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Nonrelativistic Electron Bremsstrahlung in a Strongly Magnetized Plasma

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In this paper we have calculated the emission of bremsstrahlung radiation of nonrelativistic electrons in Landau levels in a dense plasma. We have found that the radiation rate is inversely proportional to the electron momentum, which is characteristic of one-dimensional gases.

I. INTRODUCTION

In a previous paper[1] we presented the emission rate of bremsstrahlung radiation in vacuum. In view of the large plasma effect present in dense matter, we present here a nonrelativistic calculation of bremsstrahlung radiation in a dense plasma with a strong magnetic field. (That a nonrelativistic calculation is adequate in our theory for pulsars will be discussed in a separate paper.)

The problem of radiation emitted by an electron in passing through the field of a charged nucleus is a classical one in electrodynamics[2] and plasma physics.[3] In astrophysics it is known as the free-free transition,[4] while in electrodynamics it appears under the name bremsstrahlung. The main problem in this paper is concerned with bremsstrahlung radiation in such a strong field that the trajectory of the electron is no longer classical.[5] As the ion merely provides a Coulomb field, which in nonrelativistic cases can be regarded as static, we need not concern ourselves with quantization of ion orbits in intense magnetic fields. In a magnetic field the electron state can be described in terms of a principal quantum number n (= 0, 1, 2, ...) and a momentum variable p_z along the direction of the field H (which is taken to be in the z direction). The total energy of the electron $E(n, x)$ is[5]

$$\mathcal{E}(n,x) \equiv E(n,x)/mc^2 = (1+x^2+2nH/H_q)^{1/2} , \tag{1}$$

where $x = p_z/mc$, $H_q = m^2c^3/e\hbar = 4.414\times 10^{13}\,\mathrm{G}$.

The effect of the magnetic field merely quantized the x and y momenta by the following substitution[6]:

$$p_x^2 + p_y^2 \rightarrow 2n(H/H_q)(mc)^2 \ . \tag{2}$$

As a result of this quantization the density of final states is modified. A summation over all states now takes the following form[4,6]:

$$\textstyle\sum_n \omega_n \int (1/h)\, dp_z , \tag{3}$$

where ω_n is the degeneracy of the state n. For a particle of a given energy $\mathcal{E}$, n can only take values such that

$$\mathcal{E} \geqslant (1+2nH/H_q)^{1/2} \ . \tag{4}$$

In particular, if $\mathcal{E} - 1 \leqslant (1+2H/H_q)^{1/2} - 1$, the only state allowed is the state with $n=0$, and this particle behaves as a one-dimensional particle.

This paper is concerned primarily with cases such that Eq. (4) is satisfied for small values of n.

II. REGIMES OF INTEREST

We will now discuss the regime of applicability and will show that the conditions of small values of n are satisfied at the surface of neutron stars. The particle density of a zero-temperature Fermi gas in a magnetic field has been studied in previous papers.[5] The relation between the electron Fermi energy and the maximum number of particles N_m allowable in the ground state with $n=0$ is ($\lambda_c = \hbar/mc$)

$$N_m = (1/\sqrt{2}\,\pi^2)(H/H_q)^{3/2}\,\lambda_c^{-3} = 1.24\times10^{30}\,(H/H_q)^{3/2}. \tag{5}$$

The corresponding density ρ_m is

$$\rho_m = m_p N_m Z/A = 2.08\times10^{6}\,(Z/A)\,(H/H_q)^{3/2} \quad . \tag{6}$$

If we consider a field strength of 10^{12} G (which is favored in a number of theories), then the electrons behave as a one-dimensional gas when the density is less than 4×10^3 g/cm^3. The atmosphere of a neutron star has a density $\simeq 1$ g/cm^3. Hence the one-dimensional behavior of electrons is strongly exhibited in the atmosphere of neutron stars. As any radiation from a neutron star will emerge from a surface layer of one optical thickness, the bremsstrahlung process – which is the only continuum emission process – should play a dominant role in pulsar models where the pulsar radiation emerges from the surface. The association of delicate and minute variations of a pulsar period to its rotational period leaves no doubt that the pulsar radiation must emerge from the surface.

III. BASIC FORMULATION

A physically important improvement of this work, over previously considered cases,[7] is the introduction of plasma effect. We will separate the outgoing electromagnetic wave into two modes, ordinary and extraordinary,[8] each of them characterized by a well-defined refractive index. The plasma formulation of the problem calls for a dielectric tensor $\epsilon_{\alpha\beta}$ which is by its own nature an external parameter, whose form depends on the physical condition of the plasma in a magnetic field. Many different forms of dielectric tensors are available in the literature depending on whether one considers cold,[8] hot,[8,9] classical, or quantum plasma.[10,11] In our general formalism, we have not placed any assumption on the dielectric tensor: Hence, our result should be of general interest. After the general formalism is derived, we will present some simple applications in terms of an expression of $\epsilon_{\alpha\beta}$ applicable to a cold plasma.[8] The final form of I the total energy radiated per unit time integrated over all frequencies and all angles is given by

$$I = I_0 \int_{4\pi} d\Omega \int_0^\infty I(\omega, n, n', \Omega)\, d\omega \quad , \tag{7}$$

with $\quad I_0 = Z^2\alpha^3[mc^2/(\hbar/mc^2)]\,N_i\,\lambda_c^3/8\pi$. (8)

$I(\omega, \Omega, n, n')$ contains all information of the spectrum and angular dependence of radiation. In Eq. (7) the frequency ω is measured in units of $mc^2/\hbar = 7.76\times10^{20}$ sec^{-1}.

