, where

$$\tfrac{1}{2}i\gamma-\Delta\omega_0=\int_0^\infty[f(\omega)d\omega/(\omega-\omega_0-i\epsilon)] , \tag{34}$$

or

$$\gamma=2\pi f(\omega_0) , \tag{35a}$$

$$\Delta\omega_0=-P\int_0^\infty[f(\omega)d\omega/(\omega-\omega_0)] . \tag{35b}$$

Also from Eq. (27), when $D^{(0)}_{r,m-1}(\omega_1;E)$ is neglected, it follows that

$$a[E=-i(z+i\epsilon)] \simeq \frac{i}{z-(\frac{1}{2}N+m)\omega_0+C^2_{r,m-1}(\frac{1}{2}i\gamma-\Delta\omega_0)} . \tag{36}$$

The relations (33) and (36) are adopted as an approximate solution of the system of coupled integral equations (23) and Eq. (22). When these relations are set into Eqs. (20), the matrix element $A(t-t_0)$ and the functions $B^{(l)}(\omega_1,\ldots,\omega_l;t-t_0)$ are obtained. The matrix element $B_{\alpha_1\alpha_2\cdots\alpha_l}(t-t_0)$ can be evaluated then from Eq. (15). The first three matrix elements are given below:

$$A(t-t_0)=\exp\{-i[(\tfrac{1}{2}N+m)\omega_0-C^2_{r,m-1}(\tfrac{1}{2}i\gamma-\Delta\omega_0)](t-t_0)\} , \tag{37a}$$

$$B_{\vec{k}_1\lambda_1}(t-t_0)=C_{r,m-1}g_{\vec{k}_1\lambda_1}\exp\{-i[(\tfrac{1}{2}N+m)\omega_0-C^2_{r,m-1}(\tfrac{1}{2}i\gamma-\Delta\omega_0)](t-t_0)\})$$
$$\times(1-\exp\{-i[\omega_{\vec{k}_1}-\omega_0-2(m-1)(\tfrac{1}{2}i\gamma-\Delta\omega_0)](t-t_0)\})F_{m-1}(\omega_{\vec{k}_1}-\omega_0) , \tag{37b}$$

$$B_{\vec{k}_1\lambda_1;\vec{k}_2\lambda_2}(t-t_0)=\frac{1}{\sqrt{2}}C_{r,m-1}C_{r,m-2}g_{\vec{k}_1\lambda_1}g_{\vec{k}_2\lambda_2}\exp\{-i[(\tfrac{1}{2}N+m)\omega_0-C^2_{r,m-1}(\tfrac{1}{2}i\gamma-\Delta\omega_0)](t-t_0)\}$$
$$\times[(1-\exp\{-i[\omega_{\vec{k}_1}+\omega_{\vec{k}_2}-2\omega_0-2(2m-3)(\tfrac{1}{2}i\gamma-\Delta\omega_0)](t-t_0)\})F_{2m-3}(\omega_{\vec{k}_1}+\omega_{\vec{k}_2}-2\omega_0)[F_{m-1}(\omega_{\vec{k}_2}-\omega_0)$$
$$-F_{m-2}(\omega_{\vec{k}_2}-\omega_0)]+(1-\exp\{-i[\omega_{\vec{k}_1}-\omega_0-2(m-1)(\tfrac{1}{2}i\gamma-\Delta\omega_0)](t-t_0)\})F_{m-1}(\omega_{\vec{k}_1}-\omega_0)$$
$$\times F_{m-2}(\omega_{\vec{k}_2}-\omega_0)+(\exp\{-i[\omega_{\vec{k}_1}+\omega_{\vec{k}_2}-2(2m-3)(\tfrac{1}{2}i\gamma-\Delta\omega_0)](t-t_0)\}-\exp\{-i[\omega_{\vec{k}_2}-\omega_0$$
$$-2(m-1)(\tfrac{1}{2}i\gamma-\Delta\omega_0)](t-t_0)\})F_{m-1}(\omega_{\vec{k}_2}-\omega_0)F_{m-2}(\omega_{\vec{k}_1}-\omega_0)] , \tag{37c}$$

where

$$F_s(\omega)=1/[\omega-2s(\tfrac{1}{2}i\gamma-\Delta\omega_0)] . \tag{38}$$

In the above relations, use has been made of the definition [Eq. (10)] of $C_{r,m}$.

III. TRANSITION PROBABILITIES AND EXPECTATION VALUES

In Sec. II explicit expressions have been obtained for evaluating the matrix elements defined by Eq. (12). From these matrix elements transition probabilities and expectation values can be evaluated as functions of time.

In the following the cooperation number r will be limited to its maximum value $r=\frac{1}{2}N$. In this case the Dicke state $|r=\frac{1}{2}N, m\rangle$ can be given a simple physical meaning. Let $\frac{1}{2}(N+2m)$ out of N atoms be excited. A possible state which describes this situation is the following:

$$|m_1=\tfrac{1}{2},\ldots,m_{(N+2m)/2}=\tfrac{1}{2},m_{(N+2m)/2+1}=-\tfrac{1}{2},\ldots,m_N=-\tfrac{1}{2};s=1;N,m\rangle$$
$$=|r_1=\tfrac{1}{2},m_1=\tfrac{1}{2}\rangle\cdots|r_{(N+2m)/2}=\tfrac{1}{2},m_{(N+2m)/2}=\tfrac{1}{2}\rangle|r_{(N+2m)/2+1}=\tfrac{1}{2},m_{(N+2m)/2+1}=-\tfrac{1}{2}\rangle\cdots|r_N=\tfrac{1}{2},m_N=-\tfrac{1}{2}\rangle\ , \quad (39)$$

i.e., the first $\frac{1}{2}(N+2m)$ atoms with indices $1, 2, \ldots, \frac{1}{2}(N+2m)$ are excited and the rest of them are in the ground state. A permutation of the values of m_i in Eq. (39) gives another possible state in which it is given explicitly which $\frac{1}{2}(N+2m)$ out of the N atoms are excited. There are altogether $N!/[\frac{1}{2}(N+2m)]![\frac{1}{2}(N-2m)]!$ such states and they are denoted as follows:

$$|\{m_i\};s;N,m\rangle,\quad s=1,2,\ldots,N!/[\tfrac{1}{2}(N+2m)]![\tfrac{1}{2}(N-2m)]!$$

where the index s refers to each particular permutation. These states satisfy the orthonormality condition. Now using the induction method and the Clebsch-Gordan coefficients which couple a spin $\frac{1}{2}$ with a spin $\frac{1}{2}(N-1)$ it is easy to prove the identity

$$|r=\tfrac{1}{2}N,m\rangle=\left(\frac{[\frac{1}{2}(N+2m)]![\frac{1}{2}(N-2m)]!}{N!}\right)^{1/2}\times\sum_{s=1}^{N!/[(1/2)(N+2m)]![(1/2)(N-2m)]!}|\{m_i\};s;N,m\rangle. \quad (40)$$

The state on the right-hand side describes the physical situation in which any $\frac{1}{2}(N+2m)$ out of N atoms are excited with equal *a priori* probability, all possibilities being considered. The physical meaning of the Dicke state $|r=\frac{1}{2}N,m\rangle$ follows then from the identity (40).

Initially, i.e., at time $t=t_0$, the set of N atoms is assumed to be in the state $|\frac{1}{2}N,m\rangle$ and there are no photons present in the field. The probability that the system remains in that same state at a later time t is equal to

$$B_m^{(0)}(N;t-t_0)\equiv|A(t-t_0)|^2=\exp\{[\tfrac{1}{2}N(\tfrac{1}{2}N+1)-(m-1)m]\gamma(t-t_0)\}\ , \quad (41)$$

as it follows from Eqs. (37a) and (10). In particular, if all atoms are excited initially ($m=\frac{1}{2}N$) or if any one out of the N atoms is excited initially with equal *a priori* probability ($m=-\frac{1}{2}N+1$), in both cases one has

$$B^{(0)}_{(1/2)N}(N;t-t_0)=B^{(0)}_{(-1/2)N+1}(N;t-t_0)=e^{-N\gamma(t-t_0)}\ . \quad (42)$$

The maximum rate of decay $B_m^{(0)}(N;t-t_0)$ at time $t=t_0$ is attained when the expression $\frac{1}{2}N(\frac{1}{2}N+1)-(m-1)m$ in Eq. (41) becomes maximum. This happens for $m=\frac{1}{2}$.

