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Phonon-Phonon Interactions in Liquid Helium II

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The quantum hydrodynamics of Landau is here regarded as a means of calculating phonon-phonon interactions in liquid helium II. This theory is tested by applying it to the low-density Bose-Einstein gas, at low temperatures and long wavelengths, it agrees with the microscopic theory of the Bose-Einstein gas. This agreement justifies the past use of the hydrodynamic theory to calculate the Landau-Rumer sound absorption in liquid helium; however, the calculated absorption is roughly half that measured by Abraham, Eckstein, Ketterson, Kuchnir, and Vignos. Hence the higher-order terms in the hydrodynamic theory are considered; they probably cannot account for the discrepancy.

I. INTRODUCTION

Phonon-phonon interactions in liquid helium II can be discussed by means of two different theories. In the following, we shall compare the two theories with each other, and use them to examine past calculations of the sound absorption at high frequencies. The disagreement between measured and calculated sound absorption will be discussed, but not explained.

The quantum theory of anharmonic lattice vibrations in a crystal is a large and complicated subject. The anharmonic effects include the scattering and absorption of phonons and shifts in their vibration frequencies. Some simplifications are produced when the crystal is replaced by liquid helium II. At temperatures below about 0.5 deg, longitudinal phonons are the only excitations which occur naturally in the liquid. Their interactions can be calculated in the way discussed here, and can be measured directly, by experiments with rf sound.

The first theory of phonon-phonon interactions is called quantum hydrodynamics. It is the simplest way to calculate any specific property of longitudinal phonons. It is based on the equations of hydrodynamics, which were brought into quantum theory by Landau.[1] The hydrodynamic theory recognizes no atoms, and no atomic properties. Instead, it contains some constants which it does not evaluate, such as the speed and dispersion of sound at zero temperature. The omission of atoms seems a reasonable approximation at long wavelengths, and the wavelengths of thermal phonons are long when the temperature is low. But this approximation leads to divergent integrals in perturbation theory. Therefore the hydrodynamic theory is somewhat inconsistent, or at least incomplete, and it should be compared with a detailed and consistent theory.

The other theory we shall use is the detailed microscopic theory of liquid helium. The properties of liquid helium can, in principle, be calculated from the temperature, the density, and the properties of helium atoms. But the interaction between atoms in the liquid is strong enough to make this calculation rather difficult, and quantitative results can hardly be said to exist. Therefore the atomic or microscopic theory is more usefully applied to a fictitious condensed Bose-Einstein gas, whose density is rather low. This fictitious gas is a useful model because its elementary excitations have been studied, and nearly all their properties have been calculated. The elementary excitations of low energy are phonons, in both theories. Therefore the low-density Bose-Einstein gas serves here as a phonon gas from which the divergent integrals have been removed. The speed and absorption of sound in the Bose-Einstein gas will be calculated in the Appendix; they are desired only at frequencies and temperatures where comparison with the hydrodynamic theory is appropriate. The two theories agree if the frequency and temperature are sufficiently low that dispersion is small for both the incident phonons and the thermal phonons.

The simplest processes among phonons are the fission of one phonon into two, and the inverse process. Their effects will be calculated in both theories, and this will show the agreement of the theories. In either theory, the effects are simplest if ω, the frequency of the external source, is so high that the phonon-phonon interactions act directly, and the two-fluid model is inappropriate. Therefore we assume

$$\omega\tau \gg 1, \tag{1}$$

where τ is the mean lifetime of a thermal phonon. On the other hand, in all the experiments done heretofore, the frequency has been so low that

$$\hbar\omega \ll \kappa T, \tag{2}$$

where $\hbar$ is the well-known quantum, κ is Boltzmann's constant, and T is the temperature. Under these conditions, the dominant process is the fusion of an incident phonon with a thermal phonon, which is the Landau-Rumer process.[2] According to the hydrodynamic theory, this process gives a sound attenuation proportional to

$$\left(1+\frac{\rho}{c}\frac{\partial c}{\partial \rho}\right)^2 \frac{\omega(\kappa T)^4}{\rho\hbar^3 c^6} \ , \tag{3}$$

where ρ is the density and c is the speed of sound. This result, which was first obtained by Kawasaki,[3] is here confirmed by comparison with the microscopic theory. The constant of proportionality must be discussed below, in Sec. IV, but it is a fairly definite number. The recent experiments of Abraham, Eckstein, Ketterson, Kuchnir, and Roach[4] determine the Grüneisen ratio; it is

$$\frac{\rho}{c}\frac{\partial c}{\partial \rho} = 2.84 \tag{4}$$

at the vapor pressure and 0.1 deg. Therefore, the theoretical attenuation formula contains no unknown or adjustable parameters.

Recently, Abraham, Eckstein, Ketterson, Kuchnir, and Vignos[5,6] have measured the attenuation of sound at various frequencies satisfying (1) and (2), and temperatures below 0.5 deg. Their results confirm (3) in a general way, but they are about 1.5 to 2.7 times larger than the theoretical attenuation mentioned above and discussed below. They propose to fit their results to a formula containing three adjustable parameters, one of which is

$$\frac{\rho}{c}\frac{\partial c}{\partial \rho} = 5.30 \, .$$

This drastic adjustment indicates the difference between theory and experiment. It is too large to be attributed to experimental error.

This discrepancy between theory and experiment stimulates us to give a long discussion of the theory leading to (3). We shall obtain this result, and its numerical coefficient, from both the hydrodynamic and the microscopic theories. The agreement between the two theories extends to frequencies not satisfying (2), and to the temperature dependence of the speed of sound. Thus we shall obtain a theoretical relation between sound attenuation and the Grüneisen ratio, both of which have been measured in liquid helium. We shall go on to discuss the higher-order effects in the hydrodynamic theory, and to calculate some of them. They seem not to affect the relation between attenuation and Grüneisen ratio. However, the possibility that some higher-order terms do affect this relation has not been excluded. In quantum electrodynamics, the radiative corrections do not affect the relation between the electronic charge and the Thomson cross section[7]; but the corresponding theorem in quantum hydrodynamics has not been found.

We shall give some results and discussion before describing our calculations. First, the relevant results of the microscopic theory will be given in Sec. II. The hydrodynamic theory will be compared with them in Sec. III. The main results of hydrodynamic calculations will be presented and discussed in Sec. IV. In Sec. V, the measurements will be discussed in terms of the hydrodynamic theory. The hydrodynamic theory itself will be specified in Sec. VI, by giving the equations of motion and our form of perturbation theory. This theory permits various approximate calculations. In Sec. VII, the simplest approximation in quantum hydrodynamics will be used; it produces results which show the agreement of the two theories. A self-consistent calculation will be presented in Sec. VIII; it gives the Simons formula[8] for the rate of phonon fusion. A more systematic calculation will be presented and discussed in Sec. IX, but our higher-order corrections fail to affect (3). We summarize our view of the theory in Sec. X, the conclusion.

II. ATOMIC OR MICROSCOPIC THEORY

The condensed Bose-Einstein gas is a useful model for liquid helium, but one can do explicit calculations only at rather low densities. Such calculations give us two results suitable for comparison with quantum hydrodynamics: the attenuation of phonons at frequencies satisfying (1), and the temperature dependence of the speed of sound. Here we give our version of the second formula; its coefficients have not been published previously. This section also includes a few comments on the relation of the two theories. The Appendix is a more detailed account of the microscopic theory.

The first calculation of the effects of interparticle forces on the condensed Bose-Einstein gas was done by Bogoliubov in 1946.[9] For a gas of spin-zero particles, each of mass m, he found that the energy of an elementary excitation is

$$\hbar\omega(k)=\left[\frac{\hbar^2k^2}{2m}\left(\frac{\hbar^2k^2}{2m}-\frac{8\pi\hbar^2\rho s_k}{m^2}(1-d)\right)\right]^{1/2}. \tag{5}$$

Here $\hbar k$ is the momentum of the elementary excitation, s_k is the particle-particle scattering amplitude at this momentum, and $1-d$ is the fraction of particles that are in the lowest state available to a single particle in the container. The Bose condensation and d, the so-called depletion, should be defined in a way more appropriate for systems of interacting particles[10]; but they are barely relevant to this paper.

However, d must be small if the approximations of Bogoliubov are to be justified, and s_k is practically constant at the low energies which are relevant here. Hence, we shall often use the simpler and more explicit dispersion law

$$\hbar\omega(k)=\left[\frac{\hbar^2k^2}{2m}\left(\frac{\hbar^2k^2}{2m}+\frac{8\pi\hbar^2\rho a}{m^2}\right)\right]^{1/2}, \tag{6}$$

where $a=-s_0=$ minus the scattering length introduced by Fermi.[11]

According to this dispersion law, $\hbar\omega(k)$ becomes $\hbar^2k^2/2m$ at high energies. The elementary excitations of short wavelength are moving particles, on which the interaction has only a limited effect. But in the hydrodynamic theory, the strength of the interaction increases indefinitely as the wave number increases; this causes a problem.

The hydrodynamic calculations require the assumption of a dispersion law, which is conventionally written[12] as

$$\omega=c(k-\gamma\hbar^2k^3+\text{higher powers of }k). \tag{7}$$

Comparison of (6) and (7) shows that

$$c=(4\pi\hbar^2\rho a/m^3)^{1/2} \tag{8}$$

and $$\gamma=-m(32\pi\hbar^2\rho a)^{-1} \tag{9}$$

in the Bogoliubov model.

According to the calculation of Bogoliubov, the energy (6) and the speed of sound (8) are real and independent of temperature. In this model, therefore, the decay of phonons, and the variation of the speed of sound with temperature, depend on the small corrections to (5) or (6). These corrections are of the order of $(\rho a^3/m)^{1/2}$, which must be small in the Bogoliubov model; they were calculated by Lee, Huang, and Yang,[13] and more thoroughly by Beliaev.[14] Their work was for the case of zero temperature; its generalization to nonzero temperature was first studied by Mohling and Morita.[15] In such calculations, κT is compared to the energy which appears in (6); we define

$$\theta=m^2\kappa T/4\hbar^2\rho a .$$

This extension of the work of Bogoliubov can be done for arbitrary values of θ, but only for temperatures small compared to the transition temperature. The detailed calculations of Hohenberg[16] and the present author are described briefly in the Appendix.

One of the results of calculation is the speed of sound. For small values of θ, we find[17]

$$\begin{aligned}c=(4\pi\hbar^2\rho a/m^3)^{1/2}\{1+8(\rho a^3/\pi m)^{1/2}\\ \times[1-\tfrac{3}{20}\theta^4(\ln\theta+0.6367)+\tfrac{11}{14}\theta^6(\ln\theta+0.9037)-\cdots]\\ -\tfrac{3}{5}i(\pi\rho a^3/m)^{1/2}(\theta^4-\tfrac{110}{21}\theta^6+\cdots)\}.\end{aligned} \tag{10}$$

Here the imaginary part stands for the Landau-Rumer sound attenuation, which is proportional to frequency. Since we are interested only in the temperature dependence, we divide this result by the zero-temperature limit. The ratio of the speed of sound to its zero-temperature limit is

$$\begin{aligned}R=1+(3\pi^2/20)[(\kappa T)^4/\rho\hbar^3c^5]\\ \times[\ln(4\pi/\theta)-3.1677-\tfrac{110}{21}\theta^2\ln(0.405/\theta)+\cdots]\\ -i(3\pi^3/40)[(\kappa T)^4/\rho\hbar^3c^5](1-\tfrac{110}{21}\theta^2+\cdots),\end{aligned} \tag{11}$$

which is nearly the formula given by quantum hydrodynamics.

This formula displays two of the dimensionless parameters which occur in quantum hydrodynamics:

$$(\kappa T)^4/\rho\hbar^3 c^5 \tag{12}$$

is, in a sense, the small expansion parameter of quantum hydrodynamics,[18] and it is about $T^4/4000$ deg^4 in liquid helium. The other parameter is θ^2, which is approximately $-8\pi^2\gamma(kT/c)^2$, and it is small at temperatures where the Bogoliubov model is relevant to liquid helium. If it is large, the Bose-Einstein gas resembles an ideal gas more than liquid helium[19] or the fluid described by the hydrodynamic theory. The hydrodynamic theory interprets this quantity as a measure of the dispersion of thermal phonons, or as the square of the ratio of κT to the maximum phonon energy.

A further correction to Bogoliubov's results, of higher order than the $(\rho a^3/m)^{1/2}$ corrections just discussed, has been calculated by Wu,[20] and by Hugenholtz and Pines[21]; but it seems that this work has never been extended to nonzero temperatures. Therefore this work will not be considered here.

III. COMPARISON OF MICROSCOPIC AND HYDRODYNAMIC THEORIES

In hydrodynamics, the nonlinear terms in the equations of motion produce an interaction between sound waves. These equations of motion were brought into quantum theory by Landau,[1] and then used by Landau and Khalatnikov[12] to calculate phonon-phonon interactions. The simplest processes among phonons are the fission and fusion processes mentioned in the Introduction; we shall discuss them in detail. We begin, in this section, by giving the results of a simple hydrodynamic calculation of these processes. The results agree with those of the microscopic theory as closely as the different natures of the theories permit; but there is a slight disagreement between our results and those of Khalatnikov and Chernikova.[22]

This simple calculation will be described in Sec. VII. If the frequency satisfies (1) and (2), it gives the ratio of the speed of sound to its zero-temperature limit[17]:

$$R = 1 + \frac{\pi^2}{30}\frac{(\kappa T)^4}{\rho\hbar^3 c^5}\left\{\left(1+\frac{\rho}{c}\frac{\partial c}{\partial\rho}\right)^2 \times\left[\ln\left(\frac{2c^2}{|\gamma|(\kappa T)^2}\right) - 6.4835 - iS\right] + 2\frac{\rho}{c}\frac{\partial c}{\partial\rho} - \tfrac{2}{3}\right\}. \tag{13}$$

Here the Landau-Rumer attenuation is proportional to S, which is a step function of the dispersion:

$$S=\pi, \quad \text{if } d^2\omega/dk^2 > 0 \quad (\gamma<0)$$

$$S=0, \quad \text{if } d^2\omega/dk^2 < 0 \quad (\gamma>0).$$

The leading term in the theoretical temperature dependence of the speed of sound is proportional to $T^4 \ln T$, and agrees with earlier calculations.[23] The T^4 term in (13) disagrees with the result of the previous calculation, by Khalatnikov and Chernikova.[22] But, for the Bogoliubov model, (13) agrees with (11), if we use (8) and (9), to find

$$\frac{\rho}{c}\frac{\partial c}{\partial\rho} = \frac{1}{2} \tag{14}$$

and to evaluate $2c^2/\gamma$.