It is a general result of the Maxwell equation that the intensity emitted by a particle is[12,13]

$$I = \lim_{T\to\infty} \frac{1}{T} \int \frac{d^3k}{(2\pi)^3} \int_0^\infty \frac{d\omega}{2\pi} \,[\,\vec{j}(k,\omega)\,\vec{E}^*(k,\omega) + \vec{E}(k,\omega)\,\vec{j}^*(k,\omega)] \quad , \tag{9}$$

where * represents complex-conjugate quantities and $\vec{j}(k,\omega)$ and $\vec{E}(k,\omega)$ are the Fourier transform of the current $\vec{j}$ and electric field $\vec{E}$, respectively, i.e.,

$$F(x,t) = [1/(2\pi)^4] \int d^3k \int d\omega\, e^{ikx} e^{-i\omega t} F(k,\omega) \quad . \tag{10}$$

Equation (9) is usually written in two different forms, i.e.,

$$I = \lim_{T\to\infty} (1/T) \int_{4\pi} d\Omega \int_0^\infty d\omega\, I(\omega, \Omega) \quad ,$$

$$\text{or} \quad I = \lim_{T\to\infty} \int [d^3k/(2\pi)^3]\, \hbar\omega(k)\, W(k) \quad , \tag{11}$$

where $I(\omega,\Omega)$ and $W(k)$ are the emissivity and transition probability, respectively. It should be noted that in a plasma, ω is usually a complicated function of k.

A. Electron Current

In order to compute the current j we first formally change the Feynman diagram for the bremsstrahlung involving wave functions ψ_i and ψ_f into a fictitious first-order process, involving a pair of states Ψ_i, ψ_f.

The incoming electron wave function Ψ is given by[2]

$$\Psi_\alpha(x) \equiv \psi_\alpha(x) + \int G(x,x')\,\gamma_\mu A_\mu(x')\,\psi_\alpha(x')\, d^4x' \quad . \tag{12}$$

The label α signifies the set of quantum numbers necessary to describe a particle in a magnetic field; $A_\mu(x)$ is the external Coulomb potential and $G(x,x')$ is the Green's function of the electron in the magnetic field. Its general definition is[2]

$$G(x,x')=\frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\omega\, e^{i\omega(t-t')}\sum_\beta \frac{\psi_\beta(r)\bar{\psi}_\beta(r')}{E_\beta+\omega-i\Gamma} \quad . \tag{13}$$

In the nonrelativistic (NR) limit, when $A_\mu=\delta_{\mu 0}A_0$, $(A_0\equiv Ze/r)$, a simple computation leads to

$$\Psi_\alpha(x)=\psi_\alpha(x)+\psi_\alpha^{(1)}(x),\ \ \psi_\alpha^{(1)}(x)=\sum_\beta \frac{\langle\beta|A_0|\alpha\rangle}{E_\beta-E_\alpha-i\Gamma}\psi_\beta(x) \ , \tag{14}$$

which are well-known expressions from the perturbation theory. The electron current in Eq. (9) is now defined quantum mechanically as

$$\vec{j}(r,t)=(e/2m)(\Psi^*\vec{\Pi}\Psi-\Psi\vec{\Pi}\Psi^*) \ , \tag{15}$$

where $\vec{\Pi}=\vec{p}-(e/c)\vec{A}\equiv -i\hbar\vec{\nabla}-(e/c)\vec{A}$, (16)

and the vector potential $\vec{A}$ is given by

$$A_x=-yH,\quad A_y=A_z=0 \quad . \tag{17}$$

This form of $\vec{A}$ depends on the special gauge one chooses to work with. We have employed this special form of the gauge because it simplifies the form of the wave function we will use later on. Now, using Eq. (14) and retaining only the term of interest, we find

$$j(k,\omega)=2\pi\hbar j(k)\delta(E_i-E_f-\hbar\omega) \ , \tag{18}$$

$$j(k)=\frac{e}{2m}\sum_I\left(\frac{\langle f|e^{-ikr}\vec{\Pi}|I\rangle\langle I|A_0|i\rangle}{E_i-E_I-i\Gamma}+\frac{\langle f|A_0|I\rangle\langle I|e^{-ikr}\vec{\Pi}|i\rangle}{E_i-E_I-\hbar\omega-i\Gamma}\right) \ . \tag{19}$$

We have explicitly added a width factor Γ as one does in bound-state scattering problems where resonance effect can take place.[2] Its form will be discussed shortly. Before computing the various matrix elements in Eq. (19) we will discuss the electric field $\vec{E}$.

B. Electric Field

In vacuum, the electric field generated by a particle described by a current $\vec{j}(r,t)$ is given by the well-known Lienard-Wiechert formulas. Quantum electrodynamics gives a definite prescription for the field $A_\mu(x)$, at the point x, created by a current j_ν at the point y, i.e.,[2]

$$A_\mu(x)=\int D_{\mu\nu}(xy)j_\nu(y)\,d^4y \quad . \tag{20}$$

Here $D_{\mu\nu}(xy)$ is the photon propagator and satisfies Dyson's equation

$$D_{\mu\nu}=D^0_{\mu\nu}+D^0_{\mu\rho}\Pi_{\rho\beta}D^0_{\rho\nu} \ , \tag{21}$$

where $\Pi_{\rho\beta}$ is the polarization tensor related to the plasma properties of the medium through the well-known relations[14]