From Eq. (37b) it follows that the probability for a set of N atoms starting from the state $|\frac{1}{2}N,m\rangle$ with no photons present in the field at time $t=t_0$ to be in the state $|\frac{1}{2}N,m-1\rangle$ at a later time t and one photon be emitted with momentum $\vec{k}$ lying within d^3k [in the limit of an infinite volume V the expression $(2\pi)^3/V=\Delta k_x\Delta k_y\Delta k_z$ tends to $d^3k=k^2dkd\Omega_{\vec{k}}$] and polarization λ is equal to

$$|B_{\vec{k}\lambda}(t-t_0)|^2=\frac{|g_{\vec{k}\lambda}|^2[\frac{1}{2}N(\frac{1}{2}N+1)-(m-1)m]\exp\{-[\frac{1}{2}N(\frac{1}{2}N+1)-(m-1)m]\gamma(t-t_0)\}}{[\omega_{\vec{k}}-\omega_0+2(m-1)\Delta\omega_0]^2+(m-1)^2\gamma^2}$$
$$\times\{1-2\exp[-(m-1)\gamma(t-t_0)]\cos(\omega_{\vec{k}}-\omega_0+2(m-1)\Delta\omega_0)(t-t_0)+\exp[-2(m-1)\gamma(t-t_0)]\}\ . \quad (43)$$

Similarly one may obtain from Eq. (37c) the expression $2!|B_{\vec{k}_1\lambda_1;\vec{k}_2\lambda_2}(t-t_0)|^2$ which is equal to the probability that two photons $(\vec{k}_1\lambda_1)$ lying within d^3k_1 and $(\vec{k}_2,\lambda_2)$ lying within d^3k_2 be emitted at a later time t and the set of N atoms be in the state $|\frac{1}{2}N,m-2\rangle$. The expressions $l!|B_{\vec{k}_1\lambda_1;\ldots;\vec{k}_l\lambda_l}\times(t-t_0)|^2$, $l=3,4,\ldots,\frac{1}{2}N+m$ are interpreted in a similar manner.

The probability that l photons be emitted with unspecified momentum and polarization while the set of N atoms is in the state $|\frac{1}{2}N,m-l\rangle$ at time t is obtained from the relation

$$B_m^{(l)}(N;t-t_0)=\frac{(2\pi)^3}{V}\sum_{\vec{k}_1\lambda_1}\cdots\frac{(2\pi)^3}{V}\sum_{\vec{k}_l\lambda_l}|B_{\vec{k}_1\lambda_1;\ldots;\vec{k}_l\lambda_l}(t-t_0)|^2\ , \quad (44)$$

for $l=1,2,\ldots,\frac{1}{2}N+m$. Equation (6) should be taken into account for a large volume V. In particular for $l=1,2$, Eq. (44) yields

$$B_m^{(1)}(N;t-t_0)=\frac{(\frac{1}{2}N-m+1)(\frac{1}{2}N+m)}{2m-2}\times\exp[-(\tfrac{1}{2}N-m+1)(\tfrac{1}{2}N+m)\gamma(t-t_0)]\times\{1-\exp[-(2m-2)\gamma(t-t_0)]\}\ , \quad (45)$$

$$B_m^{(2)}(N;t-t_0)=\tfrac{1}{2}(\tfrac{1}{2}N-m+1)(\tfrac{1}{2}N+m)(\tfrac{1}{2}N-m+2)(\tfrac{1}{2}N+m-1)\times\exp[-(\tfrac{1}{2}N-m+1)(\tfrac{1}{2}N+m)\gamma(t-t_0)]\left(-\frac{1}{2m-2}\{1-\exp[-(2m-2)\gamma(t-t_0)]\}\right.$$

$$+\frac{1}{2m-3}\{1-\exp[-2(2m-3)\gamma(t-t_0)]\}$$
$$-\frac{1}{2m-4}\exp[-(2m-2)\gamma(t-t_0)]$$
$$\{1-\exp[-(2m-4)\gamma(t-t_0)]\}\Big). \quad (46)$$

In deriving these two expressions it has been assumed that $\Delta\omega_0$ is negligible as compared to ω_0 so that it can be ignored and also that the function $f(\omega)$ [Eq. (16)] is smooth in comparison to the function $F_s(\omega-\omega_0)$ [Eq. (38)] so that it can be set equal to $f(\omega_0)=\gamma/2\pi$. The probability that no photons be emitted ($l=0$) at time t has already been obtained [Eq. (41)]. A general expression for $B_m^{(l)}(N;t-t_0)$ is obtained in the Appendix.

When there is one atom excited initially with equal *a priori* probability, i.e., $m=-\frac{1}{2}N+1$ and $r+m=1$, only $B^{(0)}_{(-1/2)N+1}(N;t-t_0)$ and $B^{(1)}_{(-1/2)N+1}(N;t-t_0)$ are different from zero. The function $B^{(0)}_{(-1/2)N+1}(N;t-t_0)$ is given by Eq. (42) while from Eq. (45)

$$B^{(1)}_{(-1/2)N+1}(N;t-t_0)=1-e^{-N\gamma(t-t_0)} \quad . \quad (47)$$

These two functions satisfy the identity

$$B^{(0)}_{(-1/2)N+1}(N;t-t_0)+B^{(1)}_{(-1/2)N+1}(N;t-t_0)=1 \quad . \quad (48a)$$

Similarly, when there are two atoms excited initially, i.e., $m=-\frac{1}{2}N+2$ and $r+m=2$, only $B^{(0)}_{(-1/2)N+2}(N;t-t_0)$, $B^{(1)}_{(-1/2)N+2}(N;t-t_0)$, and $B^{(2)}_{(-1/2)N+2}(N;t-t_0)$ are different from zero. These three functions satisfy the identity

$$B^{(0)}_{(-1/2)N+2}(N;t-t_0)+B^{(1)}_{(-1/2)N+2}(N;t-t_0)$$
$$+B^{(2)}_{(-1/2)N+2}(N;t-t_0)=1 \quad (48b)$$

as it can be easily verified from Eqs. (41), (45), and (46). Actually it may be shown in general that

$$\sum_{l=0}^{(1/2)N+m} B_m^{(l)}(N;t-t_0)=1 \quad . \quad (49)$$

This identity is obtained with the help of the relations

$$\langle\tfrac{1}{2}N,m|U^+(t,t_0)U(t,t_0)|\tfrac{1}{2}N,m\rangle=\langle\tfrac{1}{2}N,m|\tfrac{1}{2}N,m\rangle=1 \quad ,$$
$$\langle\tfrac{1}{2}N,m|U^+(t,t_0)U(t,t_0)|\tfrac{1}{2}N,m\rangle$$
$$=\sum_f|\langle f|U(t,t_0)|\tfrac{1}{2}N,m\rangle|^2 \quad ,$$

where the summation is over a complete set of states $|f\rangle$. It expresses the fact, that the total probability that either no photons, or one photon, or two photons, ..., or $\frac{1}{2}N+m$ photons be emitted, is conserved. It should be pointed out that Eqs. (41), (45), and (46), which are only approximate, satisfy for $m=-\frac{1}{2}N+1$, $-\frac{1}{2}N+2$ the identity (49), which is exact since no approximation whatsoever was involved in its derivation. In addition these three equations satisfy the right initial conditions [Eqs. (14)].