We have omitted from (13) the corrections of order θ^2, such as the $T^6 \ln T$ term in (10) and (11). This is because the hydrodynamic theory cannot give all of these terms. We shall see that dispersion, and the parameter γ, do not occur in the hydrodynamic equations; the dispersion law can be (and will be) inserted into the single-phonon energy, but not into the phonon-interaction energy.

Another aspect of this simple hydrodynamic calculation gives a formula for the sound absorption, as a function of $\hbar\omega/\kappa T$. This formula will be Eq. (52). The Landau-Rumer absorption in (13) is one of its limiting cases, and the $\hbar k^5/\rho$ absorption of Beliaev[14] is the other limiting case. From the microscopic theory, the same formula is obtained, but again with corrections of order θ^2; see the Appendix for this result.

IV. FURTHER RESULTS OF QUANTUM HYDRODYNAMICS

In this section, we proceed to discuss various modifications and corrections of the simple result (13). First, we discuss the Simons formula for attenuation,[8] and adopt his argument. The resulting theoretical attenuation, in dB per unit length, is

$$\frac{10}{2.303c}\,\frac{\pi^3}{30}\left(1+\frac{\rho}{c}\frac{\partial c}{\partial p}\right)^2\frac{\omega(\kappa T)^4}{\rho\hbar^3 c^5}\ . \tag{15}$$

Its disagreement with experiment has been emphasized in Sec. I. This disagreement suggests that one should systematically calculate higher-order effects in quantum hydrodynamics. This section includes the results of our calculations; we give the methods and details later.

The Simons formula[8] amounts to a simple correction to the Landau-Rumer attenuation,[2] which is caused by the fusion of an incident phonon with a thermal phonon. This theoretical attenuation is described by the $-iS$ in (13). The step function S expresses the well-known result that the fission of one phonon into two, and the inverse process, are forbidden if the dispersion is negative, or γ is positive. This result is derived by the application of energy conservation

$$\omega = \omega' + \omega'' \tag{16}$$

and momentum conservation

$$\vec{k} = \vec{k}' + \vec{k}''$$

to the fission process. But Kawasaki[3] pointed out that this argument is not conclusive. The phonon frequencies satisfy (16), but (16) is not really a condition imposed on the wave vectors, because the frequency spectrum of each phonon has a certain width, connected with the lifetime. Simons also thought of this,[8] and he proposed a new attenuation formula. It amounts to the replacement of S by

$$\tfrac{1}{2}\pi - \arctan[\text{const}\,\gamma\,\omega\tau(\kappa T/c)^2], \tag{17}$$

where τ is supposed to be the same as the mean lifetime of a thermal phonon, introduced in (1). If $\tau = \infty$, this factor becomes S.

This replacement is reasonable, but not entirely defensible. The trouble is that an idea of the frequency spectrum of a thermal phonon is needed for a discussion of the condition (16). The shape of a spectral line sometimes follows the famous curve of Lorentz. This shape is assumed in the derivations below and in the literature.[24] The calculation of the correct shape appears to be quite difficult, and no such calculation is found in the literature of liquid helium. For this reason, the relaxation time which appears in (17) is perhaps not the same as that found by measurement of any other property.[25] However, if

$$\gamma\omega\tau\,(\kappa T/c)^2 \tag{18}$$

is small, the Simons formula appears to be correct. It replaces S by $\frac{1}{2}\pi$, the mean value. This procedure, which gives (15), is reasonable; it seems that the conservation laws exclude half the integral.[8] Furthermore, the recent experiments[6,26] show that the attenuation increases with frequency, but not quite in direct proportion; this behavior is given by the Simons formula (assuming $\gamma\tau$ is positive). On the other hand, if (18) is large, the shape of the frequency spectrum becomes quite important, and we doubt that it follows the curve of Lorentz. In this case, the Simons formula gives an attenuation independent of frequency, and this prediction is neither explained by theory nor confirmed by experiment.[6] We believe that the Landau-Rumer process is forbidden if ω is sufficiently large or T is sufficiently small. And the Simons formula is the only attempt to describe how this process goes away.

The thermal-phonon lifetime itself can be discussed on the basis of the hydrodynamic theory. This can be done only for an unbounded system, for the effect of a real container on sound of short wavelengths is quite hard to calculate. For thermal phonons, $\hbar\omega$ is of the order of κT, and the processes of fission and fusion are more or less forbidden. If they do determine the lifetime of thermal phonons, τ is of the order of

$$\rho\hbar^4(c/\kappa T)^5. \tag{19}$$

If not, then the absorption is due to elastic collisions between thermal phonons, and τ is of the order of

$$\gamma\rho^2\hbar^7 c^8/(\kappa T)^7. \tag{20}$$

This was shown by Landau and Khalatnikov,[12] and the numerical coefficient of (20) has lately been calculated by Eckstein.[27] Moreover, the detailed calculations of Eckstein, and our independent calculations, show that elastic collisions contribute to the attenuation of external sound an amount proportional to

$$\omega^3(\kappa T)^4/\gamma\rho^2\hbar^4 c^9, \tag{21}$$

at frequencies satisfying (1) and (2). This is proportional to ω^3, and the measured attenuation is not. Even without quoting the calculated coefficient of (21), or making a precise estimate of γ, it can be seen that this is a negligible contribution to the attenuation.

Instead of this process, we shall discuss the fusion of two phonons, which probably causes the measured attenuation at frequencies satisfying (1) and (2). This amounts to a discussion of higher-order terms in our calculations, and of the nature of quantum hydrodynamics.

All the foregoing specific results of quantum hydrodynamics are obtained only by circumventing the divergence difficulties of the theory. It is natural, at least for the purpose of calculation, to distinguish two kinds of divergences.

First, there is the divergence difficulty at large wave numbers. We resolve the motion of the fluid into Fourier components, and treat their interaction by perturbation theory. The interaction found from hydrodynamics becomes stronger and stronger as the wave numbers increase, causing a cutoff dependence in the integrals obtained from perturbation theory. This difficulty with quantum hydrodynamics is absent from the microscopic theory. It is considerably reduced by taking the imaginary parts of the integrals, or by calculating only the absorption of sound; and probably the remaining cutoff dependence can be eliminated by a program of renormalization. But this does not repair the reactive terms in the hydrodynamic theory, which are cutoff dependent and unreliable. In the simple calculation described in Sec. III, this cutoff dependence can be eliminated in a different way, by separating the zero-temperature limit of each integral from the temperature-dependent remainder. This subtraction leads to (13), although the speed of sound at zero temperature cannot usefully be calculated in the hydrodynamic theory. Since (13) shows not even a qualitative agreement with experiment, this trick may be misleading; the cutoff wave number K may be temperature dependent. We cannot say much about this quantity, so we try to eliminate it. This means, in particular, that we express results in terms of the experimental speed of sound, which differs from the "bare" value that appears in the Hamiltonian.

The other divergence difficulty appears at small angles. This means that an integrand is often infinite if, among the wave vectors in it, a pair are parallel or antiparallel. Since this difficulty is caused by energy denominators, it has a physical interpretation of sorts. The difficulty can be removed by inserting into the appropriate denominators the frequency spectrum of the phonons, or their dispersion law, or both. Only the dispersion law has been used in the derivation of (13). This is partly because the frequency spectrum of a thermal phonon is unknown, but mainly because dispersion appears in the Bogoliubov function, (5) or (6), whereas phonon-phonon interactions, and the consequent nonuniqueness of the phonon frequency, are, in the microscopic theory, effects of higher order. This use of dispersion, with the neglect of the phonon frequency spectrum, is also reasonable in quantum hydrodynamics itself; it is logical to calculate the phonon-phonon interactions, which are not necessarily of microscopic nature, in terms of the dispersion, which is a microscopic property. Therefore, the dispersion law

$$\omega = c(k - \gamma\hbar^2 k^3) \tag{22}$$

will, in general, be used to remove the small-angle divergences which appear in the hydrodynamic theory. This process will make some of the terms calculated below proportional to $1/\gamma$, and others proportional to $\ln\gamma$.

The appearance of these divergences means that perturbation theory does not actually give an expansion in powers of (12), the small parameter. The removal of divergences causes two other dimensionless parameters to appear in our results; they are

$$\gamma(\kappa T/c)^2 \tag{23}$$

and $$\hbar K^4/\rho c\ , \tag{24}$$

where K is the cutoff wave number. Of course the ratio

$$\hbar\omega/\kappa T$$

also appears; but, as it is small in the acoustic experiments, only its first power need be retained. This means that, in the hydrodynamic theory, the attenuation is a function of (12), (23), and (24). The lowest-order result is proportional to (12), and to the frequency; it has been included in (13) and discussed above. The effects of next order, in the hydrodynamic theory, include elastic collisions between phonons; this process, noted above, gives a negligible contribution to the acoustic attenuation. The next order also gives many propagator corrections and vertex corrections to (3) or (15), with their divergence difficulties. If we retain only the most severe divergences of each kind, the Landau-Rumer attenuation is multiplied by a factor of

$$1 + \alpha[(\kappa T)^2/\gamma\rho(\hbar c)^3] + \bar{\alpha}(\hbar K^4/\rho c), \tag{25}$$

where α and $\bar{\alpha}$ are real numbers whose value has not been calculated. The $\bar{\alpha}$ term drops out when the attenuation is expressed in terms of the experimental Grüneisen ratio (4), instead of the bare Grüneisen ratio, which is unknown. The α term, and other things, give us an ωT^6 term in the attenuation, which was possibly observed by Waters, Watmough, and Wilks.[28] Those authors had a mistaken notion that the ωT^6 term arises from elastic collisions among phonons. Our ωT^6 term comes from the phonon fusion process, and it drops out when phonon fusion is forbidden. This term is probably small, for the observed attenuation is nearly proportional to ωT^4. It appears that the observed attenuation is due to phonon fusion, and that it is not affected by these higher-order effects.

Another dimensionless parameter can be formed by eliminating T between (12) and (23). This gives

$$(\rho c\gamma^2\hbar^3)^{-1}, \tag{26}$$

which is the expansion parameter of the microscopic theory. But this parameter does not occur anywhere in our systematic development of the hydrodynamic theory, which goes to show that the two theories are essentially different. This, and the agreement which the two theories showed in Sec. III, suggest that quantum hydrodynamics is competent to describe real helium.

The calculations which are discussed here and presented below show that there is a simple connection (15) between the sound attenuation and the other measured properties of liquid helium. Does this connection hold to all orders in perturbation theory, or independently of perturbation theory? The question refers to a theorem in quantum hydrodynamics which has not been found, and may not exist, although the corresponding theorem in quantum electrodynamics[7] is well known. Can the question be answered in such a way as to increase the coefficient of ωT^4 in the attenuation? Probably not, for the terms of next higher order do not have this effect. We cannot find an explanation in quantum hydrodynamics for the disagreement between the theoretical attenuation (15) and the recent experiments of Abraham, Eckstein, Ketterson, Kuchnir, and Vignos.[5,6]

V. DISCUSSION OF EXPERIMENTS

The outstanding features of the measured attenuation of sound have been mentioned in the Introduction. Here we review the other features, and some other measured quantities which appear in the hydrodynamic theory. In particular, the dispersion and lifetime of thermal phonons will be discussed.

The speed of sound itself has been measured. We assume that (4), which gives its dependence on density, is approximately correct at all temperatures below 0.5 deg. Then there is a dependence on temperature, which is perhaps related to the attenuation. This dependence was measured by Whitney and Chase,[29] who also reviewed the previous measurements, and estimated that

$$c = 2.383\times 10^4 \text{ cm sec}^{-1},$$

at zero temperature. More recently, Abraham, Eckstein, Ketterson, Kuchnir, and Vignos[5,6] covered the region of temperatures from 0.1 to 0.5 deg, and frequencies satisfying (1) and (2), in detail. Their measurements of the speed of sound refute the prediction (13) of *both* theories. The theory says that the speed of sound minus the zero-temperature limit is roughly proportional to T^4. The measurements show that this difference is proportional to T^3, but relatively small, at temperatures from 0.1 deg to an upper limit which depends on the frequency. This disagreement between theory and experiment shows that the reactance produced by the phonon-phonon interactions is not understood. The reactive terms in the hydrodynamic theory all have a cutoff dependence, which suggests that none of them are understood; the result (13) is obtained from quantum hydrodynamics, but higher-order corrections to it would be cutoff dependent.

The measured attenuation of sound is nearly proportional to ωT^4, if the frequency satisfies (1) and (2), and the temperature is below 0.5 deg.[5,6,26,28] If the Simons correction is adopted, the theoretical attenuation is

$$\frac{10}{2.303c}\frac{\pi^2}{15}\frac{\omega(\kappa T)^4}{\rho\hbar^3c^5}\left(1+\frac{\rho}{c}\frac{\partial c}{\partial\rho}\right)^2 \times\left[\frac{\pi}{2}-\arctan\left(\text{const}\frac{\gamma\omega\tau}{c^2}(\kappa T)^2\right)\right]. \tag{27}$$

Abraham, Eckstein, Ketterson, Kuchnir, and Vignos used this formula to represent their measurements.[5,6] The quantitative discrepancy, which we have emphasized, required them to adjust the Grüneisen ratio. Also, they assumed

$$\gamma\tau = C/T^n$$

and used C and n as the other two adjustable parameters; this is necessary because $\gamma\tau$ is rather poorly known. They found that $n=3$, which is curious; it disagrees with (19) and (20). The various discrepancies between measured and theoretical sound propagation suggest that some unknown mechanism is present, or that the phonon-phonon coupling has been incorrectly calculated.