$$\Pi_{ik}=i\omega\sigma_{ik},\quad \Pi_{i4}=\Pi_{4i}=-k_j\sigma_{ij} \ ,$$
$$k_ik_j\sigma_{ij}=i\omega\Pi_{44},\quad \sigma_{ij}=\frac{\omega}{4\pi}(\epsilon_{ij}-\delta_{ij}) \ , \tag{22}$$

where σ_{ij} is the conductivity tensor. We will find it more useful to follow a different method which consists of writing the electric field $\vec{E}$ as a superposition of an ordinary and an extraordinary wave. This method was extensively used by Shafranov[12] and it leads to an expression for $\vec{E}$ which is the natural extension of the Lienhard-Wiechert potential in a medium. Such an expression is[12]

$$\vec{E}_l(r,t)=\int\frac{4\pi i}{\omega}\frac{\vec{a}(l)\vec{j}\cdot\vec{a}^*(l)}{k^2c^2/\omega^2-N_l^2(k,\omega)}e^{ikr}e^{-i\omega t}d^3k\,d\omega \ , \tag{23}$$

where the polarization vector $\vec{a}$ is given by

$$\vec{a}(l)=[1+\alpha^2(l)]^{-1/2}[i\alpha_x(l),\ 1,\ i\alpha_z(l)] \ , \tag{24}$$

and α_x is the ratio of the x and y components of the electric field, i.e., $\alpha_x\equiv -iE_x/E_y$ and, analogously, α_z. They are given by[12]

$$\begin{aligned}
\alpha(l)&=T_l^{-1}[(\epsilon_{12}\epsilon_{33}+\epsilon_{23}\epsilon_{13})\cos\theta+(\epsilon_{12}\epsilon_{13}+\epsilon_{23}\epsilon_{11})\sin\theta] \ ,\\
\alpha_x(l)&=T_l^{-1}[\epsilon_{12}\epsilon_{33}+\epsilon_{23}\epsilon_{13}-N_l^2(\epsilon_{12}\sin\theta-\epsilon_{23}\cos\theta)\sin\theta] \ ,\\
\alpha_z(l)&=T_l^{-1}[-\epsilon_{12}\epsilon_{13}-\epsilon_{11}\epsilon_{23}\\
&\qquad -N_l^2(\epsilon_{12}\sin\theta-\epsilon_{23}\cos\theta)\cos\theta] \ ,\\
T_l&=N_l^2[\epsilon_{11}\sin^2\theta+\epsilon_{33}\cos^2\theta+\epsilon_{13}\sin2\theta]\\
&\qquad -\epsilon_{11}\epsilon_{33}+\epsilon_{13}^2 \ ,
\end{aligned} \tag{25}$$

where $$\epsilon_{\alpha\beta}=\begin{pmatrix}\epsilon_{11} & i\epsilon_{12} & \epsilon_{13}\\ -i\epsilon_{12} & \epsilon_{22} & i\epsilon_{23}\\ \epsilon_{13} & -i\epsilon_{23} & \epsilon_{33}\end{pmatrix} \tag{26}$$

is the dielectric tensor to be specified later. N_l^2 is the refractive index for the two modes (ordinary and extraordinary) and is given by

$$N_l^2=(1/2A)[B\pm(B^2-4AC)^{1/2}] \ , \tag{27}$$

with

$$\begin{aligned}
A&=\epsilon_{11}\sin^2\theta+\epsilon_{33}\cos^2\theta+\epsilon_{13}\sin2\theta \ ,\\
B&=-(\epsilon_{12}\sin\theta-\epsilon_{23}\cos\theta)^2-\epsilon_{13}^2(\cos^22\theta+\sin^4\theta)\\
&\qquad+\epsilon_{11}\epsilon_{33}+\epsilon_{22}(\epsilon_{11}\sin^2\theta+\epsilon_{33}\cos^2\theta+\epsilon_{13}\sin2\theta) \ ,\\
C&=\epsilon_{11}\epsilon_{22}\epsilon_{33}-\epsilon_{33}\epsilon_{12}^2-\epsilon_{11}\epsilon_{23}^2-\epsilon_{22}\epsilon_{13}^2-2\epsilon_{12}\epsilon_{23}\epsilon_{13} \ .
\end{aligned} \tag{28}$$

IV. DIELECTRIC TENSOR FOR A COLD PLASMA

With Eqs. (23) and (10) substituted in Eq. (1) one obtains, in principle, the energy loss at any angle and for any form of dielectric tensor. The problem is quite involved and the final form too complicated to analyze. We will therefore study separately the propagation $\parallel$ and $\perp$ to $\vec{H}$ as usually done in magnetized plasma, and will specify the type of plasma to be worked with. We will use the tensor $\epsilon_{\alpha\beta}$ as given in the magneto-ionic theory, i.e.,[8]

$$\epsilon_{\alpha\beta}=\begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix} , \tag{29}$$

where

$$R=1-\frac{\omega_p^2}{\omega^2}\;\frac{\omega}{\omega-\omega_H-i\nu}-\frac{\Omega_p^2}{\omega^2}\;\frac{\omega}{\omega+\Omega_H-i\nu} ,$$

$$L=1-\frac{\omega_p^2}{\omega^2}\;\frac{\omega}{\omega+\omega_H-i\nu}-\frac{\Omega_p^2}{\omega^2}\;\frac{\omega}{\omega-\Omega_H-i\nu} , \tag{30}$$

$$P=1-\omega_p^2/\omega^2-\Omega_p^2/\omega^2 \;, 2S=R+L,\quad 2D=R-L.$$

The various symbols are defined in the following way:

$$\omega_p^2=4\pi n_e e^2/m,\qquad \omega_H=eH/mc \;,$$

$$\Omega_p^2=\omega_p^2(m/M_i)Z_i,\quad \Omega_H=\omega_H(m/M_i)Z_i \;,$$

where M_i is the mass of the ion, Z_i its charge, and $n_iZ_i=n_e$. With this notation, Eq. (27) for N_l^2 reduces to

$$N_l^2=(1/2A)(B\pm F) \;, \tag{31}$$

with

$$A=S\sin^2\theta+P\cos^2\theta,\quad B=RL\sin^2\theta+PS(1+\cos^2\theta) \;,$$
$$C=PRL,\quad F^2=(RL-PS)^2\sin^4\theta+4P^2D^2\cos^2\theta \;. \tag{32}$$