When the functions $B_m^{(l)}(N;t-t_0)$, $l=0,1,\ldots,\frac{1}{2}N+m$ are given, one may evaluate from them expectation values and correlations. For example, the expectation value for the total energy of the atoms is equal to

$$\mathcal{E}_m(N;t-t_0)=\hbar\omega_0[\mathcal{R}_m(N;t-t_0)+\tfrac{1}{2}N] \quad , \quad (50)$$

where

$$\mathcal{R}_m(N;t-t_0)\equiv\langle\tfrac{1}{2}N,m|R_3(t-t_0)|\tfrac{1}{2}N,m\rangle$$
$$\equiv\sum_{l=0}^{(1/2)N+m}(m-l)B_m^{(l)}(N;t-t_0) \quad . \quad (51)$$

Here $R_3(t-t_0)$ is the Heisenberg operator $U^+(t,t_0)R_3U(t,t_0)$. The relation (50) can also be written as follows:

$$\mathcal{E}_m(N;t-t_0)=\hbar\omega_0\sum_{l=0}^{(1/2)N+m}(\tfrac{1}{2}N+m-l)B_m^{(l)}(N;t-t_0) \quad , \quad (52)$$

where Eq. (49) has been used. The last expression certainly represents an average since $B_m^{(l)}(N;t-t_0)$ is not only the probability that l photons be emitted at time t but it is also the probability that $\frac{1}{2}N+m-l$ atoms be still excited at time t, each excited atom with an energy $\hbar\omega_0$.

Now one may introduce the expectation value

$$\mathcal{P}_m^{(n)}(N;t-t_0)=\langle\tfrac{1}{2}N,m|R_+^{(1)}(t-t_0)R_-^{(1)}(t-t_0)$$
$$\times\cdots R_+^{(n)}(t-t_0)R_-^{(n)}(t-t_0)|\tfrac{1}{2}N,m\rangle \quad , \quad (53a)$$

for $n=1,2,\ldots,N$ where

$$R_+^{(i)}(t-t_0)=U^+(t,t_0)R_+^{(i)}U(t,t_0) \quad ,$$
$$R_-^{(i)}(t-t_0)=U^+(t,t_0)R_-^{(i)}U(t,t_0), \quad i=1,2,\ldots,N \quad .$$

The physical meaning of this expectation value is revealed from the relation

$$\mathcal{P}_m^{(n)}(N;t-t_0)$$
$$=\sum_f|\langle f|R_-^{(1)}R_-^{(2)}\cdots R_-^{(n)}U(t,t_0)|\tfrac{1}{2}N,m\rangle|^2 \quad , \quad (53b)$$

where the summation is over a complete set of states $|f\rangle$. Hence $\mathcal{P}_m^{(n)}(N;t-t_0)$ is equal to the probability that the atoms with indices $1,2,\ldots,n$ be excited at time t irrespective of the state of the rest of the atoms and the radiation field. Actually one can easily prove that $R_+^{(1)}R_-^{(1)}R_+^{(2)}R_-^{(2)}\cdots R_+^{(n)}\times R_-^{(n)}$ is a projection operator. It is not difficult to show that

$$\mathcal{P}_m^{(n)}(N;t-t_0)=\sum_{l=0}^{(1/2)N+m}C^{(n)}(\tfrac{1}{2}N,m-l)B^{(l)}(N;t-t_0) \quad , \quad (54)$$

where

$$C^{(n)}(\tfrac{1}{2}N, m-l)$$
$$=\langle \tfrac{1}{2}N, m-l \,|\, R_+^{(1)}R_-^{(1)}\cdots R_+^{(n)}R_-^{(n)} \,|\, \tfrac{1}{2}N, m-l\rangle$$
$$=\frac{(\frac{1}{2}N+m-l)(\frac{1}{2}N+m-l-1)\cdots(\frac{1}{2}N+m-l-n+1)}{N(N-1)\cdots(N-n+1)}, \tag{55}$$

for $n=1, 2, \ldots, N$. If the indices $i=1, 2, \ldots, n$ of the operators $R_+^{(i)}$, $R_-^{(i)}$ in Eq. (55) are replaced by any other set of n different indices among the N available, Eq. (55) remains the same. Hence the function $\mathcal{P}_m^{(n)}(N; t-t_0)$ does not depend on the particular choice of the n atoms in Eqs. (53a) or (53b). When all atoms are excited initially ($m=\frac{1}{2}N$) one has from Eqs. (42), (45), (46), (54), and (55):

$$\mathcal{P}_{(1/2)N}^{(N)}(N; t-t_0)=e^{-N\gamma(t-t_0)}, \tag{56a}$$

$$\mathcal{P}_{(1/2)N}^{(N-1)}(N; t-t_0)$$
$$=e^{-N\gamma(t-t_0)}\left[1+\frac{1}{N-2}\,(1-e^{-(N-2)\gamma(t-t_0)})\right], \tag{56b}$$

$$\mathcal{P}_{(1/2)N}^{(N-1)}(N; t-t_0)$$
$$=e^{-N\gamma(t-t_0)}\left[1+\frac{1}{N-3}\,(1-e^{-2(N-3)\gamma(t-t_0)})\right.$$
$$\left.-\frac{2}{N-4}\,e^{-(N-2)\gamma(t-t_0)}(1-e^{-(N-4)\gamma(t-t_0)})\right]. \tag{56c}$$

The first relation has already been obtained by Dillard and Robl[4] [in the statement following Eq. (3.12) in their paper]. For $N=2$ Eq. (56b) gives the relation

$$\mathcal{P}_1^{(1)}(2; t-t_0)=e^{-2\gamma(t-t_0)}[1+\gamma(t-t_0)], \tag{57a}$$

and for $N=3$ Eq. (56c) gives the relation

$$\mathcal{P}_{3/2}^{(1)}(3; t-t_0)=e^{-3\gamma(t-t_0)}[4\gamma(t-t_0)-1+2e^{-\gamma(t-t_0)}]. \tag{57b}$$

These two relations coincide with Eqs. (3.23) and (3.24) in the paper by Dillard and Robl. A general expression for $\mathcal{P}_{1/2N}^{(n)}(N; t-t_0)$ is given in the Appendix.

It is worth mentioning that $\mathcal{R}_m(N; t-t_0)$ and $\mathcal{P}_m^{(1)}\times(N; t-t_0)$ are not independent of each other. It follows from Eqs. (51), (54), and (55) that

$$\mathcal{R}_m(N; t-t_0)=N[\mathcal{P}_m^{(1)}(N; t-t_0)-\tfrac{1}{2}]. \tag{58}$$

Then one has from Eqs. (50) and (58)

$$\mathcal{E}_m(N; t-t_0)=N\hbar\omega_0\mathcal{P}_m^{(1)}(N; t-t_0), \tag{59}$$

for the energy of the atoms.

The photon expectation values can also be expressed in terms of the functions $B_{\vec{k}_1\lambda_1;\ldots;\vec{k}_l\lambda_l}\times(t-t_0)$ or $B_m^{(l)}(N; t-t_0)$. For example, the number of photons with momentum $\vec{k}$ lying within d^3k and polarization λ at a given time t is given by the relation

$$\mathfrak{N}_{\vec{k}\lambda}(m, N; t-t_0)$$
$$\equiv\langle \tfrac{1}{2}N, m \,|\, a_{\vec{k}\lambda}^{+}(t-t_0)a_{\vec{k}\lambda}(t-t_0) \,|\, \tfrac{1}{2}N, m\rangle$$
$$=\sum_{l=0}^{(1/2)N+m-1}(l+1)\,\frac{(2\pi)^3}{V}\sum_{\vec{k}_1\lambda_1}\cdots\frac{(2\pi)^3}{V}$$
$$\times\sum_{\vec{k}_l\lambda_l}|B_{\vec{k}\lambda;\vec{k}_1\lambda_1;\ldots;\vec{k}_l\lambda_l}(t-t_0)|^2, \tag{60}$$

where $a_{\vec{k}\lambda}(t-t_0)=U^{+}(t, t_0)a_{\vec{k}\lambda}U(t, t_0)$.