The dispersion of phonons at thermal energies and below is quite important in our theoretical picture, but this dispersion is rather poorly known. It is expressed by the quantity γ, which was introduced by Landau and Khalatnikov[12] in 1949. They estimated

$$\gamma = 2.8\times 10^{37} \text{ gm}^{-2}\text{ cm}^{-2}\text{ sec}^2. \tag{28}$$

The experimental results obtained since then are only moderately helpful. The dispersion law for elementary excitations in liquid helium is now well known at large wave numbers, because of neutron-scattering experiments[30]; but at the small wave numbers which are relevant here, these experiments yield only the speed of sound, not the other terms in (7). However, the latest such experiment, by Henshaw and Woods,[31] does suggest that γ is smaller than (28). Moreover, some recent authors[6,22,32,33] have argued, from the measured attenuation, that the arctangent term in (27) is rather small, and hence that γ is much smaller than (28), which we regard as an upper limit. Since the attenuation does not quite increase in direct proportion to the frequency,[6,26] we may argue from (27) that γ is positive. Finally, we may perhaps assume that the dimensionless parameter (26) is less than, say, 10^6. This implies

$$5\times 10^{35}\text{ gm}^{-2}\text{ cm}^{-2}\text{ sec}^2 < \gamma < 3\times 10^{37}\text{ gm}^{-2}\text{ cm}^{-2}\text{ sec}^2,$$

which gives

$$\frac{16T^2}{\text{deg}^2} > \frac{(\kappa T)^2}{\gamma\rho(\hbar c)^3} > \frac{T^2}{4\,\text{deg}^2}. \tag{29}$$

This result suggests that the α term in (25) cannot be very large at the temperatures below 0.5 deg, which are relevant here. On the other hand, the α term in (25) might be sufficiently large, at 0.3 or 0.4 deg, to cause slow convergence of perturbation theory. We will not speculate about this possibility.

The other hydrodynamic parameter is τ, the thermal phonon lifetime. It is not so completely unknown, for there are relevant experiments and the theoretical estimates (19) and (20). Since a small value of γ, or a large value of

$$(\kappa T)^2/\gamma\rho(\hbar c)^3, \tag{30}$$

encourages the fission and fusion of thermal phonons, τ is roughly given by the greater of (19) and (20).

The most relevant experiment is that of Whitworth.[34] He found the mean free path of phonons from the thermal conductivity of the phonon gas. The phonons flowed through tubes, and the mean free path was, in effect, compared with the diameters of the tubes. However, this experiment is not sensitive to the forward peak which theoretically occurs in the elastic scattering cross section,[34] nor to the fusion of phonons moving in nearly parallel directions, nor to the inverse process. In the present state of the theory, these small-angle processes are important.[32] If they are not important in reality, the mean free path measured by Whitworth is τc. He found that this quantity is proportional to $T^{-4.3}$, in his unfortunately narrow range of temperatures. His mean free paths are of the order of c times (19); hence the experiment tends to support (19), whose numerical coefficient is unknown.

The recent measurements of sound absorption give an indirect measurement of $\gamma\tau$, if the Simons formula is accepted. Abraham, Eckstein, Ketterson, Kuchnir, and Vignos[5,6] found that τ is proportional to T^{-3}. However, the walls of the tube are a disturbing factor in this experiment; and their effect on the thermal phonons is hard to calculate. The wavelength of a thermal phonon is of the order of $\hbar c/\kappa T = 18\,\text{Å}\,\text{deg}/T$; the roughness of the walls should have a considerable and temperature-dependent effect, if it is roughness on this small scale. Such roughness is invisible, as Whitworth noted.[34] If roughness and smoothness can be measured on such a small scale, its variation can be used to show that the walls have a substantial or a negligible effect on the τ which appears in the Simons formula.

This relaxation time might also be determined by impurities; but the precautions of Abraham, Eckstein, Ketterson, Kuchnir, and Vignos have excluded this possibility.[6] The only likely impurity was He^3. Its concentration was quite small, and the effect of a small concentration was measured in a separate experiment.[35]

This concludes our discussion of the pertinent measurements on liquid helium II. The temperature-dependence of the speed of sound is quite different from the prediction (13). The attenuation is substantially larger than the theoretical estimate (15), which, being obtained by neglect of the arctangent in (27), should be an overestimate. The mean lifetime of a thermal phonon in an infinite container is not well known, but this causes no great difficulty. The dispersion, measured by γ, is poorly known, but important in the hydrodynamic theory. We shall discuss this theory further, after describing our calculations.

VI. ASSUMPTIONS AND METHODS OF QUANTUM HYDRODYNAMICS

Quantum hydrodynamics is defined by the equations of this section, which also includes our form of perturbation theory, and a formula for the vertex function. We follow Landau[1] in deriving the commutation relations for density and velocity, and writing a reasonable form of Hamiltonian; this procedure brings the equations and problems of hydrodynamics into the quantum theory. We want to study irrotational flow, and we define the appropriate variables and functions in this section.

In a system of particles of equal mass, the density is

$$\rho(\vec{r}, t) = \sum_i m\delta(\vec{r} - \vec{r}_i),$$

where $\vec{r}_i$ is the position operator for particle i. The momentum density is

$$\vec{j}(\vec{r}, t) = \tfrac{1}{2}\sum_i [\vec{p}_i\delta(\vec{r} - \vec{r}_i) + \delta(\vec{r} - \vec{r}_i)\vec{p}_i],$$

where $\vec{p}_i$ is the momentum operator for particle i. This operator gives the usual expression for current in terms of a wave function. The commutation relations at equal times are

$$[\rho(\vec{r}, t), \rho(\vec{r}', t)] = 0,$$

$$[\vec{j}(\vec{r}, t), \rho(\vec{r}', t)] = -i\hbar\rho(\vec{r}, t)\nabla\delta(\vec{r} - \vec{r}'),$$

$$[\vec{j}_x(\vec{r}, t), \vec{j}_x(\vec{r}', t)] = -i\hbar[\vec{j}_x(\vec{r}, t) + \vec{j}_x(\vec{r}', t)]\frac{\partial}{\partial x}\delta(\vec{r} - \vec{r}'),$$

$$[\vec{j}_x(\vec{r}, t), \vec{j}_y(\vec{r}', t)] = i\hbar\frac{\partial}{\partial x'}[\delta(\vec{r} - \vec{r}')\vec{j}_y(\vec{r}', t)] - i\hbar\frac{\partial}{\partial y}[\delta(\vec{r} - \vec{r}')\vec{j}_x(\vec{r}, t)],$$

and so forth. In hydrodynamics, it is customary to deal with the velocity, so Landau introduced the velocity operator:

$$\vec{v}(\vec{r}, t) = \tfrac{1}{2}\{[\rho(\vec{r}, t)]^{-1}\vec{j}(\vec{r}, t) + \vec{j}(\vec{r}, t)[\rho(\vec{r}, t)]^{-1}\}.$$

This leads to some interesting results. An immediate result of this definition is

$$[\vec{v}(\vec{r}, t), \rho(\vec{r}', t)] = -i\hbar\nabla\delta(\vec{r} - \vec{r}'),$$

which means that a simplification has been obtained; for it implies that $\nabla\times\vec{v}(\vec{r}, t)$ and $[\vec{v}(\vec{r}, t), \vec{v}(\vec{r}', t)]$ both commute with $\rho(\vec{r}'', t)$. The other equal-time commutation relations are

$$[\vec{v}_x(\vec{r}, t), \vec{v}_x(\vec{r}', t)] = 0,$$

$$[\vec{v}_x(\vec{r}, t), \vec{v}_y(\vec{r}', t)] = -i\hbar\delta(\vec{r} - \vec{r}')\frac{1}{\rho}\left(\frac{\partial\vec{v}_x}{\partial y} - \frac{\partial\vec{v}_y}{\partial x}\right),$$

and so forth.[36]

These commutation relations, being deduced from quantum mechanics, hold for any system of particles. The sweeping assumption which gives hydrodynamics its macroscopic character is the assumption that the Hamiltonian is a function of ρ and $\vec{v}$. This obliterates the microscopic motions. But it immediately gives Lagrange's theorem. [1]

It is natural to assume that

$$\tfrac{1}{2}\vec{v}\rho\vec{v} = \tfrac{1}{4}(\rho\vec{v}^{\,2} + \vec{v}^{\,2}\rho)$$

is the kinetic energy density of the fluid, and that the potential energy density is a function of ρ only. These quantitites probably cannot be identified with the kinetic and potential energies of the particles, but they do lead to a Hamiltonian:

$$H = \int\left[\frac{\vec{v}\rho\vec{v}}{2} + \frac{c_0^2}{2\rho_0}(\rho_0-\rho_0)^2 + \frac{(u-1)c_0^2}{6\rho_0^2}(\rho-\rho_0)^3 + \frac{zc_0^2}{24\rho_0^3}(\rho-\rho_0)^4 + \cdots\right]d\vec{r} - \int \mu(\vec{r},t)\,\rho(\vec{r},t)\,d\vec{r} - \int \vec{V}(\vec{r},t)\cdot\vec{v}(\vec{r},t)\,d\vec{r}. \qquad (31)$$

Here c_0 is the bare or nominal speed of sound, ρ_0 is the nominal density, and μ and $\vec{V}$ are two external disturbances. Also, u and z are two dimensionless constants introduced by Landau and Khalatnikov. [12] In this notation, u is twice the bare Grüneisen ratio; but in the notation of recent papers, [5,6,22,23,26,27,29,32,33,37] u is the experimental Grüneisen ratio (4).

The foregoing assumptions yield the equations of motion

$$\frac{\partial\rho}{\partial t} = \frac{1}{i\hbar}[\rho, H] = -\tfrac{1}{2}\nabla\cdot(\rho\vec{v}) - \tfrac{1}{2}\nabla\cdot(\vec{v}\rho) + \nabla\cdot\vec{V}$$

and
$$\frac{\partial\vec{v}_j}{\partial t} = \frac{1}{i\hbar}[\vec{v}_j, H] = -\tfrac{1}{2}\sum_{k=1}^{3}\left(\frac{\partial\vec{v}_j}{\partial x_k}\vec{v}_k + \vec{v}_k\frac{\partial\vec{v}_j}{\partial x_k}\right) - \frac{c_0^2}{\rho_0}\frac{\partial}{\partial x_j}(\rho-\rho_0) - \frac{(u-1)c_0^2}{2\rho_0^2}\frac{\partial}{\partial x_j}(\rho-\rho_0)^2 - \text{higher terms} + \text{terms in } \mu \text{ and } \vec{V},$$

where j and k refer to directions in space. In this way, Landau brings the hydrodynamic equations into quantum theory. Some later authors have preferred a canonical field-theoretic procedure. [38]

As is usual, the equations of motion cannot be solved exactly. But Lagrange's theorem serves to separate hydrodynamics into a sector with $\nabla\times\vec{v}=0$ and a sector with $\nabla\times\vec{v}\neq 0$. The latter sector supplied the name of the roton, [1] but it gives a poor model of the roton, [9,39] and it will not be considered here. If $\nabla\times\vec{v}=0$, quantum hydrodynamics becomes a means of calculating the properties of phonons and the equal-time commutation relations become canonical.

The perturbation method used here, whose results have been discussed above, treats the cubic terms in (31) as a small perturbation. This means that the zeroth approximation to the Hamiltonian is

$$H_0 = \int\left[\tfrac{1}{2}\rho_0\vec{v}^{\,2} + \tfrac{1}{2}c_0^2(\rho-\rho_0)^2/\rho_0\right]d\vec{r},$$

which describes an infinite assembly of harmonic oscillators. The cubic terms constitute

$$H_3 = \int\left[\tfrac{1}{2}\vec{v}(\rho-\rho_0)\vec{v} + \tfrac{1}{6}(u-1)c_0^2(\rho-\rho_0)^3/\rho_0^2\right]d\vec{r}.$$

The quadratic and higher terms are not very interesting. It is convenient, although artificial, to introduce the coupling constant

$$g = c_0\rho_0^{-1/2}$$

by defining new field variables. In accordance with the usual sound practice, we introduce a velocity potential:

$$\vec{v} = -g\nabla\phi_1.$$

Also, we let

$$\phi_2 = g(\rho-\rho_0)$$

and $\hbar=1$. Then the equal-time commutation relations are

$$[\phi_1(\vec{r}), \phi_1(\vec{r}\,')] = [\phi_2(\vec{r}), \phi_2(\vec{r}\,')] = 0$$

and $[\phi_1(\vec{r}), \phi_2(\vec{r}\,')] = i\delta(\vec{r}-\vec{r}\,')$.