Equation (31) can also be written in a more transparent form, i.e.,

$$\tan^2\theta=-P(N_x^2-R)(N_0^2-L)/(SN_x^2-RL)(N_0^2-P) \;, \tag{33}$$

from which the dispersion relations at $\theta=0$ and $\theta=\frac{1}{2}\pi$ are easily obtained as

$$\theta=\tfrac{1}{2}\pi,\quad N_0^2=P \text{ (ordinary)},$$
$$N_x^2=RL/S \text{ (extraordinary)}, \tag{34}$$

$$\theta=0\;,\quad N_0^2=L,\quad N_x^2=R \;. \tag{35}$$

The terminology, ordinary and extraordinary, is well established at $\theta=\frac{1}{2}\pi$, where it is seen that the minus sign in Eq. (33) gives rise to $N_-^2=P$, independent of the magnetic field, i.e., this mode propagates as it would in the absence of H (ordinary). The plus sign gives rise to $N_+^2=RL/S$, i.e., it does depend on H (extraordinary). The same terminology is retained for $\theta=0$. The parameters α, α_x, and α_z are easily transformed to $(l=\pm)$:

$$\alpha(l)=-PD\cos\theta\,(AN_l^2-PS)^{-1},$$

$$\alpha_x(l)=D(N_l^2\sin^2\theta-P)(AN_l^2-PS)^{-1} \;, \tag{36}$$

$$\alpha_z(l)=N_l^2D\sin\theta\cos\theta\,(AN_l^2-PS)^{-1} \;.$$

A. Propagation at $\theta=0$

From the previous equations one can easily see that at $\theta=0$, one obtains

$$A\to P,\quad B\to 2PS,\quad C\to PRL,\quad N_0^2\to L \;,$$
$$N_x^2\to R,\quad \alpha(O)=-\alpha(X)=1 \;, \tag{37}$$
$$\alpha_z(O)=\alpha_z(X)=0\,,\quad \alpha_x(O)=-\alpha_x(X)=1 \;,$$

i.e.,[13]

$$\vec{a}(O)=(1/\sqrt{2})[i,1,0],\quad \vec{a}(X)=(1/\sqrt{2})[-i,1,0] \;. \tag{38}$$

From here it follows that since $iE_x/E_y=-\alpha_x$, the ordinary wave is left-hand circularly polarized. This means that at $\theta=0$, only circularly polarized waves can propagate in a plasma. However, the emerged radiation may still be unpolarized or linearly polarized as a result of a combination of both R- and L-polarization states.

Substituting the polarization vector a into the expression for the electric field, Eq. (23), and using some algebra, we obtain $(\alpha=1,2,3)$:

$$E_\alpha j_\alpha^*+E_\alpha^* j_\alpha=(4\pi^2/\omega)\,\delta(\Lambda)\,|j_x-ij_y|^2 \tag{39}$$

for the extraordinary wave and

$$E_\alpha j_\alpha^*+E_\alpha^* j_\alpha=(4\pi^2/\omega)\delta(\Lambda)|j_x+ij_y|^2 \tag{40}$$

for the ordinary wave. The δ function comes from the fact that

$$1/\Lambda=P(\Lambda)-i\pi\delta(\Lambda),\quad \Lambda=k^2c^2/\omega^2-N_l^2 \;, \tag{41}$$

where P stands for the principal value. We shall give the computation of the energy loss for the extraordinary wave in some detail; the corresponding expression for the ordinary wave is similar. The general expression to be computed is

$$j_x-ij_y=\frac{Ze^3}{2m}\sum_I\left(\frac{\langle n'|e^{-ikr}(\Pi_x-i\Pi_y)|I\rangle\langle I|r^{-1}|n\rangle}{E_n-E_I-i\Gamma}\right.$$
$$\left.+\frac{\langle n'|r^{-1}|I\rangle\langle I|e^{-ikr}(\Pi_x-i\Pi_y)|n\rangle}{E_n-E_I-\hbar\omega-i\Gamma}\right) , \tag{42}$$

where each electron state is characterized by three quantum numbers n, p_3, p_1 indicating, respectively, the principal quantum number, the momentum in the z direction, and the location of the orbit in the x-y plane. The analytic form of each $|n\rangle$ is given by[5,15]

$$|n\rangle\equiv\psi(np_3p_1)=(1/L)\,e^{ip_1x}\,e^{ip_3z}\,e^{-(1/2)\xi^2}\,\overline{H}_n(\xi) \;,$$
$$\xi=y(\gamma)^{1/2}+p_1\gamma^{1/2},\quad \gamma\equiv(H/H_q)\lambda\!\!\!^{-}_c{}^{-2} \;, \tag{43}$$

$$\overline{H}_n(x) = \gamma^{1/4}\pi^{-1/4}(2^n n!)^{-1/2}H_n(x) \quad . \tag{44}$$