It follows from Eq. (15) that

$$\mathfrak{N}_{\vec{k}\lambda}(m, N; t-t_0)=|g_{\vec{k}\lambda}|^2 h_m(N, \omega_{\vec{k}}; t-t_0), \tag{61}$$

where $h_m(N, \omega_{\vec{k}}; t-t_0)$ does not depend on the direction of $\vec{k}$. Hence the radiation pattern of the number of photons emitted in $d^3k=k^2dkd\Omega_{\vec{k}}$ is determined by $|g_{\vec{k}\lambda}|^2$. If account is taken of the fact that in the electric-dipole approximation

$$g_{\vec{k}\lambda}=\frac{-i\omega_0}{(2\pi)^{3/2}\hbar c}\left(\frac{4\pi\hbar c^2}{2\omega_{\vec{k}}}\right)^{1/2}\vec{\mu}\cdot\vec{\epsilon}^{\,*}(\vec{k}, \lambda), \tag{62}$$

where

$$\vec{\mu}=e\int u_1^{*}(\vec{x})\,\vec{x}\,u_2(\vec{x})\,d^3x, \tag{63}$$

is the electric-dipole moment of the atoms and $\vec{\epsilon}(\vec{k}, \lambda)$ is the polarization vector of a given mode of the electromagnetic field, it is seen that the radiation pattern depends explicitly on the sort of transition the atoms make,[7] as expressed through the wave functions $u_1(\vec{x}), u_2(\vec{x})$.

The number of photons present in the field at a given time t is equal to

$$\mathfrak{N}_m^{(1)}(N; t-t_0)$$
$$\equiv\langle \tfrac{1}{2}N, m \,|\, [(2\pi)^3/V]\sum_{\vec{k}\lambda}a_{\vec{k}\lambda}^{+}(t-t_0)a_{\vec{k}\lambda}(t-t_0) \,|\, \tfrac{1}{2}N, m\rangle$$
$$=\sum_{l=1}^{(1/2)N+m} lB_m^{(l)}(N; t-t_0). \tag{64a}$$

From Eqs. (51), (58), and (64a) it is easy to establish the relation

$$\mathfrak{N}_m^{(1)}(N; t-t_0)=\tfrac{1}{2}N+m-N\mathcal{P}_m^{(1)}(N; t-t_0). \tag{64b}$$

When $N=2, 3$ and all atoms are excited, $\mathfrak{N}_{N/2}^{(1)}\times(N; t-t_0)$ can be obtained by substituting Eqs. (57) into Eq. (64b).

The statistical properties of the radiation field are described by the higher moments of the photon number operator

$$\mathfrak{N}_m^{(n)}(N; t-t_0)\equiv\langle \tfrac{1}{2}N, m \,|\, \{[(2\pi)^3/V]$$
$$\times\sum_{\vec{k}\lambda}a_{\vec{k}\lambda}^{+}(t-t_0)a_{\vec{k}\lambda}(t-t_0)\}^n \,|\, \tfrac{1}{2}N, m\rangle$$
$$=\sum_{l=1}^{(1/2)N+m} l^n B_m^{(l)}(N; t-t_0). \tag{65}$$

The last relation clearly indicates that $B_m^{(l)}(N; t-t_0)$

is equal to the probability that l photons be present in the field at time t. To prove this relation one should make use of the induction method and the following two identities:

$$\mathfrak{N}_m^{(n)}(N;t-t_0)-\mathfrak{N}_m^{(n-1)}(N;t-t_0)$$
$$=\sum_{l=2}^{(1/2)N+m}\sum_{s=2}^{l}A_s^{(n)}\frac{l!}{(l-s)!}B_m^{(l)}(N;t-t_0)\ , \quad (66a)$$

where

$$A_s^{(n)}=\sum_{r=0}^{s-2}\frac{(-1)^r}{r!(s-r-2)!}(s-r)^{n-2}\ , \quad (66b)$$

$$l^n-l^{n-1}=\sum_{s=2}^{l}A_s^{(n)}\frac{l!}{(l-s)!}\ . \quad (67)$$

It is interesting to note that only the first $\frac{1}{2}N+m$ moments in Eq. (65) are independent. As an example, let $m=-\frac{1}{2}N+1$, i.e., only one out of the N atoms is excited initially. Then from Eqs. (47), (64a), and (65) one has the relation

$$\mathfrak{N}^{(n)}_{(-1/2)N+1}(N;t-t_0)=1-e^{-N\gamma(t-t_0)} \quad (68)$$

for $n=1,2,\ldots$, i.e., the emitted radiation is Poissonian in its statistical nature. After a time $t-t_0\gg 1/(N\gamma)$ one photon will be present in the field, since only one atom was excited initially. Also, the more atoms N are contained in the system, the faster the radiation field will build up.

Now for $l=0,1,2$ and $m=-\frac{1}{2}N+l$ one has from Eqs. (41), (45), and (46) that $B^{(l)}_{(-1/2)N+1}(N;t-t_0)\to 1$, as $t-t_0\to\infty$, while $B_m^{(l)}(N;t-t_0)\to 0$, as $t-t_0\to\infty$ for any other value of m. This result can actually be generalized, i.e., $B_m^{(l)}(N;t-t_0)\to 1$, as $t-t_0\to\infty$ only when l takes its maximum value $l=\frac{1}{2}N+m$, while for all other values of $l=0,1,\ldots,\frac{1}{2}N+m-1$ the function $B_m^{(l)}(N;t-t_0)$ vanishes as $t-t_0$ tends to infinity. It follows then from Eqs. (54), (59), and (64b) that $\mathcal{P}_m^{(1)}(N;\infty)=0$, $\mathcal{E}_m(N;\infty)=0$, and $\mathfrak{N}_m^{(1)}(N;\infty)=\frac{1}{2}N+m$, while at time $t=t_0$ one has $\mathcal{P}_m^{(1)}(N;0)=\frac{1}{2}+m/N$, $\mathcal{E}_m(N;0)=\hbar\omega_0(\frac{1}{2}N+m)$ and $\mathfrak{N}_m^{(1)}(N;0)=0$. Hence the total amount of energy that the atoms have originally is transferred to the radiation field by the creation of $\frac{1}{2}N+m$ photons, as time evolves. The energy of the radiation field at time t is equal to

$$\mathcal{W}_m(N;t-t_0)=\langle\tfrac{1}{2}N,m\big|[(2\pi)^3/V]$$
$$\sum_{\vec{k}\lambda}\hbar\omega_{\vec{k}}a^{\dagger}_{\vec{k}\lambda}(t-t_0)a_{\vec{k}\lambda}(t-t_0)\big|\tfrac{1}{2}N,m\rangle\ . \quad (69)$$

The right-hand side of this relation can be expressed in terms of the functions

$$[(2\pi)^3/V]\sum_{\vec{k}_1\lambda_1}\cdots[(2\pi)^3/V]\sum_{\vec{k}_l\lambda_l}\left|B_{\vec{k}\lambda;\vec{k}_1\lambda_1;\ldots;\vec{k}_l\lambda_l}(t-t_0)\right|^2\ ,$$

which are centered around ω_0 and have a very small width compared to ω_0. Hence Eq. (69) becomes approximately

$$\mathcal{W}_m(N;t-t_0)\simeq\hbar\omega_0\mathfrak{N}_m^{(1)}(N;t-t_0)\ . \quad (70)$$

Now from Eqs. (59), (64b), and (70) it follows that at any time t

$$\mathcal{E}_m(N;t-t_0)+\mathcal{W}_m(N;t-t_0)\simeq\hbar\omega_0(\tfrac{1}{2}N+m)\ . \quad (71)$$

On the other hand, the total energy of the system of atoms and radiation, including the interaction part of the energy, is conserved [Eq. (1)], so that

$$\mathcal{H}_m(N;t-t_0)\equiv\langle\tfrac{1}{2}N,m\big|H(t-t_0)\big|\tfrac{1}{2}N,m\rangle$$
$$=\hbar\omega_0(\tfrac{1}{2}N+m)\ , \quad (72)$$

where $H(t-t_0)=U^{+}(t,t_0)HU(t,t_0)$. From the last two relations it follows that

$$\mathcal{H}_m^{(\mathrm{int})}(N;t-t_0)$$
$$\equiv\mathcal{H}_m(N;t-t_0)-\mathcal{E}_m(N;t-t_0)-\mathcal{W}_m(N;t-t_0)$$
$$\simeq 0\ , \quad (73)$$

i.e., the total energy of the system at any time t is divided among the atoms and the photons while the interaction part of the energy is almost zero.