The essential parts of the Hamiltonian are

$$H_0 = \int\left[\tfrac{1}{2}c_0^2(\nabla\phi_1)^2 + \tfrac{1}{2}\phi_2^2\right]d\vec{r},$$
$$H_3 = g\int\left[\tfrac{1}{2}(\nabla\phi_1)\phi_2(\nabla\phi_1) + \tfrac{1}{6}(u-1)\phi_2^3/c_0^2\right]d\vec{r}.$$

It is convenient to include external sources, or linear terms, in the total Hamiltonian, which is

$$H = H_0 + H_3 - \textstyle\sum_j \int S_j(\vec{r},t)\phi_j(\vec{r},t)d\vec{r}.$$

This gives the equations of motion

$$\frac{\partial\phi_1}{\partial t} = \phi_2 + \tfrac{1}{2}g(\nabla\phi_1)^2 + \tfrac{1}{2}g(u-1)\phi_2^2/c_0^2 - S_2$$

and
$$\frac{\partial\phi_2}{\partial t} = c_0^2\nabla^2\phi_1 + \tfrac{1}{2}g[(\nabla^2\phi_1)\phi_2 + \phi_2(\nabla^2\phi_1)] + \tfrac{1}{2}g(\nabla\phi_1\cdot\nabla\phi_2 + \nabla\phi_2\cdot\nabla\phi_1) + S_1.$$

They can be combined into one equation, by introducing some matrices and two-component vectors:

$$\frac{\partial\phi_j(1)}{\partial t_1} = C_{jk}\phi_k(1) + \tfrac{1}{2}g\bar{D}_{jkl}(123)\phi_k(2)\phi_l(3) - i\sigma_{jk}^{(y)}S_k(1). \qquad (32)$$

Here 1, 2, and 3 stand for points in space time; 1 stands for $\vec{r}_1, t_1$, and so forth. Summation over k and l and integration over 2 and 3 are understood. Three matrices appear here; $C_{11}=\sigma_{11}^{(y)}=0$,

$\bar{D}_{111} = \delta(12)\delta(13)(\nabla_2 \cdot \nabla_3)$, and the other components are obvious.

Of course Green's functions, or correlation functions, will be used.[40] They form a matrix:

$$\tilde{G}_{jk}(11') = (-i)[\langle(\phi_j(1)\phi_k(1'))_+\rangle - \langle\phi_j(1)\rangle\langle\phi_k(1')\rangle],$$

where $(\ \ )_+$ denotes the ordering according to time arguments, and $\langle\ \ \rangle$ denotes an average over a sort of canonical ensemble, which depends on the source terms. Since the Green's functions tend to be periodic, all time variables will be temporarily confined to the interval from 0 to $-i\beta$, where

$$\beta \equiv (\kappa T)^{-1}, \quad \mathrm{Re}\beta > 0, \quad \text{and } \mathrm{Im}\beta < 0 .$$

The generalized Boltzmann factor is

$$\exp[-\beta(H_0+H_3)]\{\exp[-i\int_0^{-i\beta}(-S_1\phi_1 - S_2\phi_2)dt]\}_+ . \tag{33}$$

All time-dependent quantities are included in the same time ordering. Hence, for example, the average of $\phi_j(1)$ is

$$\langle\phi_j(1)\rangle = \frac{\mathrm{Tr}\exp[-\beta(H_0+H_3)]\{\exp[-i\int_0^{-i\beta}(-S_1\phi_1 - S_2\phi_2)dt]\,\phi_j(1)\}_+}{\mathrm{Tr}\exp[-\beta(H_0+H_3)]\{\exp[-i\int_0^{-i\beta}(-S_1\phi_1 - S_2\phi_2)dt]\}_+} .$$

When the source terms vanish, this becomes the ordinary average over the canonical ensemble. The source terms can be varied; functional differentiation of this formula gives

$$\frac{1}{i}\frac{\delta}{\delta S_k(1')}\langle\phi_j(1)\rangle = i\tilde{G}_{jk}(11'). \tag{34}$$

The ensemble average of (32) is

$$\left(\delta_{jk}\frac{\partial}{\partial t_1} - C_{jk}\right)\langle\phi_k(1)\rangle = \tfrac{1}{2}ig\bar{D}_{jkl}(123)\tilde{G}_{kl}(23)$$
$$+\tfrac{1}{2}g\bar{D}_{jkl}(123)\langle\phi_k(2)\rangle\langle\phi_l(3)\rangle - i\sigma_{jk}^{(y)}S_k(1).$$

Functional differentiation of this equation, using (34), yields the equation of motion for $\tilde{G}$, which can be written

$$G_{0\,jk}^{-1}(12)\tilde{G}_{kl}(21') = \delta_{jl}\delta(11') - \tfrac{1}{2}igD_{jkm}(123)$$
$$\times\frac{\delta\tilde{G}_{km}(23)}{\delta S_l(1')} + \tfrac{1}{2}gD_{jkm}(123)[\tilde{G}_{kl}(21')\langle\phi_m(3)\rangle$$
$$+\tilde{G}_{ml}(31')\langle\phi_k(2)\rangle] . \tag{35}$$

Here $G_0^{-1}(12) = \begin{pmatrix} c_0^2\nabla_1^2 & -\dfrac{\partial}{\partial t_1} \\ \dfrac{\partial}{\partial t_1} & -1 \end{pmatrix}\delta(12)$;

and $D_{111} = 0$, $D_{112} = -\delta(12)\delta(13)(\nabla_2^2 + \nabla_2\cdot\nabla_3)$,

$D_{121} = -\delta(12)\delta(13)(\nabla_3^2 + \nabla_2\cdot\nabla_3)$, $D_{122} = 0$,

$D_{211} = \delta(12)\delta(13)(\nabla_2\cdot\nabla_3)$, $D_{212} = D_{221} = 0$,

$D_{222} = \delta(12)\delta(13)(u-1)/c_0^2$.

Now Green's function is supposed to be the unique periodic solution of (35). Its inverse is supposed to exist; this means

$$\tilde{G}_{jk}^{-1}(12)\tilde{G}_{kl}(21') = \tilde{G}_{jk}(12)\tilde{G}_{kl}^{-1}(21') = \delta_{jl}\,\delta(11') .$$

Also, $\tilde{G}^{-1}$ can be considered as a functional of $\langle\phi_j\rangle$, rather than S_j. Therefore

$$\tilde{G}_{jk}^{-1}(16) = G_{0\,jk}^{-1}(16) - \Sigma_{jk}(16)$$
$$-\tfrac{1}{2}gD_{jlm}(123)[\delta_{lk}\delta(26)\langle\phi_m(3)\rangle$$
$$+\delta_{mk}\delta(36)\langle\phi_l(2)\rangle] , \tag{36}$$

where $\Sigma_{jk}(16) = -\tfrac{1}{2}igD_{jlp}(123)\tilde{G}_{lm}(24)$
$$\times[\delta\tilde{G}_{mn}^{-1}(45)/\delta\langle\phi_k(6)\rangle]\tilde{G}_{np}(53) \tag{37}$$

is the mass operator.

Mass-operator perturbation theory is obtained by expanding the mass operator in powers of g. Equation (36) suggests that $\delta\tilde{G}^{-1}/\delta\langle\phi\rangle$ is of order g. Then (37) shows that Σ is of order g^2, and differentiation of (36) gives

$$\delta\tilde{G}_{mn}^{-1}(45)/\delta\langle\phi_k(6)\rangle = -g\,D_{mnk}(457)\,\delta(76) + 0(g^3) . \tag{38}$$

Substitution into (37) gives

$$\Sigma_{jk}(16) = \tfrac{1}{2}ig^2 D_{jlp}(123)\tilde{G}_{lm}(24)D_{mnk}(457)$$
$$\times\delta(76)\tilde{G}_{np}(53) + 0(g^4) . \tag{39}$$

Functional differentiation of this gives

$$\frac{\delta\tilde{G}_{mn}^{-1}(45)}{\delta\langle\phi_k(6)\rangle} = -gD_{mnk}(457)\delta(76)$$
$$-ig^3D_{mpq}(478)\tilde{G}_{qr}(89)D_{rsk}(901)\tilde{G}_{st}$$
$$\times(02)D_{tun}(235)\tilde{G}_{up}(37)\delta(16) + 0(g^5) . \tag{40}$$

Substitution into (37) then gives the g^4 term in (39) explicitly; functional differentiation of the result gives the g^5 term in (40), and so on. In this way, each term of the expansion

$$\Sigma_{jk}=g^2\Sigma_{jk}^{(2)}+g^4\Sigma_{jk}^{(4)}+\cdots \tag{41}$$

can be found in terms of D and $\tilde{G}$. When there are no external sources, the last term in (36) is supposed to vanish (except for terms in $\langle\rho\rangle-\rho_0$), so that $\tilde{G}^{-1}$ is $G_0^{-1}-\Sigma$.

In the present calculations, the terms in (41) after the first or second will be dropped, but even this approximation amounts to a complicated integral equation for $\tilde{G}^{-1}$. Various approximate solutions will be discussed in the following sections. The g^4 terms in (41) will be neglected in Secs. VII and VIII, but not in Sec. IX.

VII. FIRST APPROXIMATION TO MASS OPERATOR

The simplest approximation to Σ will be found in this section, and used to show the agreement of the two theories. This will require a specific treatment of divergences and dispersion in quantum hydrodynamics. It will give us an attenuation formula, and the result (13), which disagrees slightly with Khalatnikov and Chernikova.[22]

Equations (39) and (41) give

$$\Sigma_{jk}^{(2)}(16)=\tfrac{1}{2}i\,D_{jlp}(123)\tilde{G}_{lm}(24)D_{mnk}\times(457)\delta(76)\tilde{G}_{np}(53)\ . \tag{42}$$

In the physical system to be studied, the source terms in the equations of motion are absent, and Green's function is supposed to depend only on the space and time differences. Since it is a periodic function of the time difference, we introduce its Fourier coefficient:

$$\tilde{G}(\vec{k},\omega_\nu)=\int d\vec{r}\,e^{-i\vec{k}\cdot(\vec{r}_1-\vec{r}_1')}\int_0^{-i\beta}dt_1\,e^{i\omega_\nu(t_1-t_1')}\tilde{G}(11')\ .$$

Here ω_ν is any frequency such that $-i\beta\omega_\nu$ is a multiple of 2π. The Fourier coefficient of Σ is defined similarly. Using the values of D_{jkl}, (42) becomes

$$\Sigma_{11}^{(2)}(\vec{k},\omega_\nu)=\frac{i}{2}\int\frac{d\vec{k}'}{(2\pi)^3}\;\frac{1}{-i\beta}\times\sum_{\nu'}[(\vec{k}\cdot\vec{k}')^2\tilde{G}_{11}(\ )\tilde{G}_{22}(\ )-2(\vec{k}\cdot\vec{k}')(\vec{k}\cdot\vec{k}'')\times\tilde{G}_{12}(\ )\tilde{G}_{12}(\ )+(\vec{k}\cdot\vec{k}'')^2\tilde{G}_{22}(\ )\tilde{G}_{11}(\ )]\ ,$$

$$\Sigma_{12}^{(2)}(\vec{k},\omega_\nu)=\frac{i}{2}\int\frac{d\vec{k}'}{(2\pi)^3}\;\frac{1}{-i\beta}\times\sum_{\nu'}[(\vec{k}\cdot\vec{k}')(\vec{k}'\cdot\vec{k}'')\tilde{G}_{11}(\ )\tilde{G}_{12}(\ )+(\vec{k}\cdot\vec{k}'')(\vec{k}'\cdot\vec{k}'')\times\tilde{G}_{12}(\ )\tilde{G}_{11}(\ )+(\vec{k}\cdot\vec{k}')(u-1)c_0^{-2}\tilde{G}_{12}(\ )\tilde{G}_{22}(\ )+(\vec{k}\cdot\vec{k}'')(u-1)c_0^{-2}\tilde{G}_{22}(\ )\tilde{G}_{12}(\ )]\ ,$$

$$\Sigma_{21}^{(2)}(\vec{k},\omega_\nu)=-\Sigma_{12}^{(2)}(\vec{k},\omega_\nu)\ ,$$

$$\Sigma_{22}^{(2)}(\vec{k},\omega_\nu)=\frac{i}{2}\int\frac{d\vec{k}'}{(2\pi)^3}\;\frac{1}{-i\beta}\times\sum_{\nu'}[(\vec{k}'\cdot\vec{k}'')^2\tilde{G}_{11}(\ )\tilde{G}_{11}(\ )-2(\vec{k}'\cdot\vec{k}'')(u-1)c_0^{-2}\times\tilde{G}_{12}(\ )\tilde{G}_{12}(\ )+(u-1)^2c_0^{-4}\tilde{G}_{22}(\ )\tilde{G}_{22}(\ )]\ . \tag{42$'$}$$

Here $\vec{k}''=\vec{k}'-\vec{k}$; also, in each term, the arguments of the first $\tilde{G}$ are $\vec{k}',\omega_\nu'$, and those of the second are $\vec{k}''$, $\omega_\nu'-\omega_\nu$.