Remembering that[16, 17]

$$(\Pi_x - i\Pi_y)|n\rangle = (2nH/H_q)^{1/2}mc\,|n-1\rangle\,, \tag{45}$$

we easily obtain

$$j_x - ij_y = \frac{Ze^3(2H/H_q)^{1/2}}{2mc}\left((n'+1)^{1/2}\frac{\langle n'+1, p_1', P_3|r^{-1}|n, p_1, p_3\rangle}{\omega - \omega_H - \frac{1}{2}\omega^2 N_x^2 - \omega N_x x'} - n^{1/2}\frac{\langle n' p_1' p_3'|r^{-1}|n-1, p_1, \tilde{P}_3\rangle}{\omega - \omega_H + \frac{1}{2}N_x^2\omega^2 - \omega N_x x}\right) \quad . \tag{46}$$

The energy-momentum conservation condition gives

$$P_3 = \hbar k_3 + p_3', \quad \tilde{P}_3 = p_3 - \hbar k_3\ , \ x \equiv p_3/mc \quad . \tag{47}$$

The frequencies ω are in units of $mc^2/\hbar$, i.e., ω means $\omega/(mc^2/\hbar)$ and N_x is the refractive index of the extraordinary wave: Henceforth the subscript x will be dropped. In our units ω_H means $\omega_H(mc^2/\hbar)^{-1} = H/H_q$. In general we now have[15] ($r \equiv r - R_\alpha$):

$$\left\langle n' p_2' p_3' \left| \frac{1}{r} \right| n p_1 p_3 \right\rangle = \frac{4\pi}{\Omega}\frac{1}{L^2}\sum\nolimits_q \frac{1}{q^2}\, e^{-iq\cdot R_\alpha}$$
$$\times\delta\left(\frac{p_1}{\hbar} - \frac{p_1'}{\hbar} - q_1\right)\cdot\delta\left(\frac{p_3}{\hbar} - \frac{p_3'}{\hbar} - q_3\right)$$
$$\times H(n,n')\, \mathcal{I}(n,n',q_1^2+q_2^2)\ , \tag{48}$$

$$H(n,n') = (-)^{n'} e^{-i(n-n')\phi}\,\Theta(n-n') + (-)^n e^{i(n-n')\phi}$$
$$\times\Theta(n'-n) - (-)^n \delta n, n',$$

$$\Theta(x) = 1, \ \text{for}\ x \geq 0\,; \ \Theta(x) = 0\ , \ \text{for}\ x < 0$$

$$\mathcal{I}(n,n',q_1^2+q_2^2) = (n!n'!)^{-1/2} e^{-t/2}$$
$$\times t^{(n+n')/2}\,{}_2F_0(-n', -n; -1/t)\ ,$$

$$2\gamma t = q_1^2 + q_2^2, \quad \phi \equiv \operatorname{arccot}(q_2/q_1) \quad . \tag{49}$$

Inserting Eq. (48) into Eq. (46), taking the modulus squared, and summing over the ions considered to be uncorrelated[3]

$$\sum_{\alpha=1}^{N_i}\sum_{\beta=1}^{N_i} e^{-i(q\cdot R_\alpha - q'\cdot R_\beta)} = \delta_{q-q'}N_i\ , \tag{50}$$

we finally obtain ($n_i = N_i/\Omega$)

$$|j_x - ij_y|^2 = \frac{Z^2e^6 8\pi^2\omega_H n_i}{m^2c^2\Omega}$$
$$\times\sum_{q_2}\left(\frac{(1+n')^{1/2}\mathcal{I}(n,n'+1)}{E_1[(p_1-p_1')^2 + q_2^2 + (p_3 - P_3)^2]}\right.$$
$$\left. + \frac{(n)^{1/2}\mathcal{I}(n-1,n')}{E_2[(p_1-p_1')^2 + q_2^2 + (p_3' - \tilde{P}_3)^2]}\right)^2 \ , \tag{51}$$

$$E_1 = \omega - \omega_H - \tfrac{1}{2}\omega^2N^2 - \omega N x',$$

$$E_2 = \omega - \omega_H + \tfrac{1}{2}\omega^2N^2 - \omega N x' \quad .$$

Inserting Eqs. (39) and (51) in Eq. (1) and integrating on k^2 with the aid of the δ function, we obtain

$$\frac{dI(\omega,\Omega)}{d\omega\, d\Omega} = \omega^2\tilde{I}\int_{-\infty}^{+\infty}dx_1'\int_{-\infty}^{+\infty}dx_2\int_{-\infty}^{+\infty}dx_3'$$
$$\times\left(\frac{1}{E_1}\frac{(n'+1)^{1/2}\mathcal{I}(n,n'+1)}{[(x_1-x_1')^2 + x_2^2 + (x-x'-N\omega)^2]}\right.$$
$$\left. + \frac{1}{E_2}\frac{(n)^{1/2}\mathcal{I}(n-1,n')}{[(x_1-x_1')^2 + x_2^2 + (x-x'-N\omega)^2]}\right)^2$$
$$\times\delta(\mathcal{E}_i - \mathcal{E}_f - \omega)\ , \tag{52}$$

where we have summed over p_1' and p_3', $p_1'/mc \equiv x_2$, $p_3'/mc \equiv x_3'$. The integrals on x_1' and x_2 can be performed immediately since the integrand depends only on $x_1'^2 + x_2^2$, i.e.,

$$\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\ \mathfrak{F}(x^2+y^2) = \pi\int_0^\infty ds\ \mathfrak{F}(s) \quad . \tag{53}$$