A general result can be obtained concerning the rate at which the radiation field is generated at time $t=t_0$. From Eqs. (58) and (64b) one has the relation

$$\frac{d\mathfrak{N}_m^{(1)}(N;t-t_0)}{dt}=-\frac{d\mathcal{R}_m(N;t-t_0)}{dt}\ . \quad (74)$$

Using this expression and the equation of motion

$$\frac{dR_3(t-t_0)}{dt}=-\frac{(2\pi)^3}{V}\sum_{\vec{k}\lambda}[g_{\vec{k}\lambda}a^{\dagger}_{\vec{k}\lambda}(t-t_0)R_-(t-t_0)$$
$$-g^{*}_{\vec{k}\lambda}a_{\vec{k}\lambda}(t-t_0)R_+(t-t_0)]\ , \quad (75)$$

one may obtain the relation

$$\left.\frac{d\mathfrak{N}_m^{(1)}(N;t-t_0)}{dt}\right|_{t=t_0}=-2C_{(1/2)N,m-1}$$
$$\times\mathrm{Im}\left[A(t-t_0)\frac{(2\pi)^3}{V}\sum_{\vec{k}\lambda}g_{\vec{k}\lambda}B^{*}_{\vec{k}\lambda}(t-t_0)\right]_{t=t_0}\ , \quad (76)$$

or substituting $A(t-t_0)$, $B_{\vec{k}\lambda}(t-t_0)$ from Eqs. (37a) and (37b) and performing the summation, the last relation yields

$$\left.\frac{d\mathfrak{N}_m^{(1)}(N;t-t_0}{dt}\right|_{t=t_0}=[\tfrac{1}{2}N(\tfrac{1}{2}N+1)-(m-1)m]\gamma\ . \quad (77)$$

If all atoms are excited, the rate at which the radiation field is generated at time t_0 is proportional to the number of atoms N in the system.

IV. PREPARATION OF SYSTEM

Up to now exclusive use has been made of the

Dicke states because they are most appropriate from the mathematical point of view in expressing and solving the system of equations (13). But it is the superposition states that can be easily prepared experimentally by applying an external electromagnetic field on the system of atoms which are initially in their ground state. In this case the Hamiltonian describing the interaction is

$$\hat{H}(t)=\hbar\omega_0(R_3+\tfrac{1}{2}N)+\hbar G(t)R_+ +\hbar G^*(t)R_- \ , \quad (78)$$

where

$$G(t)=(i\omega_0/\hbar c)\vec{\mu}^*\cdot\vec{A}^{(e)}(\vec{0},t) \ . \quad (79)$$

Here $\vec{A}^{(e)}(\vec{0},t)$ is the external vector potential at the point $\vec{X}=\vec{0}$, around which the N atoms are clustered, at time t. For simplicity, the electromagnetic field is taken to be at resonance with the transition frequency of the atoms, so that

$$\vec{A}^{(e)}(\vec{0},t)=\vec{a}e^{-i\omega_0 t}+\vec{a}^* e^{i\omega_0 t} \ . \quad (80)$$

When this expression for the vector potential is substituted into Eqs. (78) and (79), and the nonresonant terms are dropped, since their contribution is small for not too strong external fields, the Hamiltonian in Eq. (78) reduces to the following effective Hamiltonian:

$$\hat{H}_{\rm eff}(t)=\hbar\omega_0(R_3+\tfrac{1}{2}N)+\hbar G\,e^{-i\omega_0 t}R_+ +\hbar G^* e^{i\omega_0 t}R_- \ , \quad (81)$$

where

$$G=(i\omega_0/\hbar c)\vec{\mu}^*\cdot\vec{a} \ . \quad (82)$$

The evolution operator $\hat{U}(t)$ can be explicitly evaluated for this Hamiltonian. It satisfies the differential equation

$$i\hbar\frac{d\hat{U}(t)}{dt}=\hat{H}_{\rm eff}(t)\hat{U}(t) \ , \quad (83)$$

and the initial condition $\hat{U}(0)=1$. The solution is

$$\hat{U}(t)=\prod_{i=1}^{N}\left[e^{-i\omega_0(R_3^{(i)}+\frac{1}{2})t}(\cos|G|t+e^{i(\Phi-\frac{1}{2}\pi)}R_+^{(i)}\right.$$
$$\left.\sin|G|t+e^{-i(\Phi+\frac{1}{2}\pi)}R_-^{(i)}\sin|G|t)\right] \ , \quad (84)$$

where Φ is defined by the equation

$$G=|G|e^{i\Phi} \ . \quad (85)$$

Also $R_3^{(i)}$, $R_+^{(i)}$, $R_-^{(i)}$ are the spin $-\frac{1}{2}$ operators referring to the ith atom [Eqs. (2) and (4)]. If initially, i.e., at time $t=0$, the system is in its ground state

$$|g\rangle=\prod_{j=1}^{N}|r_j=\tfrac{1}{2},m_j=-\tfrac{1}{2}\rangle \ , \quad (86)$$

then at time t_0 the state of the system is the superposition state

$$|\alpha(t_0),\beta(t_0);N\rangle\equiv\hat{U}(t_0)|g\rangle$$
$$=\prod_{i=1}^{N_e}[\alpha(t_0)|r_i=\tfrac{1}{2},m_i=\tfrac{1}{2}\rangle$$
$$+\beta(t_0)|r_i=\tfrac{1}{2},m_i=-\tfrac{1}{2}\rangle] \ , \quad (87)$$

where

$$\alpha(t_0)=e^{i(\Phi-\frac{1}{2}\pi-\omega_0 t_0)}\sin|G|t_0 \ , \quad (88a)$$

$$\beta(t_0)=\cos|G|t_0 \ . \quad (88b)$$

The absolute values of the coefficients $\alpha(t_0)$, $\beta(t_0)$ are not independent of each other, i.e.,

$$|\alpha(t_0)|^2+|\beta(t_0)|^2=1 \ . \quad (89)$$

The probability then for each individual atom to be either in its ground state or excited at time t_0 is equal to 1.

Now, with the help of Eqs. (40) and (87), the following relation can be established:

$$|\alpha(t_0),\beta(t_0);N\rangle=\sum_{m=-(1/2)}^{(1/2)N}\alpha(t_0)^{(N+2m)/2}\beta(t_0)^{(N-2m)/2}$$
$$\times\left(\frac{N!}{[\frac{1}{2}(N+2m)]![\frac{1}{2}(N-2m)]!}\right)^{1/2}|\tfrac{1}{2}N,m\rangle \ , \quad (90)$$

i.e., the superposition state may be expressed as a linear combination of Dicke states with their cooperation number r taking its maximum value $r=\frac{1}{2}N$. It is Eq. (90) then that allows the evaluation of expectation values with respect to superposition states if the expectation values with respect to Dicke states are given. For example, the total energy of the atoms is equal to

$$\mathcal{E}(N;|\alpha(t_0)|;t-t_0)$$
$$\equiv\hbar\omega_0\langle\alpha(t_0),\beta(t_0);N|(R_3(t-t_0)+\tfrac{1}{2}N)|\alpha(t_0),\beta(t_0);N\rangle$$
$$=\hbar\omega_0\sum_{m=-(1/2)N}^{(1/2)N}|\alpha(t_0)|^{N+2m}|\beta(t_0)|^{N-2m}$$
$$\times\frac{N!}{[\frac{1}{2}(N+2m)]![\frac{1}{2}(N-2m)]!}\mathcal{E}_m(N;t-t_0) \ . \quad (91)$$