These formulas can be evaluated only by substituting something definite for $\tilde{G}$. It is natural to substitute the known function

$$G_0(k,\omega_\nu)=\frac{1}{\omega_\nu^2-c_0^2k^2}\begin{pmatrix}1 & i\omega_\nu\\ -i\omega_\nu & c_0^2k^2\end{pmatrix}\ .$$

But the result of this is that the integrals diverge; in particular, $\Sigma_{22}^{(2)}$ is proportional to K^4. This indicates a substantial difference between $\tilde{G}_{22}^{-1}$ and $G_0^{-1}{}_{22}$. Therefore, let

$$\bar{G}^{-1}(\vec{k},\omega_\nu)=\begin{pmatrix}-c_0^2k^2 & i\omega_\nu\\ -i\omega_\nu & -Z\end{pmatrix} \tag{43}$$

be the initial approximation to $\tilde{G}^{-1}$. Here Z must be positive, that there may be sound. The inverse of this matrix is

$$\bar{G}(\vec{k},\omega_\nu)=\frac{1}{\omega_\nu^2-c^2k^2}\begin{pmatrix}Z & i\omega_\nu\\ -i\omega_\nu & c_0^2k^2\end{pmatrix}\ , \tag{44}$$

where $c=Z^{1/2}c_0$ is the experimental speed of sound. Substitution into (42$'$) gives

$$\Sigma_{jk}^{(2)}(\vec{k},\omega_\nu)=\frac{(-i)^{j-k}}{2^4(2\pi)^3}\sum_{\sigma_1}\int\frac{d\vec{k}'}{k'k''}\,(\sigma_1A_jA_k)\times\left[\frac{1}{\omega_\nu-ck'-\sigma_1ck''}-\frac{(-1)^{j-k}}{\omega_\nu+ck'+\sigma_1ck''}\right]\times[\coth(\tfrac{1}{2}\beta ck')+\sigma_1\coth(\tfrac{1}{2}\beta ck'')]\ , \tag{45}$$

where $\vec{k}''=\vec{k}-\vec{k}'$ and σ_1 takes on the two values ±1. Furthermore,

$$A_1=(\vec{k}\cdot\vec{k}')k''+\sigma_1(\vec{k}\cdot\vec{k}'')k'$$

and $A_2=(1/c)[\sigma_1Z(\vec{k}'\cdot\vec{k}'')+(u-1)k'k'']\ .$

Such concise expressions for the mass operator are a practical necessity in this theory, but they are not very convenient. A more convenient form of (45), in terms of double integrals, is obtained by using

$$x=(k'+k'')/k\quad\text{and}\quad y=(k'-k'')/k$$

as integration variables. With or without this simplification, it is evident that $\Sigma^{(2)}$ is the integral of a rational function of ω_ν, so that analytic continuation to all complex frequencies[40,41] amounts to replacing ω_ν by z. In the same way, (44) is continued to z equal to any complex frequency.

The simplest approximation to Σ has now been found explicitly, and the inverse of Green's function is

$$\tilde{G}^{-1}(\vec{k}, z) = \bar{G}^{-1}(\vec{k}, z) - \Sigma(\vec{k}, z) + \text{term in } (\langle\rho\rangle - \rho_0) \;. \quad (46)$$

This is a function of a complex variable, from which all the correlation functions may be found.[40, 42] To invert this matrix, the determinant is needed; to find the frequencies of oscillation, only the determinant is needed. Neglecting higher powers of g^2, the determinant of (46) is

$$-[z - ck - B^-(k,z)][z + ck + B^+(k,z)] \;,$$

where
$$B^{\mp}(k,z) = \frac{ck^6}{2^{12}\pi^2\rho_0} \sum_\sigma \int_1^\infty dx \int_{-1}^{1} dy \frac{\sigma}{z - \sigma ckx}\left[\left(\frac{u-1}{Z} - 1\right)x^2 - \left(\frac{u-1}{Z} + 1\right)y^2 + 2 \pm 2\sigma x(1-y^2)\right]^2$$
$$\times\{\coth[\tfrac{1}{4}\beta ck(x-y)] + \coth[\tfrac{1}{4}\beta ck(x+y)]\} + \frac{ck^6}{2^{12}\pi^2\rho_0}\sum_\sigma \int_1^\infty dx \int_{-1}^{1} dy \frac{\sigma}{z - \sigma cky}\left[\left(\frac{u-1}{Z} + 1\right)x^2\right.$$
$$\left. - \left(\frac{u-1}{Z} - 1\right)y^2 - 2 \pm 2\sigma y(x^2 - 1)\right]^2 \{\coth[\tfrac{1}{4}\beta ck(x-y)] - \coth[\tfrac{1}{4}\beta ck(x+y)]\} \;. \quad (47)$$

Here another sign factor appears; σ is ± 1. The Fourier transform of the retarded commutators is $\tilde{G}(k, \omega + i0^+)$, and it shows the frequencies of oscillation. The positive member of the pair of frequencies is

$$ck + B^-(k, ck + i0^+) \;. \quad (48)$$

The integrals which appear here show the divergences which were discussed in Sec. IV. The divergence at large wave numbers has been mentioned in connection with (43). It has been taken into account by the introduction of Z, and it will be eliminated from the following calculations, by various devices. The speed of sound is the sum of its zero-temperature limit and a temperature-dependent part; the former part has been discussed, and will be dropped. This means that, for our immediate purpose, we replace the sum of cotangents in (47) by

$$\coth(\tfrac{1}{4}\beta ck(x-y)) + \coth(\tfrac{1}{4}\beta ck(x+y)) - 2 \;.$$

The small-angle divergence shows up in (47), in the form of logarithmic divergences at $x = 1$ and $y = \pm 1$. It was argued in Sec. IV that such divergences should be removed by using (22). This means that, in the denominators of (45),

$$ck \text{ is replaced by } c(k - \gamma k^3) \;, \qquad ck' \text{ is replaced by } c(k' - \gamma k'^3) \;, \quad (49)$$

and so forth. We express this statement in terms of x and y, and apply it to (47). Since only the speed of sound is sought now, γk^3 is negligible, and only the x^2 terms in the large [ ]'s in (47) are retained. The result of integration is

$$R = 1 + \frac{\pi^2}{120}\frac{(\kappa T)^4}{\rho_0 c^5}\left\{\left(\frac{u-1}{Z} + 3\right)^2\left[\ln\left(\frac{2\beta^2 c^2}{|\gamma|}\right) - 5.9835 - iS\right] - \frac{8}{3} - \frac{1}{2}\left(\frac{u-1}{Z} - 1\right)^2\right\} \;. \quad (50)$$

Now the derivation of (13) will be completed by a calculation of the physical Grüneisen ratio. The Σ in (36) is supposed to be a small correction, which we neglect temporarily. The lowest approximation to Green's function is (44), whose inverse is (43). Moreover, $\langle\rho\rangle$ is constant; suppose that $\langle\rho\rangle - \rho_0$ is small. The $k = 2$ components of (38) give the corrections to (43), to first order in g and $\langle\rho\rangle - \rho_0$. In this way, $\tilde{G}^{-1}$ and its variation with density are found approximately; and $\tilde{G}^{-1}$ gives c^2. The result is

$$\frac{\rho}{c}\frac{\partial c}{\partial \rho} = \frac{1}{2}\left(\frac{u-1}{Z} + 1\right) \;. \quad (51)$$

This result has been used to express (50) in terms of the experimental Grüneisen ratio.

The other result which is contained in (48) is the imaginary part, which gives the attenuation. We go back to (47) and calculate the discontinuity across the real axis; this eliminates divergences of both kinds. But the small-angle difficulty shows up in that $\mathrm{Im}\tilde{G}^{-1}(\vec{k}, \omega + i0^+)$ has a jump discontinuity at $\omega = \pm ck$; this amounts to a discontinuity in the absorption at $\gamma = 0$. Inspection of (45) shows that the absorption is proportional to S, the step function. If $d^2\omega/dk^2$ is positive or γ is negative, the imaginary part of (48) is

$$-\frac{k^5}{2^9\pi\rho_0}\left(1 + \frac{\rho}{c}\frac{\partial c}{\partial\rho}\right)^2\left(\int_{-1}^{1}(1-y^2)^2 \coth[\tfrac{1}{4}\beta ck(y+1)]\,dy\right.$$

$$+\int_1^\infty (x^2-1)^2\{\coth[\tfrac{1}{4}\beta ck(x-1)] - \coth[\tfrac{1}{4}\beta ck(x+1)]\}dx\Big) \quad . \tag{52}$$

If $\beta ck \ll 1$, this is proportional to kT^4, and gives the damping which has been included in (13). If $\beta ck \to \infty$, the two integrals approach $\frac{16}{15}$ and 0, and this imaginary part agrees with that calculated by Beliaev.[14,43] Approximately the result of the microscopic theory is obtained by substituting (14), the Grüneisen ratio, into (52). However, the microscopic theory also gives some corrections, of relative order (23) (or θ^2). The hydrodynamic theory does suggest the existence of these terms. For, dispersion had to be inserted into (47); the correction of the energy denominators led to the logarithm in (50), and the step function in the attenuation. But there is no reasonable basis for correcting the numerators or matrix elements in (47). The interaction terms in the Hamiltonian, and their matrix elements, are furnished by the hydrodynamic theory but without the small dispersion corrections. Therefore the terms of order (23) cannot be calculated consistently in the hydrodynamic theory; and they have been ignored in the derivation of (50) and (52). Such corrections are related to the microscopic details which are not described by the hydrodynamic theory.

In this section, (42) and a simple propagator have been used to do the simplest calculation in quantum hydrodynamics. The results, (13) and (52), show the extent of agreement with the microscopic theory.

VIII. SELF-CONSISTENT CALCULATION OF SOUND ABSORPTION

The simple calculation which has just been discussed gave an attenuation proportional to S. This step function will now be smoothed out by inserting the frequency spectrum of the thermal phonons. This introduces τ, the thermal-phonon lifetime, into the attenuation formula. The theory based on this relaxation time has been worked out in detail by Pethick and Ter Haar.[37] Here, only the Simons formula[8] will be derived. It is a generalization of the Landau-Rumer attenuation formula, obtained by putting the phonon's frequency spectrum into the $\tilde{G}$ which is inserted in (42). Since the phonon's frequency spectrum is not known, a simple assumption is used. And vertex corrections are neglected, for (42) does not include them.

The simplest assumption which includes some recognition of the phonon's frequency spectrum is

$$\tilde{G}^{-1}(\vec{k}, z) = \begin{pmatrix} -\dfrac{c^2k^2}{Z} & i(z+i\Gamma\,\mathrm{Im}z/|\mathrm{Im}z|) \\ -i(z+i\Gamma\,\mathrm{Im}z/|\mathrm{Im}z|) & -Z \end{pmatrix}, \tag{53}$$

where Γ is a function of k. This is a generalization of (43), and its inverse is

$$\tilde{G}(\vec{k}, z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{A(k,\omega)}{z-\omega} , \tag{54}$$

where the spectral function is

$$A(k,\omega) = \sum_{\sigma=-1}^{+1} \frac{\sigma\Gamma}{(\omega-\sigma ck)^2+\Gamma^2} \begin{pmatrix} \dfrac{Z}{ck} & i\sigma \\ -i\sigma & \dfrac{ck}{Z} \end{pmatrix} .$$

This shows that the imagniary part of the frequency is $-\Gamma$, or that a width 2Γ has been assigned to each peak.

The substitution of (54) into (42′) gives terms of the form

$$\int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} \frac{A_{ij}(k',\omega')A_{kl}(k'',\omega'')}{\omega_\nu - \omega' + \omega''} \times [\coth(\tfrac{1}{2}\beta\omega') - \coth(\tfrac{1}{2}\beta\omega'')] .$$

Then the process of analytic continuation[40,41] replaces ω_ν by z. If Γ is small compared to κT, the last factor may be replaced by

$$\coth(\pm\tfrac{1}{2}\beta ck') - \coth(\pm\tfrac{1}{2}\beta ck''),$$

and then the integrations over ω' and ω'' are easy. This calculation gives $\tilde{G}^{-1}(\vec{k}, z)$, whose determinant is

$$-z^2 + c^2k^2 + \frac{c^2}{2^4(2\pi)^3 Z^2\rho_0} \int \frac{d\vec{k}'}{k'k''} \times \Big\{ \text{terms in}\,[\coth(\tfrac{1}{2}\beta ck') + \coth(\tfrac{1}{2}\beta ck'')] + \sum_\sigma \frac{E[\coth(\tfrac{1}{2}\beta ck') - \coth(\tfrac{1}{2}\beta ck'')]}{z + \sigma(ck'-ck'') + i(\Gamma'+\Gamma'')\mathrm{Im}z/|\mathrm{Im}z|} \Big\}, \tag{55}$$

where $\vec{k}'' = \vec{k}' - \vec{k}$

and
$$E = \sigma[Z(\vec{k}\cdot\vec{k}')k'' + Z(\vec{k}\cdot\vec{k}'')k']^2 + \sigma[Z(\vec{k}'\cdot\vec{k}'')k + (u-1)kk'k'']^2 - 2zc^{-1}[Z(\vec{k}\cdot\vec{k}')k'' + Z(\vec{k}\cdot\vec{k}'')k'] \times [Z(\vec{k}'\cdot\vec{k}'') + (u-1)k'k''] .$$

In Sec. VII, the (coth + coth) terms, or phonon-fission terms, did not contribute to the imaginary part of the speed of sound, nor to the $T^4 \ln T$ term in the real part. Therefore they will be dropped from (55). This approximation is valid only in the region (2). It makes the integral converge and gives a formula for the determinant. The discontinuity of this determinant across the real axis, at $z = ck$, can be found from this formula, or from (53), which gives $-4ick\Gamma$. The calculation is self-consistent if these two quantities are equal, or if

$$\Gamma = \frac{c}{2^5(2\pi)^3 Z^2 \rho_0} \int \frac{d\vec{k}'}{kk'k''} (\Gamma' + \Gamma'')$$
$$\times \sum_\sigma \frac{\sigma F^2}{(\sigma ck + ck' - ck'')^2 + (\Gamma' + \Gamma'')^2}$$
$$\times [\coth(\tfrac{1}{2}\beta ck') - \coth(\tfrac{1}{2}\beta ck'')] .$$

Here $F = Z(\vec{k}\cdot\vec{k}')k'' + Z(\vec{k}\cdot\vec{k}'')k'$
$$- \sigma[Z(\vec{k}'\cdot\vec{k}'')k + (u-1)kk'k''] ,$$

and the dispersion should be put into the denominator, using (49). The γk^3 term is negligible, and the angles between the wave vectors are supposed to be small. This small-angle approximation gives

$$\Gamma = \frac{(\beta ck)}{2^7\pi^2\rho_0} \left(\frac{u-1}{Z} + 3\right)^2 \int_0^\infty k'^4 \left[\tfrac{1}{2}\pi - \arctan\left(\frac{3c\gamma k k'^2}{2\Gamma'}\right)\right]$$
$$\times \operatorname{csch}^2(\tfrac{1}{2}\beta ck')\, dk' ,$$

which gives the lifetime of a low-frequency phonon in terms of the properties of thermal phonons. One of these properties is Γ', whose dependence on k' is unknown; but it is of the order of $1/\tau$. Hauling the second factor of the integrand outside the integral is a reasonable approximation. This second factor becomes (17), and the other factors reproduce the result of Sec. VII. This is to say that the Simons formula, in the limit $\tau = \infty$, gives the Landau-Rumer attenuation found in Sec. VII.