Analogously, the integral on x' can be performed by using the δ function on the energy. The final result is

$$\frac{dI(\omega,\Omega)}{d\omega\, d\Omega} = I_0\frac{\omega^2 N}{(\epsilon - \omega - n'\omega_H)^{1/2}}\int_0^\infty\frac{dt}{(t+\lambda)^2}$$
$$\times\left(\frac{1}{E_1}(1+n')^{1/2}\mathcal{I}(n,n'+1,t)\right.$$
$$\left. + \frac{1}{E_2}(n')^{1/2}\mathcal{I}(n-1,n')\right)^2 , \tag{54}$$

$$I_0 = \frac{Z^2\alpha^3}{\sqrt{2}\,8\pi}\frac{mc^2}{\hbar/mc^2}n_i\lambda\!\!\!^{-3}_c, \quad N^2 = R, \quad \epsilon = E_i/mc^2\ ,$$

$$\omega \equiv \omega/(mc^2/\hbar), \quad \omega_H = H/H_q\,;$$

$$\lambda = \frac{1}{2\omega_H}\{[2(\epsilon - n\omega_H)]^{1/2}$$
$$-\omega N - \gamma[2(\epsilon - \omega - n'\omega_H)]^{1/2}\}^2, \tag{55}$$

$\gamma = \pm 1$, (forward or backward scattering)

$$\mathscr{I}(n,n',t)=(n!n'!)^{-1/2}e^{-t/2}\times t^{(n+n')/2}{}_2F_0(-n,-n',-1/t)\ ,$$

$$E_1=\omega-\omega_H-\tfrac{1}{2}\omega^2N^2-\omega\gamma N[2(\epsilon-\omega-n'\omega_H)]^{1/2}\ ,$$

$$E_2=\omega-\omega_H+\tfrac{1}{2}\omega^2N^2-\omega\gamma N[2(\epsilon-n\omega_H)]^{1/2}\ .$$

A perfectly analogous computation gives the following result for the ordinary wave:

$$\frac{dI(\omega,\Omega)}{d\omega\,d\Omega}=I_0\frac{\omega^2N}{(\epsilon-\omega-n'\omega_H)^{1/2}}\int_0^\infty\frac{dt}{(t+\lambda)^2}\times\left(\frac{1}{E_3}(n')^{1/2}\mathscr{I}(n'-1,n,t)+\frac{1}{E_4}(n+1)^{1/2}\mathscr{I}(n+1,n',t)\right)^2\ , \qquad (56)$$

$$E_3=\omega+\omega_H-\tfrac{1}{2}\omega^2N^2-\omega\gamma N[2(\epsilon-\omega-n'\omega_H)]^{1/2}\ , \qquad (57)$$

$$E_4=\omega+\omega_H+\tfrac{1}{2}\omega^2N^2-\omega\gamma N[2(\epsilon-n\omega_H)]^{1/2},\quad N^2=L\ .$$

Simple cases of n, and n' will be discussed later.

B. Propagation at $\theta=\frac{1}{2}\pi$

In this case the only wave we consider is the extraordinary one since the ordinary will not propagate if

$$N_0^2=P=1-\omega_p^2/\omega^2-\Omega_p^2/\omega^2<0\ . \qquad (58)$$

(58) is satisfied in the radio-wave region, wherein, we are most interested. The density we consider at the surface of a neutron star is of the order of $N_e\approx10^{20}/\mathrm{cm}^3$. The extraordinary mode has the following polarization vector:

$$\vec{a}(X)=(0,1,0)\ . \qquad (59)$$

This form has been deduced from Eq. (25), omitting the longitudinal component, since our unit module's normalization is valid for pure transverse waves. In this case the quantity of interest is given by [see Eq. (42)]

$$E_\alpha j_\alpha^*+j_\alpha E_\alpha^*=(8\pi^2/\omega)\,|j_y(k,\omega)|^2\delta(\Lambda)\ . \qquad (60)$$

We will find it more useful to work with wave functions which exhibit the cyclindrical symmetry of the problem. They are of course nothing but a different way of rewriting Eq. (43). Their form and geometrical interpretation can be found in Chap. II of Ref. 16. Using the fact that[17]

$$\pi_y|n,l,s\rangle=i(\tfrac{1}{2}\omega_H)^{1/2}\times[(n+i)^{1/2}|n+1,l+1,s\rangle-n^{1/2}\,|n-1,l-1,s\rangle]\quad (61)$$

and Eqs. (4.5)–(4.8) of Ref. 16, we easily obtain $(j_0=Ze^3/2mc)$:

$$j_y=j_0(j_1+j_2)(\tfrac{1}{2}\omega_H)^{1/2},$$

$$j_1=\sum_I\frac{1}{E_1}[(N+1)^{1/2}I_{N+1,n'}+N^{1/2}I_{N-1,n'}]\times i^\Lambda\langle NL|r^{-1}|nl\rangle I_{\Sigma,s}$$

$$j_2=\sum_I\frac{1}{E_2}[(n+1)^{1/2}I_{n+1,N}+n^{1/2}I_{n-1,N}]\times i^{\Lambda_1}\langle n'l'|r^{-1}|NL\rangle I_{s,\Sigma}\ , \qquad (62)$$

with $(I=NL\Sigma P)$

$$E_1=\omega-(N-n')\omega_H,\qquad E_2=-\omega-(N-n)\omega_H,$$

$$\Lambda=L-l'+1\ ,\qquad \Lambda_1=l-L+1\ .$$

The argument of the function I is $k^2/2\gamma$. Because of the spherical symmetry of the Coulomb potential the last matrix element gives $L=l'$ for j_2 and $l=L$ for j_1, i.e., $[\alpha\equiv\Delta p(2H_q/H)^{1/2}]$:

$$\langle NL\Sigma P|1/r|nlsp\rangle=\delta_{n-s}^{N-\Sigma}\int_0^\infty 2K_0(\alpha\rho^{1/2})I_{N,\Sigma}(p)I_{n,s}(p)\,dp\equiv\delta_{n-s}^{N-\Sigma}\,T(N\Sigma|ns)\ .$$