Here $R_3(t-t_0)$ is the Heisenberg operator $U^+(t,t_0)\times R_3 U(t,t_0)$, as before. The probability that n atoms be excited at time t if the atoms are in a superposition state at time t_0 is equal to

$$\mathrm{P}^{(n)}(N,|\alpha(t_0)|;t-t_0)$$
$$\equiv\langle\alpha(t_0),\beta(t_0);N|R_+^{(1)}(t-t_0)R_-^{(1)}(t-t_0)$$
$$\times\cdots R_+^{(n)}(t-t_0)R_-^{(n)}(t-t_0)|\alpha(t_0),\beta(t_0);N\rangle$$
$$=\sum_{m=(-1/2)N}^{(1/2)N}|\alpha(t_0)|^{N+2m}|\beta(t_0)|^{N-2m}$$
$$\times\frac{N!}{[\frac{1}{2}(N+2m)]![\frac{1}{2}(N-2m)]!}\mathcal{P}_m^{(n)}(N;t-t_0) \ , \quad (92)$$

where $n=1,2,\ldots,N$. It is a simple matter to establish the relation

$$\mathcal{E}(N,|\alpha(t_0)|;t-t_0)=N\hbar\omega_0\,\mathcal{P}^{(1)}(N,|\alpha(t_0)|;t-t_0)\ . \tag{93}$$

The energy of the atoms at time $t=t_0$, i.e., immediately after the system has been prepared to a superposition state, is equal to $N\hbar\omega_0|\alpha(t_0)|^2$, since $\mathcal{P}^{(n)}(N,|\alpha(t_0)|;0)=|\alpha(t_0)|^n$ for $n=1,2,\ldots,N$. When $N=2,3$, the probability $\mathcal{P}^{(1)}(N,|\alpha(t_0)|;t-t_0)$ is given by the following expressions:

$$\mathcal{P}^{(1)}(2,|\alpha(t_0)|;t-t_0)=|\alpha(t_0)|^2e^{-2\gamma(t-t_0)}\times[1+|\alpha(t_0)|^2\gamma(t-t_0)]\ , \tag{94a}$$

$$\mathcal{P}^{(1)}(3,|\alpha(t_0)|;t-t_0)=|\alpha(t_0)|^2e^{-3\gamma(t-t_0)}\times[1+2|\alpha(t_0)|^2(1-2|\alpha(t_0)|^2)\times(1-e^{-\gamma(t-t_0)})+4|\alpha(t_0)|^4\gamma(t-t_0)]\ . \tag{94b}$$

The number of photons with given momentum and polarization at time t is given by the relation

$$\begin{aligned}\mathfrak{N}_{\vec{k}\lambda}(N,|\alpha(t_0)|;t-t_0)&\equiv\langle\alpha(t_0),\beta(t_0);N|\,a^+_{\vec{k}\lambda}(t-t_0)\,a_{\vec{k}\lambda}(t-t_0)\,|\alpha(t_0),\beta(t_0);N\rangle\\&=\sum_{(1/2)N}^{(1/2)N}|\alpha(t_0)|^{N+2m}|\beta(t_0)|^{N-2m}\times\frac{N!}{[\frac{1}{2}(N+2m)]![\frac{1}{2}(N-2m)]!}\,\mathfrak{N}_{\vec{k}\lambda}(m,N;t-t_0)\ .\end{aligned} \tag{95}$$

From this relation and Eq. (61) it follows that the radiation pattern of the number of photons emitted in $d^3k=k^2dk\,d\Omega_{\vec{k}}$ is proportional to $|g_{\vec{k}\lambda}|^2$.

The number of photons present in the radiation field at time t as well as its statistical properties can be determined from the relation

$$\begin{aligned}\mathfrak{N}^{(n)}(N,|\alpha(t_0)|;t-t_0)&\equiv\langle\alpha(t_0),\beta(t_0);N|\,\{[(2\pi)^3/V]\sum_{\vec{k}\lambda}a^+_{\vec{k}\lambda}(t-t_0)\\&\quad\times a_{\vec{k}\lambda}(t-t_0)\}^n|\alpha(t_0),\beta(t_0);N\rangle\\&\sum_{m=-(1/2)N}^{(1/2)N}|\alpha(t_0)|^{N+2m}|\beta(t_0)|^{N-2m}\\&\times\frac{N!}{[\frac{1}{2}(N+2m)]![\frac{1}{2}(N-2m)]!}\,\mathfrak{N}^{(n)}_m(N;t-t_0)\ ,\end{aligned} \tag{96}$$

where $n=1,2,3,\ldots$. It is not difficult to prove that

$$\mathfrak{N}^{(1)}(N,|\alpha(t_0)|;t-t_0)=N[|\alpha(t_0)|^2-\mathcal{P}^{(1)}(N,|\alpha(t_0)|;t-t_0)] \tag{97}$$

with the help of Eqs. (64b) and (95). The energy of the radiation field is approximately given by the expression

$$\mathcal{W}(N,|\alpha(t_0)|;t-t_0)\simeq\hbar\omega_0\,\mathfrak{N}^{(1)}(N,|\alpha(t_0)|;t-t_0)\ . \tag{98}$$

Also the rate at which photons are emitted at time $t=t_0$ can be obtained from Eqs. (77) and (95). It is equal to

$$\left.\frac{d\mathfrak{N}^{(1)}(N,|\alpha(t_0)|;t-t_0)}{dt}\right|_{t=t_0}=N|\alpha(t_0)|^2[N-(N-1)|\alpha(t_0)|^2]\gamma\ . \tag{99}$$

This expression coincides with that of Dillard and Robl. Its maximum occurs when $|\alpha(t_0)|^2=\frac{1}{2}+\frac{1}{2}[(N-1)^{-1}]$. For this value of $|\alpha(t_0)|^2$, Eq. (99) becomes equal to $\{\frac{1}{4}(N-1)N+\frac{1}{2}N+\frac{1}{4}[N/(N-1)]\}\gamma$, i.e., for large N it is approximately proportional to N^2. But if all atoms are excited, i.e., $|\alpha(t_0)|^2=1$, then Eq. (99) is proportional to N.

The state where all atoms are excited can be prepared experimentally by choosing the time t_0 during which the atoms are under the influence of the external electromagnetic field such that $|G|t_0=n\pi+\frac{1}{2}\pi$, where $n=1,2,3,\ldots$. Of course, caution must be taken so that t_0 is much smaller than the decay time of the system of atoms. Also it may be noted that the expectation values given above are independent of the phase Φ, defined by Eq. (85). Hence, in general one needs to know only the intensity of the external electromagnetic field in order to evaluate expectation values.

V. ASSUMPTION OF A LARGE INTERATOMIC DISTANCE

The treatment in Secs. II–IV was based on the assumption that the N atoms are clustered together within a volume with linear dimensions much smaller than the mean wavelength λ_0 of the emitted radiation. When the atoms lie at distances which are not negligible compared to λ_0, the Hamiltonian introduced in Eq. (1) should be modified into the following one:

$$\begin{aligned}H&=\hbar\omega_0(R_3+\tfrac{1}{2}N)+\frac{(2\pi)^3}{V}\sum_{\vec{k}\lambda}\hbar\omega_{\vec{k}}\,a^+_{\vec{k}\lambda}a_{\vec{k}\lambda}\\&\quad-\hbar\sum_{i=1}^{N}\frac{(2\pi)^3}{V}\sum_{\vec{k}\lambda}(g_{\vec{k}\lambda}e^{-i\vec{k}\cdot\vec{X}_i}a^+_{\vec{k}\lambda}R^{(i)}_-\\&\quad+g^*_{\vec{k}\lambda}e^{i\vec{k}\cdot\vec{X}_i}a_{\vec{k}\lambda}R^{(i)}_+)\ .\end{aligned} \tag{100}$$