The condition (2) has been used in this calculation; the correction factor which should be applied to the general formula (52) is unknown. The Simons formula itself is not well established, for the reasons mentioned above. One trouble is that (53) is an arbitrary assumption; it would be better to use a propagator like (44) and to compute the propagator corrections and vertex corrections. Some such calculations are done in Sec. IX, where we give further comments on the Simons formula. But it would be rather hard to calculate $A(k, \omega)$ by perturbation theory. Therefore this calculation has been presented, and the Simons formula must serve to describe the inhibition of phonon fusion by the dispersion.

IX. HIGHER-ORDER TERMS

The theoretical problem is to solve (36) and (37) for Σ or $\tilde{G}$. In this section, we shall consider and evaluate the g^4 terms in (41), using (44) as the propagator. This will permit us to discuss the higher-order corrections to the results of Sec. VII, and the lack of connection between perturbation theory and the Simons formula. We shall conclude that the theoretical expression for attenuation is (15), in which c rather than c_0 appears, and the measured Grüneisen ratio rather than the bare Grüneisen ratio.

But first, we note some symmetries and allege some general properties of the mass operator and vertex function. They will simplify our calculations, and perhaps justify the choice of (43).

The simplest symmetry is

$$\tilde{G}(12) = \tilde{G}(21), ,$$

and this implies

$$\Sigma(12) = \Sigma(21) .$$

Functional differentiation gives (34), and then

$$\delta/\delta S(3)\tilde{G}(12) \,..$$

This three-point function is symmetric under permutations of 1, 2, and 3. It follows that

$$\delta\tilde{G}^{-1}(12)/\delta\langle\phi(3)\rangle$$

has the same symmetry. These remarks simplify the evaluation of higher-order terms. The subscripts, which have been dropped from this paragraph, are to be permuted along with the space-time coordinates.

It also seems useful to add a constant to the velocity-potential.

Consider the substitution

$$\phi_1 \to \phi_1 + f(t), \quad S_1 \to S_1,$$
$$\phi_2 \to \phi_2, \qquad S_2 \to S_2 - \frac{df}{dt} ,$$

where $f(t)$ is a c number and a periodic function of the time. This is contrived so that the equal-time commutators and the equations of motion are invariant. But there are changes in the Hamiltonian, the Boltzmann factor (33), and $\langle\phi_j\rangle$. These considerations yield

$$\int \frac{\delta\tilde{G}^{-1}_{ij}(12)}{\delta\langle\phi_1(3)\rangle}\, d\vec{r}_3 = 0 . \tag{56}$$

The vertex function can, in the case of no external disturbance, be written as

$$\frac{\delta\tilde{G}^{-1}_{mn}(45)}{\delta\langle\phi_k(6)\rangle} = \int \frac{d\vec{k}'}{(2\pi)^3} e^{i\vec{k}'\cdot(\vec{r}_4 - \vec{r}_6)} \frac{1}{-i\beta}$$
$$\times \sum_{\nu'} e^{-i\omega'_\nu(t_4 - t_6)} \int \frac{d\vec{k}''}{(2\pi)^3} e^{ik''\cdot(\vec{r}_5 - \vec{r}_6)} \frac{1}{-i\beta}$$
$$\times \sum_{\nu''} e^{-i\omega''_\nu(t_5 - t_6)} V_{mnk}(\vec{k}', \omega'_\nu; \vec{k}'', \omega''_\nu) .$$

Then (56) presumably means that $V_{mn1}(\vec{k}', \omega'_\nu; \vec{k}'', \omega''_\nu)$ is of order $|\vec{k}' + \vec{k}''|$ as $\vec{k}' + \vec{k}'' \to 0$. Using (37), this implies that

$$\Sigma_{12}(\vec{k}, \omega_\nu) \text{ is of order } k, \tag{57a}$$

as $k \to 0$. Because of the dot products in D_{112} and D_{121},

$$\Sigma_{11}(\vec{k}, \omega_\nu) \text{ is of order } k^2, \quad \text{as } k \to 0 . \tag{57b}$$

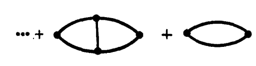

FIG. 1. The leading terms in the mass operator. The heavy dots represent gD, and the heavy lines represent $\tilde{G}$, the phonon propagator.

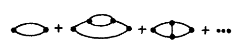

FIG. 2. The leading terms in the mass operator, written in terms of $\overline{G}$, the monochromatic propagator. This propagator is represented by thin lines, and the heavy dots represent gD. These three diagrams are named $g^2\Sigma^{(2)}$, $g^4\Sigma^{(4b)}$, and $g^4\Sigma^{(4c)}$.

However, these results are somewhat weaker than what is needed. We believe the properties of Σ are such that the exponents in (57) could be somewhat increased. If so,

$$\tilde{G}^{-1}_{11}(\vec{k}, \omega_\nu) = -c_0^2 k^2 + \text{higher powers of } k, \qquad (58a)$$

and this justifies the choice (43). This claim is somewhat technical, but not unimportant. If it were wrong, then $\overline{G}^{-1}_{11}$ would be a different multiple of k^2, (44) would be replaced by a more general propagator, and the alleged relation between the Landau-Rumer coefficient and the physical Grüneisen ratio would be upset. Similarly,

$$V_{112}(\vec{k}', \omega_\nu'; \vec{k}'', \omega_\nu'') = g(k' \circ k'') + \text{terms in } k'^2 \text{ and } k''^2 \qquad (58b)$$

is claimed. If the constant here were different from g, the alleged relation would be upset. These conjectured properties are related to the connection between the Landau-Rumer coefficient and the Grüneisen ratio, but perhaps they have not been stated in a useful or perspicuous form. The claims (58) have been verified to lowest order in perturbation theory, but nothing has been calculated to all orders, inside or outside of perturbation theory.

However, the g^4 terms in Σ have been calculated,[44] and can be discussed. The mass operator need not be written out in the style of Sec. VI; instead it is shown in Fig. 1. We ignore the quartic term in (31), which gives four-phonon vertices, for it does not lead to any effects of much interest. The heavy lines in Fig. 1 stand for $\tilde{G}$, which is not really known. The approximation (44) does not include the phonon frequency spectrum, and (53) probably does not express it correctly. An expansion in powers of g fits into this work better than either approximation. The g^2 term in the mass operator is known, and

$$\tilde{G}_{jk}(12) = G_{ojk}(12) + g^2 G_{ojl}(13)\,\Sigma^{(2)}_{lm}(34) G_{omk}(42) + O(g^4).$$

It would be logical to substitute this into Fig. 1, but the divergence of the real parts has previously led to the use of $\overline{G}$, given by (44), in place of G_0. This practice will be continued, and it will serve as an excuse for neglecting all corrections to the speed of sound. This means that the mass operator is given by Fig. 2. These three diagrams will be called $g^2\Sigma^{(2)}$, $g^4\Sigma^{(4b)}$, and $g^4\Sigma^{(4c)}$, respectively. The effects of the first diagram were calculated in Sec. VII. Figure 2 shows that $\Sigma^{(4b)}$ will give the propagator corrections to (45), and $\Sigma^{(4c)}$ will give the vertex corrections.

A simple calculation gives

$$\Sigma^{(4b)}_{jk}(12) = iD_{jlp}(134)\overline{G}_{lm}(35)\Sigma^{(2)}_{mn}(56)\overline{G}_{nq}(67) \times D_{qrk}(789)\overline{G}_{rp}(84)\delta(92) \ .$$

The Fourier coefficient of this is

$$\Sigma^{(4b)}(\vec{k}, \omega_\nu) = \int d\vec{k}' d\vec{k}'' \{\text{terms in } [\omega_\nu \pm c(k' + \sigma q)]^{-2} + \text{terms in } [\omega_\nu \pm c(k' + \sigma q)]^{-1} + \text{terms in } [\omega_\nu \pm c(k'' + \sigma_1 q + \sigma_2 q')]^{-1}\}, \qquad (59)$$

where $\vec{q} = \vec{k}' - \vec{k}$ and $\vec{q}' = \vec{k}'' - \vec{k}'$. Here the squared denominator arises because the wave vectors of two lines in the diagram are equal. It could be removed by partial integration, and the possibility is enough to show that all the terms in $\omega_\nu \pm c(k' + \sigma q)$ correspond to the fission and fusion processes discussed here. They will be called three-phonon terms and three-phonon processes. The terms in $\omega_\nu \pm c(k'' + \sigma_1 q + \sigma_2 q')$ correspond to the elastic collision of two phonons, the fission of one phonon into three, and the fusion of three phonons into one. They are called four-phonon terms and four-phonon processes; none of them are studied here, so such terms may be dropped.

The other Fourier coefficient which is needed is $\Sigma^{(4c)}(\vec{k}, \omega_\nu)$, a very lengthy expression. It is calculated from the Fourier coefficients of (40); the use of (44) gives

$$V_{222}(\vec{k}', \omega_\nu'; \vec{k}'', \omega_\nu'') = -gZ(u-1)c^{-2} - \left(\frac{g}{4\pi c}\right)^3 \int \frac{d\vec{q}}{qq'q''} \sum_{\sigma_1\sigma_2} \frac{\sigma_1\sigma_2[Z(\vec{q}\cdot\vec{q}') - \sigma_2(u-1)qq']}{[(\omega_\nu' + \omega_\nu'')^2 - c^2(q + \sigma_2 q')^2]}$$
$$\times \left(\frac{[\omega_\nu'(\omega_\nu' + \omega_\nu'') + c^2(q + \sigma_2 q')(q + \sigma_1 q'')]}{[\omega_\nu'^2 - c^2(q + \sigma_1 q'')^2]} W(\vec{q}, \vec{q}') + \frac{[\omega_\nu''(\omega_\nu' + \omega_\nu'') + c^2(\sigma_2 q + q')(q' + \sigma_1 q'')]}{[\omega_\nu''^2 - c^2(q' + \sigma_1 q'')^2]} W(\vec{q}', \vec{q}) \right) + O(g^5), \qquad (60)$$

where $\vec{q}'=\vec{q}-\vec{k}'-\vec{k}''$, $\vec{q}''=\vec{q}-\vec{k}'$,

$$W(\vec{q},\vec{q}')=[Z(\vec{q}\cdot\vec{q}'')-\sigma_1(u-1)qq''][Z(\vec{q}'\cdot\vec{q}'')+\sigma_1\sigma_2(u-1)q'q''][\coth(\tfrac{1}{2}\beta cq)+\sigma_1\coth(\tfrac{1}{2}\beta cq'')] \quad .$$

The other seven components of V have also been computed.[44] The whole thing is substituted into the Fourier transform of (37). This gives $\Sigma^{(2)}$ and

$$\Sigma^{(4c)}(\vec{k},\omega_\nu)=\int d\vec{k}'d\vec{q}\{\text{terms in }[\omega_\nu\pm c(k'+\sigma k'')]^{-1}$$
$$+\text{terms in }[\omega_\nu\pm c(q'+\sigma q'')]^{-1}$$
$$+\text{terms in }[\omega_\nu\pm c(k'+\sigma_1 q+\sigma_2 q'')]^{-1}$$
$$+\text{terms in }[\omega_\nu\pm c(k''+\sigma_1 q+\sigma_2 q')]^{-1}\}, \quad (61)$$

where $\vec{k}''$, $\vec{q}'$, and $\vec{q}''$ are dependent on the other wave vectors. The denominators here correspond to the various ways of cutting the crossbar diagram into two parts, each of which contains one end. Clearly, the first two denominators give three-phonon effects, and the last two denominators give four-phonon effects.

These fourth-order terms in Σ include numerous three-phonon terms; they are corrections to (45), which must finally appear as corrections to (52), the attenuation formula. On the other hand, these calculations also give corrections to the Grüneisen ratio, or corrections to (51). The Grüneisen ratio is closely related to V, the vertex function; for gV_{jk2} is the derivative of $\tilde{G}^{-1}$ with respect to $\langle\rho\rangle$. Formula (51) is easily derived from this remark and the leading terms in V. But V_{jk2} also contains terms in g^3; they are integrals which show the usual divergence at large wave numbers. Computation shows that the only integral which goes as K^4 is in (60) or V_{222}; the other integrals are not so strongly divergent. This means that the only K^4 term changes the effective value of u, or that

$$V_{222}(\vec{k}',\omega_\nu';\vec{k}'',\omega_\nu'')=-gZ(u'-1)c^{-2}$$
$$-(\text{subtracted integral}) \quad ,$$

where $u'-u$ is proportional to (24). From this formula and (58b), it is found that

$$\frac{\rho}{c}\frac{\partial c}{\partial\rho}=\frac{1}{2}\left(\frac{u'-1}{Z}+1\right) .$$

The same vertex function goes into the calculation of $\Sigma^{(4c)}$ and the attenuation. The computations, which are briefly sketched in this section, show that u' replaces u in the attenuation formula. Hence (50) and all other results should contain $(u'-1)/Z$ in place of $(u-1)/Z$. But when the results are written in terms of the physical Grüneisen ratio, this change has no effect.

The K^4 term in the vertex correction has been discussed here. It is connected with the difference between bare and physical Grüneisen ratios, but it does not affect the connection between the Landau-Rumer coefficient and the physical Grüneisen ratio. The discussion of the Grüneisen ratio having been concluded, it is convenient to set

$$\langle\rho\rangle-\rho_0=0 \quad ,$$

so that $\tilde{G}^{-1}=\overline{G}^{-1}-\Sigma$.