Summing over the index $NI\equiv NL\Sigma$ and retaining only terms linear in x (long wavelength approximations), we obtain

$$j_1+j_2=2[(\omega+x\omega_H)/(\omega^2-\omega_H^2)]I_{1+s,s}\,T(1,1+s\,|\,0,s)\ . \qquad (63)$$

The cancellation effect which takes place for the $N=0$ term is the exact analog of what happens in Compton scattering. The lowest order in ω is obtained by considering $s=0$, $I_{1,0}=x^{1/2}$, and we obtain $j\simeq\omega^2\omega_H^{-2}$.

Comparing j_1 with Eq. (46) for $n=0$, we conclude that

$$j_\parallel\simeq\omega_H^{-1},\quad j_\perp\simeq\omega^2\omega_H^{-2},\quad j_\perp\ll j_\parallel\ . \qquad (64)$$

The corresponding decay probabilities have the same ω dependence as the current and therefore it is concluded that $W_\parallel\gg W_\perp$.

V. WIDTH FACTOR AND COLLISION FREQUENCY

As it is well known from quantum electrodynamics, the bremsstrahlung (or Compton) cross sections for a bound system are divergent at certain frequencies. This pole is usually avoided by modifying the electron Green's function by adding an imaginary part to the denominator as done in Eq. (5). This imaginary part Γ is related to the mass operator of the Dirac equation. In a plasma, various forms of Γ are known and an extensive study has been carried out by Tsitovich,[14] who gives various forms for Γ depending on the specific process one deals with. In the present paper we are mostly concerned with the region in which $\omega_H \gg \omega$, i.e., far from the resonance. We will not, therefore, worry about any specific form of Γ. A second factor which one should, in principle, include in the dielectric tensor $\epsilon_{\alpha\beta}$ is the collision frequency ν. This part is introduced also to avoid the poles at $\omega = \pm\omega_H$, which can occur. A simple formula for the collisional frequency[3]

$$\nu = (5.5N_i/T^{3/2}) \ln (220T/N_i^{3/2})$$

shows again that $\nu \ll \omega_H \simeq 1$.

VI. BREMSSTRAHLUNG EMISSION IN THE LOW-QUANTUM-NUMBER REGION

In this section we will give the explicit form of the radiation intensity for a few quantum numbers, namely, $n=0$, $n'=0$, $n=0$, $n'=1$, $n=1$, $n'=0$. Using Eqs. (54)–(56) and the following notation:

$$\tilde{I}(n,n';\,O,X) \equiv (1/I_0)\, dI(\omega,\Omega)/d\omega\, d\Omega \; , \tag{65}$$

we obtain the emissivity along the direction of the field, $\theta = 0$:

$$\tilde{I}(0,0;O) = (L)^{1/2}[\omega^2/(\epsilon-\omega)^{1/2}]\,(1/E_4^2)\,C_1(\lambda); \tag{66}$$

$$\begin{aligned}
&\tilde{I}(0,0;X) = (R)^{1/2}[\omega^2/(\epsilon-\omega)^{1/2}]\,(1/E_1^2)\,C_1(\lambda)\,,\\
&\lambda \equiv (1/2\omega_H)\,[(2\epsilon)^{1/2} - \omega(R,L)^{1/2} - \gamma[2(\epsilon-\omega)]^{1/2}]^2 \; ,\\
&E_1 = \omega - \omega_H - \tfrac{1}{2}\omega^2 R - \gamma\omega[2R(\epsilon-\omega)]^{1/2} \; ,
\end{aligned} \tag{67}$$

$$E_4 = \omega + \omega_H + \tfrac{1}{2}\omega^2 L - \gamma\omega(2L\epsilon)^{1/2} \; ;$$

$$\begin{aligned}
&\tilde{I}(1,0;O) = (L)^{1/2}[\omega^2/(\epsilon-\omega)^{1/2}]\,(1/E_4^2)\,C_3(\lambda) \; ,\\
&E_4 = \omega + \omega_H + \tfrac{1}{2}\omega^2 L - \omega\gamma[2L(\epsilon-\omega_H)]^{1/2} \; ,
\end{aligned} \tag{68}$$

$$\lambda \equiv (1/2\omega_H)\{[2(\epsilon-\omega_H)]^{1/2} - \omega(L)^{1/2} - \gamma[2(\epsilon-\omega)]^{1/2}\}^2 ;$$

$$\begin{aligned}
&\tilde{I}(1,0;X) = (R)^{1/2}[\omega^2/(\epsilon-\omega)^{1/2}]\{(1/E_1^2)[C_2 + C_3 - 2C_1]\\
&\quad + (1/E_2^2)\,C_2 + (1/E_1E_2)\,[2C_1 - 2C_2]\} \; ,\\
&E_1 = \omega - \omega_H - \tfrac{1}{2}\omega^2 R - \omega\gamma[2R(\epsilon-\omega)]^{1/2} \; ,\\
&E_2 = \omega - \omega_H + \tfrac{1}{2}\omega^2 R - \omega\gamma[2R(\epsilon-\omega_H)]^{1/2} \; ,
\end{aligned} \tag{69}$$