Here $\vec{X}_i$ is the position of the ith atom. This Hamiltonian reduces to that of Eq. (1), when the atoms lie close together around the point $\vec{X}=0$. Also it commutes with the operator defined by Eq. (8). Hence, the total number of photons and excited atoms is conserved by this Hamiltonian too. But the cooperation number is not conserved in general, as the following commutation relation shows:

$$[R^2,H]=4i\hbar\frac{(2\pi)^3}{V}\sum_{\vec{k}\lambda}g_{\vec{k}\lambda}a^{\dagger}_{\vec{k}\lambda}\sum_{\substack{i,j\\ i\neq j}}R_3^{(i)}R_-^{(j)}$$
$$\times e^{-\frac{1}{2}i\vec{k}\cdot(\vec{X}_i+\vec{X}_j)}\sin\tfrac{1}{2}\vec{k}\cdot(\vec{X}_i-\vec{X}_j)$$
$$-4i\hbar\frac{(2\pi)^3}{V}\sum_{\vec{k}\lambda}g^{*}_{\vec{k}\lambda}a_{\vec{k}\lambda}\sum_{\substack{i,j\\ i\neq j}}R_3^{(i)}R_+^{(j)}$$
$$\times e^{\frac{1}{2}i\vec{k}\cdot(\vec{X}_i+\vec{X}_j)}\sin\tfrac{1}{2}\vec{k}\cdot(\vec{X}_i-\vec{X}_j)\ . \quad (101)$$

When $|\vec{X}_i-\vec{X}_j|\ll\lambda_0$, the above commutator is almost zero due to the presence of the sine terms. Also, when $|\vec{X}_i-\vec{X}_j|\gg\lambda_0$, the right-hand side of the above relation is almost zero due to the validity of the Riemann-Lebesque lemma, which states that the sine (or cosine) Fourier transform tends to zero if its argument tends to infinity. Of course, no attempt for a rigorous proof will be made here. It is only after one knows the exact solution to the problem that he can check that the matrix elements which do not conserve the cooperation number tend to zero as $|\vec{X}_i-\vec{X}_j|$ tends to infinity.

It will be assumed then in the following that the cooperation number r is approximately conserved and the matrix elements which do not conserve r can be neglected. Then it can be easily shown that the matrix elements defined by Eqs. (12) with $r=\frac{1}{2}N$ satisfy exactly the same set of equations of motion (13), except that $g_{\vec{k}\lambda}$ ought to be replaced by

$$G_{\vec{k}\lambda}=g_{\vec{k}\lambda}\frac{1}{N}\sum_{i=1}^{N}e^{-i\vec{k}\cdot\vec{X}_i}\ , \quad (102)$$

in these equations. To prove the last statement, one should make use of the identities

$$\langle\tfrac{1}{2}N,m|R_+^{(i)}|\tfrac{1}{2}N,m\rangle=N^{-1}C_{\frac{1}{2}N,m}\ , \quad (103a)$$
$$\langle\tfrac{1}{2}N,m|R_-^{(i)}|\tfrac{1}{2}N,m\rangle=N^{-1}C_{\frac{1}{2}N,m+1}\ , \quad (103b)$$

for $i=1,2,\ldots,N$. $C_{\frac{1}{2}N,m}$ is defined by Eq. (10). Now, $f(\omega_{\vec{k}})$, defined by Eq. (16), ought also to be replaced by

$$F(\omega_{\vec{k}})=\frac{\omega_{\vec{k}}^2}{c^3}\int\sum_{\lambda}|g_{\vec{k}\lambda}|^2\frac{1}{N}$$
$$\left(1+\frac{2}{N}\sum_{\substack{i,j\\ i<j}}\cos\vec{k}\cdot(\vec{X}_i-\vec{X}_j)\right)d\Omega_{\vec{k}}\ . \quad (104)$$

Hence, the approximate solution (33) and all the relations obtained in the previous sections are valid, except that γ and $\Delta\omega_0$, defined by Eqs. (35), should be replaced by Γ and $\Delta\Omega_0$, given by the following expressions:

$$\Gamma=2\pi F(\omega_0)\ , \quad (105a)$$
$$\Delta\Omega_0=-P\int_0^{\infty}\frac{F(\omega)d\omega}{\omega-\omega_0}\ . \quad (105b)$$

In particular, $|g_{\vec{k}\lambda}|^2$ in Eqs. (61) and (95) should be replaced by

$$|G_{\vec{k}\lambda}|^2=|g_{\vec{k}\lambda}|^2\frac{1}{N}\left(1+\frac{2}{N}\sum_{\substack{i,j\\ i<j}}\cos\vec{k}\cdot(\vec{X}_i-\vec{X}_j)\right)\ . \quad (106)$$

It is seen then that the directional properties of the emitted radiation as well as the decay constant and frequency shift explicitly depend on the position of the atoms.

There are two interesting cases for which the cooperation number is approximately conserved: (i) When $|\vec{X}_i-\vec{X}_j|\ll\lambda_0$. Then

$$\sum_{\substack{i,j\\ i<j}}\cos\vec{k}\cdot(\vec{X}_i-\vec{X}_j)\simeq\sum_{\substack{i,j\\ i<j}}1=\tfrac{1}{2}(N-1)N\ ,$$

so that $|G_{\vec{k}\lambda}|^2\simeq|g_{\vec{k}\lambda}|^2$, $\Gamma\simeq\gamma$, and $\Delta\Omega_0\simeq\Delta\omega_0$. This is the case studied in the previous sections. (ii) When $|\vec{X}_i-\vec{X}_j|\gg\lambda_0$, $N\gg1$ and the atoms are randomly distributed, so that $\sum_{i<j}\cos\vec{k}\cdot(\vec{X}_i-\vec{X}_j)=0$. Then $|G_{\vec{k}\lambda}|^2=(1/N)|g_{\vec{k}\lambda}|^2$, $\Gamma=(1/N)\gamma$, and $\Delta\Omega_0=(1/N)\Delta\omega_0$. In this case the system will decay much slower as compared to case (i).

VI. DISCUSSION

There is some advantage in the method employed here, i.e., in expressing and solving the equations of motion of the matrix elements of the evolution operator rather than those of the operators themselves. As mentioned earlier, the former are linear, and hence easier to solve, while the latter are nonlinear. Also, one has a better feeling in the approximations performed on c-number functions (the matrix elements) rather than on operators. Or, assumptions may have to be introduced for the equations of motion of the matrix elements, which do not arise at all for the operators. For example, the assumption made in Sec. II that the set of atoms form an unstable system [Eq. (31) and the statement following it] is necessary if the atoms are to decay in their ground state as time evolves. Another advantage of the method employed here is that it provides much more detailed information about the behavior of the system and its evolution as compared to that obtained when one evaluates the expectation values of the time-dependent operators. The method employed here allows one to evaluate transition probabilities as functions of time. Then the expectation values of time-dependent operators can be easily evaluated by summing up appropriately these transition probabilities. Both transition probabilities and expectation values are measurable physical quantities. Hence, one should distinguish whether the photodetector is capable of measuring a single photon or a multiphoton process, i.e., whether it can measure a transition probability or the ex-

pectation value of the photon number operator.

Now let the following definition of a superradiant state be adopted here. A state of the set of N atoms at time t_0 will be called superradiant if the rate of emission per atom at time t_0 is proportional to N^α, where $\alpha \geq 1$ (for large N). In this context, the superposition state is a superradiant state when $|\alpha(t_0)|^2 \simeq \frac{1}{2}$, i.e., when t_0 is chosen such that $|G|t_0 \simeq n\pi + \frac{1}{4}\pi$, $n = 0, 1, 2, \ldots$, or $|G|t_0 \simeq n\pi - \frac{1}{4}\pi$, $n = 1, 2, 3, \ldots$ [Eq. (99)]. Superradiance can be associated with the fact that the atoms are clustered together, but this is not always the case. For example, when all the atoms are excited at time t_0 superradiance does not occur. When the atoms are far apart, then γ should be replaced by $\Gamma = N^{-1}\gamma$ in Eq. (99), and again superradiance does not occur. The Dicke state with $m = 0$ is also a superradiant state [Eq. (77)], but it should not be confused with the superposition state with $|\alpha(t_0)|^2 = |\beta(t_0)|^2 = \frac{1}{2}$. These two states are completely different to each other as Eq. (90) shows explicitly.

It is worth mentioning again that the directional properties of the emitted radiation as well as the decay constant and the frequency shift depend on the position of the atoms. When they are far apart and fixed in space, for example, when they form a lattice, then interference from the radiating atoms will take place and it will be detected in the radiation pattern, as it follows from Eq. (106).