The leading terms in (59) and (61), in another sense, are those which diverge most strongly at small angles. The resulting propagator corrections and vertex corrections are proportional to $1/\gamma$. The Fourier coefficient (59) contains

$$ck'\pm c(k''+\sigma_2 q') \quad (62)$$

and its square, as denominators, although they are not explicitly shown. This quantity can vanish if $\vec{k}'$ and $\vec{k}''$ are parallel or antiparallel; hence it causes the small-angle divergence which has been discussed in Sec. IV. This divergence is strongest when the square of (62) occurs in the denominator. Therefore the leading terms are those with the square of (62) and a certain combination of signs. When (59) is written out explicitly,[44] it is a lengthy expression from which these leading terms can be picked out. Similar denominators occur in (61), where the leading terms contain two small denominators. The replacement (49) keeps all such denominators from vanishing, assuming $\gamma>0$. The leading terms in the propagator corrections and vertex corrections are proportional to $1/\gamma$; the leading terms in the rate of a four-phonon process[27] are also proportional to $1/\gamma$.

The calculation of the corrections to (45) will be outlined here. The relevant terms in $\Sigma_{jk}^{(4b)}(\vec{k},\omega_\nu)$ are

$$-\frac{(-i)^{j-k}}{2^8(2\pi)^6Zc}\int\frac{d\vec{k}'d\vec{q}}{k'^2k''qq'}\sum_{\sigma_1\sigma_2\sigma_3}\frac{\sigma_1\sigma_3A_jA_k}{(ck'-\sigma_2cq-\sigma_3cq')^2}$$
$$\times\left[\frac{1}{\omega_\nu-c(k'+\sigma_1k'')}-\frac{(-1)^{j-k}}{\omega_\nu+c(k'+\sigma_1k'')}\right]$$
$$\times[\coth(\tfrac{1}{2}\beta ck')+\sigma_1\coth(\tfrac{1}{2}\beta ck'')]$$
$$\times[\coth(\tfrac{1}{2}\beta cq)+\sigma_2\sigma_3\coth(\tfrac{1}{2}\beta cq')]$$
$$\times[Z(\vec{q}\cdot\vec{q}')k'+\sigma_3Z(\vec{q}\cdot\vec{k}')q'+\sigma_2Z(\vec{q}'\cdot\vec{k}')q$$

$$+\sigma_2\sigma_3(u-1)qq'k']^2 \quad , \tag{63}$$

where $\vec{k}''=\vec{k}-\vec{k}'$, $\vec{q}'=\vec{k}'-\vec{q}$, and $\sigma_1, \sigma_2, \sigma_3$ are three sign factors. This has been put into the form of (45) as far as possible. It is evident that the terms with $\sigma_2=\sigma_3=-1$ are negligible; otherwise there is a divergence when $\vec{q}$ and $\vec{k}'$ are parallel or antiparallel. This divergence is removed by (49). But the integrand continues to vary rapidly with angle. Hence the relatively slow angular variation of the dot products can be neglected; this simplifies the last factor and the A's.

If the small-angle contribution is dominant, and the frequency is positive, Σ is proportional to

$$M=\begin{pmatrix} 4c^2k^2 & 2ick(u-1+Z) \\ -2ick(u-1+Z) & (u-1+Z)^2 \end{pmatrix} .$$

This can be seen from (45), the simple case. In this small-angle approximation, Σ has a term in

$$X_+=\delta(\vec{k}-\vec{k}'-\vec{k}'')k'k''\left(\frac{M}{\omega_\nu-c(k'+k'')} - \frac{M^*}{\omega_\nu+c(k'+k'')}\right)[\coth(\tfrac{1}{2}\beta ck')+\coth(\tfrac{1}{2}\beta ck'')]$$

and a term in

$$X_-=\delta(\vec{k}-\vec{k}'+\vec{k}'')k'k''\left(\frac{M}{\omega_\nu-c(k'-k'')} - \frac{M^*}{\omega_\nu+c(k'-k'')}\right)[\coth(\tfrac{1}{2}\beta ck')-\coth(\tfrac{1}{2}\beta ck'')],$$

besides the four-phonon terms.

With the indicated small-angle approximations, (63) is proportional to

$$\frac{1}{\gamma c^5}\int d\vec{k}'\,d\vec{k}''(X_+-X_-)\times\left(k'^2+\frac{4\pi^2}{\beta^2c^2}-\frac{24}{\beta^3c^3k'}\sum_{n=1}^{\infty}\frac{1-e^{-n\beta ck'}}{n^3}\right).$$

Obviously this part of Σ is a correction to the $\vec{k}'$ propagator. It might well be written symmetrically in $\vec{k}'$ and $\vec{k}''$, for both propagators require correction. The integration over $\vec{k}'$ and $\vec{k}''$ diverges, but the similar divergence of (45) has been discussed. This function gives a certain correction to the integrand in (45) or (47).

Another correction proportional to $1/\gamma$ is obtained from $\Sigma^{(4c)}$, but it is somewhat more complicated. The three-phonon terms[44] are simplified by the use of (49) and the small-angle approximations:

$$\Sigma^{(4c)}(\vec{k},\omega_\nu)=-\frac{(u-1+3Z)^2}{2^7(2\pi)^6Zc^5}\times\int d\vec{k}'\,d\vec{k}''\,d\vec{q}\,q\,(X_+\Delta_++X_-\Delta_-)+\text{irrelevant terms},$$

where
$$\Delta_\pm=\left(\frac{1}{a'_\mp a''_+}+\frac{1}{a'_\pm a''_-}\right)\coth(\tfrac{1}{2}\beta cq)\mp\frac{(k''+q)}{a'_\mp}\,\frac{\coth[\frac{1}{2}\beta c(k'\mp q)]}{(k'\mp q)a'_\mp-(k''+q)a''_+}-\frac{(k'\mp q)}{a''_+}\,\frac{\coth[\frac{1}{2}\beta c(k''+q)]}{(k'\mp q)a'_\mp-(k''+q)a''_+}$$
$$\pm\frac{(k''-q)}{a'_\pm}\,\frac{\coth[\frac{1}{2}\beta c(k'\pm q)]}{(k'\pm q)a'_\pm-(k''-q)a''_-}+\frac{(k'\pm q)}{a''_-}\,\frac{\coth[\frac{1}{2}\beta c(k''-q)]}{(k'\pm q)a'_\pm-(k''-q)a''_-} . \tag{64}$$

Here the abbreviations

$$a_\pm'=(k'q-\vec{k}'\cdot\vec{q})/(k'\pm q)^2+3\gamma k'q$$

and $$a_\pm''=(k''q-\vec{k}''\cdot\vec{q})/(k''\pm q)^2+3\gamma k''q$$

have been used. The thorough mixture of $\vec{k}'$ and and $\vec{k}''$ in (64) shows that this is the vertex-correction function. It can be shown that (64) falls off exponentially for large q, so the indicated integration over $\vec{q}$ certainly converges. It can also be shown that the result of this integration is proportional to $1/\gamma$, if the angle between $\vec{k}'$ and $\vec{k}''$ is small.[45] However, this integration over $\vec{q}$ is quite complicated, and has not been done.

In these sections, we have carried these calculations far enough to show the nature of the higher terms in quantum hydrodynamics. Two terms in K^4 have appeared in the calculations, and have been eliminated. One such term appeared in $\Sigma_{22}^{(2)}$, and it was removed by expressing results in terms of c, the experimental speed of sound. Another such term appeared in

$$\delta\tilde{G}_{22}^{-1}/\delta\langle\phi_2\rangle \quad\text{or}\quad \partial\tilde{G}_{22}^{-1}/\partial\langle\rho\rangle \quad ,$$

and it was removed by expressing the results in terms of the experimental Grüneisen ratio. There are also terms in K^2, which have hardly been noticed; their effects should be proportional to

$$(\kappa T)^2K^2/(\rho_0c^3) \quad ,$$

which is dimensionless. Then the small-angle terms, which have just been calculated, give effects proportional to (30). Also, there are terms in (23) which have been discussed, although they are omitted from the results of the hydrodynamic theory. All these terms should appear as an ωT^6 term in the sound attenuation. We have not calculated this ωT^6 term, but experiment shows that it cannot be large.[5,6,26,28]

The Simons formula[8] is supposed to describe the sound attenuation caused by the fusion of an incident phonon with a thermal phonon. It attempts to describe how this process goes away at low temperatures and high frequencies. We can now see a great gap between this formula and our perturbation theory. The problem is to replace the step function of dispersion, which comes from the simple calculation of Sec. VII, by a reasonably smooth function. The step function comes from the factor

$$1/[\omega_\nu \pm (ck' + \sigma_1 ck'')] \quad , \tag{65}$$

which occurs in (45) and many higher-order terms. Higher powers of (65) also occur in perturbation theory – in (59) for example. One might argue that the sum of all terms in perturbation theory is some function of (65) which, when integrated, yields the replacement for the step function. We have not attempted the difficult process of adding up all these terms; but it is unlikely to yield any function of (18), the argument of the alleged arctangent. For, using (19) or (20), the argument (18) can be written as the first or second power of

$$\mathrm{const}(\omega/\kappa T)^p \gamma \rho_0 c^3/(\kappa T)^2 ,$$

where $p = 1$ or $\frac{1}{2}$. This quantity can hardly be expected to occur as the reciprocal of the ratio of successive terms in perturbation theory; and it cannot otherwise occur in perturbation theory. However, the Simons formula must serve until some other description is proposed, for the effect of dispersion on three-phonon processes. We believe that the process of fusion can be eliminated by going to higher frequencies and lower temperatures; but neither theory nor experiment gives a real explanation of how this happens.

In this section, we have studied the higher-order corrections to the three-phonon processes, or to (45). First, some symmetries and conjectured general properties were stated. Then the divergent terms in the vertex function and propagator corrections were calculated, and shown to be innocuous; this result is expressed by (25) and discussed at that place. Finally, we have explained why we cannot find a connection between our systematic calculations and the Simons formula.

X. CONCLUSION

In order to justify the use of quantum hydrodynamics, the speed and attenuation of sound have been calculated in two different theories. The two theories both apply to the low-density Bose-Einstein gas, or the Bogoliubov model; and this model shows that the two theories agree at low energies. The correction terms of relative order

$$\gamma(\kappa T/c)^2 \tag{23}$$

are omitted from our hydrodynamic results, because they cannot be calculated consistently; we argue that the hydrodynamic theory determines phonon-phonon interactions only at long wavelengths.

The hydrodynamic theory, unlike the microscopic theory, appears to be useful for calculations on liquid helium, where the interactions between particles are moderately strong. Therefore the hydrodynamic theory has been compared with experiments on liquid helium. The theoretical temperature dependence of the speed of sound is given by (13), which shows not even a qualitative agreement with experiment.[5,6] Formula (13) might be corrected by reactive terms of higher order; but such terms in the hydrodynamic theory are strongly cutoff dependent, and in the microscopic theory have never been computed. Therefore the reactance arising from phonon-phonon interactions is not understood.

We have discussed at length the attenuation caused by phonon fusion. We believe that it can be eliminated by going to higher frequencies and lower temperatures; but neither theory nor experiment gives any real explanation of how this happens. The Simons formula, which attempts to describe how this process goes away, has been criticized above. But we use the argument of Simons[8] to justify the replacement of the step function in (13) by the average of its two values. This replacement gives (15), which contains no unknown parameters.

The measured attenuation is about 1.5 to 2.7 times this theoretical estimate. Either part of the attenuation is produced by an unknown mechanism, or we have underestimated the strength of the three-phonon vertex. Both possibilities seem unlikely. Here we may remark that any mysterious contribution to the attenuation must be roughly proportional to ωT^4, for the total attenuation is nearly proportional to ωT^4. On the other hand, the measurements of Abraham, Eckstein, Ketterson, Kuchnir, and Vignos[6] do show, both in the attenuation and in the speed of sound, a change in slope at some critical frequency, which increases with the temperature. This phenomenon is partly described, but not explained, by the Simons formula; and it does suggest an unknown mechanism.

We have a quantitative disagreement between measured and theoretical attenuation, and we cannot propose a plausible explanation of it.

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APPENDIX: CALCULATIONS IN MICROSCOPIC THEORY

Here we calculate the attenuation of sound, and the temperature dependence of the speed of sound. The outline of this calculation is provided by the unpublished work of Hohenberg,[16] but here we derive the needed results from the formulas of Hohenberg and Martin.[46]

In the microscopic theory of the condensed Bose-Einstein gas, it is usual to study the correlation functions of $\psi(\vec{r}, t)$ and $\psi^\dagger(\vec{r}, t)$, the particle annihilation and creation operators. Here they will be calculated approximately, to find the real and imaginary parts of the frequency of oscillation. Hohenberg and Martin[46] show that the same frequency must occur in the density autocorrelation function; so the frequency calculated here gives the speed and attenuation of sound.

These two operators connect a quantum state to another state having one particle more or less. This makes it convenient to put systems having different numbers of particles into the ensemble, fixing the phases so that

$$\langle \psi(\vec{r}, t)\rangle = \langle \psi^\dagger(\vec{r}, t)\rangle = n_0^{1/2} \tag{66}$$

is real. Here n_0 is the number density of the Bose condensate; $mn_0 = \langle\rho\rangle(1-d)$. It is independent of space and time when there is no external disturbance, which is the case of interest.