$$\lambda = (1/2\omega_H)\{[2(\epsilon-\omega_H)]^{1/2} - \omega(R)^{1/2} - \gamma\,[2(\epsilon-\omega)]^{1/2}\}^2 ;$$

$$\begin{aligned}
&\tilde{I}(0,1;O) = (L)^{1/2}[\omega^2/(\epsilon-\omega-\omega_H)^{1/2}]\\
&\quad \times\{(1/E_4^2)\,[C_2 + C_3 - 2C_1] + (1/E_3^2)\,C_2 + (1/E_3E_4)\\
&\quad \times [2C_1 - 2C_2]\} \; ;\\
&E_3 = \omega + \omega_H - \tfrac{1}{2}\omega^2 L - \omega\gamma\,[2L(\epsilon-\omega-\omega_H)]^{1/2} \; ,\\
&E_4 = \omega + \omega_H + \tfrac{1}{2}\omega^2 L - \omega\gamma(2L\epsilon)^{1/2} \; ,\\
&\lambda \equiv (1/2\omega_H)\,\{(2\epsilon)^{1/2} - \omega(L)^{1/2} - \gamma[2(\epsilon-\omega-\omega_H)]^{1/2}\}^2;
\end{aligned} \tag{70}$$

$$\begin{aligned}
&\tilde{I}(0,1;X) = (R)^{1/2}\,[\omega^2/(\epsilon-\omega-\omega_H)^{1/2}]\,(1/E_1^2)\,C_3(\lambda) \; ,\\
&E_1 = \omega - \omega_H - \tfrac{1}{2}\omega^2 R - \omega\gamma\,[2R(\epsilon-\omega-\omega_H)]^{1/2} \; ,\\
&\lambda \equiv (1/2\omega_H)\,\{(2\epsilon)^{1/2} - \omega(R)^{1/2} - \gamma[2(\epsilon-\omega-\omega_H)]^{1/2}\}^2;
\end{aligned} \tag{71}$$

$$C_1(\chi) = e^\chi(1+\chi)E(\chi) - 1 \; , \tag{72}$$

$$C_2(\chi) = (1/\chi) - e^\chi E(\chi)\,, \tag{73}$$

$$\begin{aligned}
&C_3(\chi) = 1 + \chi - (2\chi + \chi^2)\,e^\chi E(\chi)\,,\\
&E(\chi) = \int_\chi^\infty (e^{-s}/s)\,ds \; .
\end{aligned} \tag{74}$$

VII. DISCUSSION

The present calculation differs from the previous one[1] in several significant ways. First, the Green's function used here is an exact one, while in the previous case the free-particle Green's function has been used. Second, the plasma effect has been incorporated into our calculation. The final result, for example, Eq. (66), can be interpreted easily. The factor $C_1(\lambda)$ arises from the Coulomb field of the nucleus, the factor ω^2 arises from the density state of the photon, and finally, the factor E_4^{-2} comes from the Green's function, and the factor $1/(\mathcal{E}-\omega)^{1/2}$ comes from the density of the final state of the electron. As discussed in Ref. 5, in an intense magnetic field an electron exhibits one-dimensional behavior; instead of the usual expression $p^2 dp/dE$ the density of the final state for a one-dimensional particle is just $dp/dE \sim 1/p$. The effect of the refractive medium on the photon is to alter the relation between ω and k, and this is taken into account throughout the calculation.

Applications of this process to emission from pulsars will be presented in a separate paper.

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Non-Markovian Irreversible Behavior in a Simple Model

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The soluble model of an oscillator coupled to a scalar field is used as an example of irreversible behavior. By studying the reduced density operator for the oscillator, a generalized Fokker-Planck-Kramers-Chandrasekhar equation for the Wigner distribution function is derived. The diagonal matrix elements of the reduced density operator satisfy a generalized non-Markovian master equation. By application of a time-averaged method, the ordinary Pauli master equation is derived. The time evolution of the occupation probabilities of the oscillator levels has been numerically computed, and compared with the solutions of the Pauli equation.

I. INTRODUCTION

Great progress in the understanding of irreversible phenomena in quantum statistics was made by van Hove and his co-workers,[1] and by Prigogine and his school.[2] An important landmark in this context was the derivation by these workers of the generalized master equation (GME). This equation governs the time evolution of the probability distribution of the system over states of the unperturbed Hamiltonian H_0. For infinite systems, it predicts the evolution towards statistical equilibrium.[1,2]

The GME, an equation to infinite order in the perturbation, is a direct consequence of the Schrödinger equation,[3] and the only statistical hypothesis used in its derivation is the assumption of random phases at the initial time.

It is known that in the weak-coupling limit, the GME goes over into the much simpler Pauli master equation (PME), which has been solved for a number of different physical situations. The GME, on the other hand, has been explicitly written down and solved only in very few cases.[4,5] As a consequence, very little is known to date about the time evolution of the probability distribution for arbitrary coupling strength. It would seem then that a number of simple examples are still needed in order to clarify and illustrate the general features of the approach to equilibrium in cases where the coupling constant cannot be considered small.

We present here a study of irreversible behavior for the simple model of an harmonic oscillator linearly coupled to a scalar field.[6,7] Similar models have been considered by other authors.[8-11]

The dynamical equations for the system can be exactly solved, and this fact has been exploited throughout. We take the density operator at the initial time as the product of a canonical ensemble density operator for the field and an arbitrary density operator for the oscillator. Consequently the approach to equilibrium described by the model is meant in a restricted sense, because at the initial time all but one of the degrees of freedom are already in equilibrium. As has been pointed out