The method of approach used in this paper to study the spontaneous emission from N identical atoms could also be employed when the initial state contains one or more photons. The cooperation number and the total number of photons and excited atoms are still conserved, when the atoms are close together or far apart. One should expect, of course, a more complicated set of equations of motion. When the distance between the atoms is of the same order of magnitude as the mean wavelength of the emitted radiation, the solutions given above, based on the conservation of the cooperation number, are not valid.

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APPENDIX

A general expression for $B_m^{(l)}(N;\varphi)$ and $\mathcal{P}_{(1/2)N}^{(n)} \times (N;\varphi)$, where $\varphi = \gamma(t - t_0)$, will be obtained here.

By direct inspection of Eqs. (45) and (46) one may deduce that, in general, $B_m^{(l)}(N;\varphi)$ is of the form

$$B_m^{(l)}(N;\varphi) = \prod_{r=1}^{l} (\tfrac{1}{2}N + m - r + 1)(\tfrac{1}{2}N - m + r) \times e^{-(\frac{1}{2}N - m + 1)(\frac{1}{2}N + m)\varphi} Q_m^{(l)}(\varphi) \quad , \tag{A1}$$

where $l = 1, 2, 3, \ldots$ and $Q_m^{(l)}(\varphi)$ is independent of N, the number of atoms. That this is the general form of $B_m^{(l)}(N;\varphi)$ could be proved using Eqs. (15), (20b), (33b), and (44). Since the author has not actually tried to prove the stated result, Eq. (A1) may be taken as a conjecture.

Once the form of $B_m^{(l)}(N;\varphi)$ is assumed, then it is a trivial matter to prove, using Eq. (49), that $Q_m^{(l)}(\varphi)$ is given by the following recurrence relation:

$$Q_m^{(l)}(\varphi) = \frac{(-1)^{l+1}}{l!(2m-l-1)(2m-l-2)\cdots(2m-2l)}(1 - e^{-l(2m-l-1)\varphi}) + (-1)^{l+1} \times \sum_{p=1}^{l-1} \frac{(-1)^p}{(l-p)!(2m-l-p-1)(2m-l-p-2)\cdots(2m-2l)} Q_m^{(p)}(\varphi) \quad . \tag{A2}$$

For example, from this relation and Eqs. (45) and (46) one may obtain for $Q_m^{(3)}(\varphi)$ the expression

$$Q_m^{(3)}(\varphi) = \frac{1}{2!3!}\left(\frac{1}{2m-2}(1 - e^{-(2m-2)\varphi}) - \frac{2}{2m-3}(1 - e^{-2(2m-3)\varphi}) + \frac{1}{2m-4}(1 - e^{-3(2m-4)\varphi}) + \frac{3}{2m-4}e^{-(2m-2)\varphi}(1 - e^{-(2m-4)\varphi}) - \frac{2}{2m-5}e^{-(2m-2)\varphi}(1 - e^{-2(2m-5)\varphi}) + \frac{1}{2m-6}e^{-2(2m-3)\varphi}(1 - e^{-(2m-6)\varphi})\right). \tag{A3}$$

Now let $Q_m^{(l)}(\varphi)$ be assumed to be of the form

$$Q_m^{(l)}(\varphi) = \frac{1}{(l-1)!l!}\sum_{r=1}^{l}\sum_{s=0}^{[(1/2)(r-1)]}(-1)^{l+r}\frac{A_{rs}^{(l)}}{2m-r-1}e^{-s(2m-s-1)\varphi}(1 - e^{-(r-2s)(2m-r-1)\varphi}) + \frac{1}{(l-1)!l!}\sum_{r=l+1}^{2l-1}\sum_{s=0}^{[(1/2)(2l-r-1)]}(-1)^{l+r}\frac{A_{2l-r,s}^{(l)}}{2m-r-1}e^{-(r-l+s)(2m-r+l-s-1)\varphi}(1 - e^{-(2l-r-2s)(2m-r-1)\varphi}) \, . \tag{A4}$$

Here the symbol $[x]$ means the integral part of the positive number x. If this expression is substituted into Eq. (A2), after a rather lengthy manipulation, it leads to the following recurrence relation for the coefficients $A_{rs}^{(l)}$

$$A_{rs}^{(l)} = \sum_{p=r}^{l-1} \frac{(l-1)!l!}{(2l-r-1)!}(-1)^{l+p+1}\frac{(l-r+p-1)!}{(p-1)!p!(l-p)!}A_{rs}^{(p)} + \sum_{p=r-s}^{r-1}\frac{(l-1)!l!}{(2l-r-1)!}(-1)^{l+p+1}\frac{(l-r+p-1)!}{(p-1)!p!(l-p)!}A_{2p-r,p-r+s}^{(p)}\ , \tag{A5}$$

where the first sum should be omitted when $r=l$, and the second sum should be omitted, when $s=0$. Also, one should take $A_{10}^{(l)}=1$. The index r takes the values $r=0, 1, 2, \ldots, l$, so that $r \leq l$, while the index s takes the values $s=0, 1, 2, \ldots, [\frac{1}{2}(r-1)]$, for given r. The solution of the above recurrence relation [Eq. (A5)] is

$$A_{rs}^{(l)} = (r-2s)\frac{(l-1)!}{(r-s)!(l-r+s)!}\frac{l!}{s!(l-s)!}\ . \tag{A6}$$

It is easy to check that

$$A_{2l-r,\,l-r+s}^{(l)} = A_{rs}^{(l)}\ . \tag{A7}$$

Once the coefficients $A_{rs}^{(l)}$ are given explicitly by Eq. (A6), both $Q_m^{(l)}(\varphi)$ and $B_m^{(l)}(N;\varphi)$ are known for any values of m, l, and N. Hence expectation values can be evaluated.

As an example, the probability $\mathcal{P}_{(1/2)N}^{(n)}(N;\varphi)$ that n atoms out of N be excited at time t, when originally all N atoms are excited, will be evaluated. From Eqs. (54) and (A1) with $m=\frac{1}{2}N$ one may obtain the following expression for $\mathcal{P}_{(1/2)N}^{(n)}(N;\varphi)$:

$$\mathcal{P}_{(1/2)N}^{(n)}(N;\varphi) = e^{-N\varphi}\left(1+\sum_{l=1}^{N-n}\frac{(N-n)!l!}{(N-n-l)!}Q_{(1/2)N}^{(l)}(\varphi)\right)\ . \tag{A8}$$

When $Q_{1/2N}^{(l)}(\varphi)$ is substituted into this relation from Eq. (A4) and after a rearrangement of the summations one obtains the expression

$$\mathcal{P}_{(1/2)N}^{(n)}(N;\varphi) = e^{-N\varphi}\left(1+\sum_{r=N-n}^{2(N-n)-1}\sum_{s=r-(N-n)}^{[(1/2)(r-1)]}\frac{C_{rs}^{(N-n)}}{N-r-1}\times e^{-s(N-s-1)\varphi}(1-e^{-(r-2s)(N-r-1)\varphi})\right)\ , \tag{A9}$$

where

$$C_{rs}^{(k)} = \frac{(r-2s)k!}{s!(r-s)!}\sum_{l=r-s}^{k}(-1)^{l+r}\frac{l!}{(k-l)!(l-s)!(l-r+s)!}\ . \tag{A10}$$

Equations (A6) and (A7) were used in obtaining the coefficients $C_{rs}^{(k)}$. After the summation is performed in Eq. (A10), the coefficients $C_{rs}^{(k)}$ are equal to

$$C_{rs}^{(k)} = (-1)^{k+r}(r-2s)\frac{k!}{(r-k)!(k-r+s)!(k-s)!}\ . \tag{A11}$$

The expressions obtained from Eqs. (A9) and (A11), for $n=1$ and $N=2, 3, 4, 8$ coincide with those of Dillard and Robl. For $n=1$ one can easily evaluate the energy of the N atoms and the number of photons (or the rate of emission of radiation) as time evolves, using Eqs. (59) and (64b) with $m=\frac{1}{2}N$.

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