It is convenient to include a chemical-potential term in the Hamiltonian of the system of particles, so that the "energy" is independent of the number of particles. This Hamiltonian gives the equation of motion

$$\left(i\frac{\partial}{\partial t_1} + \mu m + \frac{\nabla_1^2}{2m}\right)\psi(1) = \int v(12)\psi^\dagger(2)\psi(2)\psi(1)d2,$$

where $v(12) = v(|\vec{r}_1 - \vec{r}_2|)\delta(t_1 - t_2)$

is the interatomic potential. There is a similar equation of motion for $\psi^\dagger(1)$. It is expedient to combine them, in the way suggested by Nambu.[47] Let

$$\Psi_1(1) = \psi(1) \quad \text{and} \quad \Psi_2(1) = \psi^\dagger(1).$$

Then $G_0^{-1}{}_{jk}(12)\Psi_k(2)$

$$= \tfrac{1}{2}v(12)\Psi_k^\dagger(2)\Psi_j(1)\Psi_k(2),$$

where summation and integration are understood, and

$$G_0^{-1}{}_{jk}(12) = \left[\frac{(-1)^j}{i}\frac{\partial}{\partial t_1} + \left(\mu m + \frac{\nabla_1^2}{2m}\right)\right]\delta_{jk}\delta(12).$$

In this theory, Green's functions are the one-point functions

$$G_{1/2}(1) = (-i)^{1/2}\langle\Psi(1)\rangle \quad ,$$

$$G_{1/2}^\dagger(1) = (-i)^{1/2}\langle\Psi^\dagger(1)\rangle \quad ,$$

and the matrix depending on two points:

$$\tilde{G}_1(11') = (-i)\{\langle[\Psi(1)\Psi^\dagger(1')]_+\rangle - \langle\Psi(1)\rangle\langle\Psi^\dagger(1')\rangle\}.$$

Here the subscripts are suppressed. In order to generate various approximations by functional differentiation, it is useful to add a disturbance

$$\overline{U}(11')\psi(1)\psi^\dagger(1')$$

to the Hamiltonian. The formalism is similar to that in Sec. VI; in the case of interest, $\overline{U} = 0$ and (66) holds. Hohenberg and Martin[46] define

$$\Sigma(11') = G_0^{-1}(11') - \tilde{G}_1^{-1}(11') - \overline{U}(11'),$$

and they give various approximate formulas for Σ and μ, which together determine $\tilde{G}_1^{-1}$.

The zeroth approximation in this theory is that of Bogoliubov.[9] Hohenberg and Martin express it by

$$\Sigma(11') = \tfrac{1}{2}i\,v(12)G_{1/2}(2)G_{1/2}^\dagger(2)\delta(11') + i\,v(11')G_{1/2}(1)G_{1/2}^\dagger(1')$$

and a corresponding formula for μ. The subscripts are to be inserted before summation and integration; v does not have any, but the δ function includes a Kronecker δ. The Fourier coefficient is

$$\Sigma_{ij}(k, \omega_\nu) = n_0 v(0)\delta_{ij} + n_0 v(k),$$

where $\quad v(k) = \int d\vec{r}\, e^{-i\vec{k}\cdot\vec{r}} v(|\vec{r}|).$

The corresponding approximation to μ is $n_0 v(0)/m$. It follows that

$$\tilde{G}_1^{-1}(k, \omega_\nu) = \begin{pmatrix} \omega_\nu - k^2/(2m) - n_0 v(k) & -n_0 v(k) \\ -n_0 v(k) & -\omega_\nu - k^2/(2m) - n_0 v(k) \end{pmatrix}. \tag{67}$$

This matrix must be inverted, and continued analytically until z is any complex frequency. The resulting $\tilde{G}_1(k, z)$ contains most of Bogoliubov's results.

However, it is useful to express the results in terms of the scattering amplitude; this permits application of the theory to atoms, which have a hard-core potential. This was done by Bogoliubov[9] and many of his successors. The first Born approximation is consistent with the Bogoliubov approximation; hence

$$v(k)=4\pi a/m \quad (68)$$

is substituted into (67), and an explicit formula for $\tilde{G}_1(k,z)$ is obtained. It shows that the frequency of oscillation is

$$\omega(k)=[(k^2/2m)(k^2/2m+8\pi n_0 a/m)]^{1/2}\ . \quad (69)$$

In the general theory, Hugenholtz and Pines[21] proved that $\tilde{G}_1(k,z)$ has a pole at $k=z=0$, and they claimed that the gap between positive- and negative-frequency poles of $\tilde{G}_1(k,z)$ must tend to zero as $k\to 0$. This property is possessed by the Bogoliubov approximation, and all other approximations which are useful at long wavelengths. The next "gapless" approximation is that of Beliaev.[14] Hohenberg and Martin[46] express it by

$$\begin{aligned}\Sigma(11')=&\tfrac{1}{2}iv(12)[G_{1/2}(2)G_{1/2}^{\dagger}(2)+\tilde{G}_1(22)]\delta(11')+iv(11')[G_{1/2}(1)G_{1/2}^{\dagger}(1')+\tilde{G}_1(11')]\\&-\tfrac{1}{2}v(13)v(21')\tilde{G}_1(11')[\tilde{G}_1(23)G_{1/2}(3)G_{1/2}^{\dagger}(2)+G_{1/2}(2)G_{1/2}^{\dagger}(3)\tilde{G}_1(32)]\\&-v(13)v(41')\tilde{G}_1(14)[\tilde{G}_1(43)G_{1/2}(3)G_{1/2}^{\dagger}(1')+G_{1/2}(4)G_{1/2}^{\dagger}(3)\tilde{G}_1(31')]\\&-\tfrac{1}{2}v(12)v(31')G_{1/2}(1)G_{1/2}^{\dagger}(1')\tilde{G}_1(23)\tilde{G}_1(32)-v(12)v(31')G_{1/2}(1)G_{1/2}^{\dagger}(3)\tilde{G}_1(32)\tilde{G}_1(21')\end{aligned} \quad (70)$$

and a formula for μ. These formulas are to be evaluated by using (66) and appropriate estimates of $v(k)$ and $\tilde{G}_1(k,\omega_\nu)$. The Bogoliubov estimate of $\tilde{G}_1$ is sufficiently accurate for use on the right-hand sides; this can be shown[16] by using (1). But (68) is not sufficiently accurate; this means that some scattering theory is needed, to expand $v(k)$ in powers of the scattering amplitude. In the Beliaev approximation, and even the higher approximation mentioned in Sec. II, the results may be expressed in terms of the scattering length only.[21] At sufficiently low temperatures, the other terms in the low-energy expansion of the scattering amplitude do not enter. But at higher temperatures, or in higher approximations, the details of the interatomic force begin to come in.

Since low energies and low temperatures are considered here, we express $\tilde{G}_1^{-1}$ in terms of a. The terms linear in a are indicated by (67) and (68). The higher-order terms are some lengthy integrals,[44] obtained from (70) and a little scattering theory. These integrals all converge, unlike those in quantum hydrodynamics.

As in Sec. VII, only the determinant of $\tilde{G}^{-1}(k,z)$ is needed. It gives the frequency of the oscillations in the retarded commutator:

$$\omega(k)+\Lambda^-[k,\omega(k)+i0^+]\ . \quad (71)$$

Here $\omega(k)$ is given by (69), and

$$\begin{aligned}\Lambda^-(k,z)=&\frac{ak}{m^2}\left(\frac{n_0 a}{\pi(1+\theta^2)}\right)^{1/2}\int_0^\infty\left(2m-\frac{k'^2}{\omega(k')}\coth[\tfrac{1}{2}\beta\omega(k')]\right)dk'+\frac{n_0a^2k}{2m(1+\theta^2)^{1/2}}\sum_\sigma\int_1^\infty dx\int_{-1}^1 dy\\&\times[E_1(F_- - F_+)+\theta^2 Y]\mathfrak{C}-\frac{n_0a^2k}{2m}\sum_\sigma\int_1^\infty dx\int_{-1}^1 dy\,E_2(F_- + F_+)\mathfrak{C}\\&+\frac{n_0a^2k}{4m(1+\theta^2)^{1/2}}\sum_\sigma\int_1^\infty dx\int_{-1}^1 dy\,[E_3(F_- - F_+)+Y]\mathfrak{C}\ ,\end{aligned}$$

where $\theta=\tfrac{1}{4}k(\pi n_0 a)^{-1/2}$,

$$\mathfrak{C}=\coth\{\tfrac{1}{4}\beta ck(x-y)[1+\tfrac{1}{4}\theta^2(x-y)^2]^{1/2}\}+\sigma\coth\{\tfrac{1}{4}\beta ck(x+y)[1+\tfrac{1}{4}\theta^2(x+y)^2]^{1/2}\}\ ,$$

$$E_1=[1+\theta^2(x^2+y^2)+\tfrac{3}{4}\theta^4(x^2-y^2)^2][1+\tfrac{1}{2}\theta^2(x^2+y^2)+\tfrac{1}{16}\theta^4(x^2-y^2)^2]^{-1/2}-\sigma\theta^2(x^2-y^2),$$

$$E_2=(x+y)[1-\tfrac{1}{2}\theta^2(x-y)^2][1+\tfrac{1}{4}\theta^2(x-y)^2]^{-1/2}+\sigma(x-y)[1-\tfrac{1}{2}\theta^2(x+y)^2][1+\tfrac{1}{4}\theta^2(x+y)^2]^{-1/2}\ ,$$

$$E_3=[x^2+y^2+\tfrac{1}{4}\theta^2(x^2-y^2)^2][1+\tfrac{1}{2}\theta^2(x^2+y^2)+\tfrac{1}{16}\theta^4(x^2-y^2)^2]^{-1/2}+\sigma(x^2-y^2)\ ,$$

$$1/F_\pm=2z(ck)^{-1}\pm(x+y)[1+\tfrac{1}{4}\theta^2(x+y)^2]^{1/2}\pm\sigma(x-y)[1+\tfrac{1}{4}\theta^2(x-y)^2]^{1/2}\ ,$$

$$Y=(x+y)[1+\tfrac{1}{4}\theta^2(x-y)^2]^{-1/2}+\sigma(x-y)[1+\tfrac{1}{4}\theta^2(x+y)^2]^{-1/2}\,.$$

Now an expansion[44] of (71) in powers of k shows that the speed of sound is

$$(4\pi n_0 a)^{1/2}m^{-1}+4n_0a^2m^{-1}\int_0^\infty [w^{-1/2}-(w+1)^{-1/2}\coth+\tfrac{4}{3}(w+1)^{-3/2}\coth]dw$$

$$+\frac{8n_0a^2}{m}\int_0^\infty \frac{(4w^2+3w)^2}{(2w+1)^4}\left(\frac{2w+1}{(w+1)^{1/2}}-\operatorname{arctanh}\frac{(w+1)^{1/2}}{2w+1}\right)\left(\frac{d}{dw}\coth\right)dw$$

$$-\frac{8n_0a^2}{3m}\int_0^\infty \frac{w}{(2w+1)(w+1)^{1/2}}\left(\frac{d}{dw}\coth\right)dw+4\pi i\,\frac{n_0a^2}{m}\int_0^\infty\frac{(4w^2+3w)^2}{(2w+1)^4}\left(\frac{d}{dw}\coth\right)dw\,.$$

Here the argument of each coth is

$$4\pi n_0 a\beta(w^2+w)^{1/2}/m=\pi(w^2+w)^{1/2}/\theta\,,$$

where θ is now the parameter of Sec. II. Expansion of these integrals in powers of θ gives (10).

An approximate evaluation of (71) gives the attenuation formula of Sec. VII. The functions E_j and $1/F_\pm$ are expanded in powers of θ, ignoring the possibility that x may be large. Then the leading imaginary term in (71) is of order $k\theta^4$, and it gives (52), if corrections of order d are neglected.[43] But large values of x can occur in Λ^-, and they give corrections proportional to (23), which have been discussed in Secs. III and VII.

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Nonrelativistic Electron Bremsstrahlung in a Strongly Magnetized Plasma

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In this paper we have calculated the emission of bremsstrahlung radiation of nonrelativistic electrons in Landau levels in a dense plasma. We have found that the radiation rate is inversely proportional to the electron momentum, which is characteristic of one-dimensional gases.

I. INTRODUCTION

In a previous paper[1] we presented the emission rate of bremsstrahlung radiation in vacuum. In view of the large plasma effect present in dense matter, we present here a nonrelativistic calculation of bremsstrahlung radiation in a dense plasma with a strong magnetic field. (That a nonrelativistic calculation is adequate in our theory for pulsars will be discussed in a separate paper.)

The problem of radiation emitted by an electron in passing through the field of a charged nucleus is a classical one in electrodynamics[2] and plasma physics.[3] In astrophysics it is known as the free-free transition,[4] while in electrodynamics it appears under the name bremsstrahlung. The main problem in this paper is concerned with bremsstrahlung radiation in such a strong field that the trajectory of the electron is no longer classical.[5] As the ion merely provides a Coulomb field, which in nonrelativistic cases can be regarded as static, we need not concern ourselves with quantization of ion orbits in intense magnetic fields. In a magnetic field the electron state can be described in terms of a principal quantum number n (= 0, 1, 2, ...) and a momentum variable p_z along the direction of the field H (which is taken to be in the z direction). The total energy of the electron $E(n, x)$ is[5]

$$\mathcal{E}(n,x) \equiv E(n,x)/mc^2 = (1+x^2+2nH/H_q)^{1/2} , \qquad (1)$$

where $x = p_z/mc$, $H_q = m^2c^3/e\hbar = 4.414\times10^{13}\,\text{G}$.

The effect of the magnetic field merely quantized the x and y momenta by the following substitution[6]:

$$p_x^2 + p_y^2 \to 2n(H/H_q)(mc)^2 \quad . \qquad (2)$$

As a result of this quantization the density of final states is modified. A summation over all states now takes the following form[4,6]:

$$\textstyle\sum_n \omega_n \int (1/h)\, dp_z , \qquad (3)$$

where ω_n is the degeneracy of the state n. For a particle of a given energy $\mathcal{E}$, n can only take values such that

$$\mathcal{E} \geqslant (1+2nH/H_q)^{1/2} \quad . \qquad (4)$$

In particular, if $\mathcal{E}-1 \leqslant (1+2H/H_q)^{1/2}-1$, the only