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Coherent Spontaneous Emission*

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Coherent spontaneous emission from a system of N two-level atoms interacting with a quantized radiation field is treated for the case in which Dicke's "cooperation number" r is macroscopically large. It is shown how to modify the quasiclassical approach to this problem to incorporate quantum effects that are lost in the self-consistent-field approximation. The statistics of the emitted radiation are found to vary markedly with the initial state of the system of atoms. The photon statistics tend to that typical of blackbody radiation when the initial state of the atomic system is that which would result from incoherent pumping ($m \sim r$). When, on the other hand, the atoms are initially in a superradiant state ($m \ll r$), the emitted radiation may be represented approximately by a coherent state.

I. INTRODUCTION

Several years ago Dicke[1] introduced the concept of coherence in spontaneous radiative emission from a system of N "molecules" which undergo transitions between two energy levels. In this paper we present a detailed discussion of a recent paper of ours,[2] in which a nonperturbative solution of the problem has been given.

The main feature of coherent spontaneous radiation processes is the possibility, in certain configurations, of having the radiation rate proportional to N^2 rather than to N, as one would expect when the molecules radiate "incoherently," i.e., independently of each other. Such an anomalously large radiation rate occurs as a consequence of a highly correlated motion of the N-molecules system, which, as a consequence, radiates as a single quantum-mechanical system. These exceptional states were called superradiant states by Dicke. They belong to a class of correlated states of the N-molecules system which are best defined in the framework of angular momentum theory.[3]

By representing the single two-energy-level molecule as a spin-$\frac{1}{2}$ particle, one can define a "superradiant" state as a particular eigenstate of the *total* angular momentum $|r, m\rangle$. The total angular momentum quantum number r was called the "cooperation" number by Dicke, and is obviously related to the degree of correlation among the spins.

On the other hand, $m = \frac{1}{2}(n_+ - n_-)$ (n_+, n_- give the number of excited and nonexcited molecules, respectively) is proportional to the energy stored in the system, and is such that $|m| \leq r \leq \frac{1}{2}N$.

Obviously the $|r, m\rangle$ states are not a complete basis for the N-spin system. However, from degeneracy considerations, Dicke[1] has shown that, if

the system is in thermal equilibrium, both m and r are well defined, and satisfy $|m| \simeq r$. The physical meaning of the choice $|m| = r$ is that the macroscopic transverse dipole moment proportional to $r^2 - m^2$ has to be zero when no phase relations exist between different spins. This is the case not only for a system at thermal equilibrium ($m < 0$), but also for an incoherently pumped system ($m > 0$). The states with $m = r$, which we shall refer to as "fully excited" states, radiate proportionally to r. On the other hand, it is possible to prepare the system in a state with $r \gg m$, which radiates proportionally to r^2 and is called "superradiant".[1] As we shall demonstrate, the radiation process from a fully excited state requires a thorough quantum-mechanical treatment, whereas the superradiant states radiate *essentially* in a classical way.

We shall address ourselves to the problem of the spontaneous radiation, once the gaseous system has been prepared in a state with definite and positive values of r and m.

In order to explore the nonlinear quantum features of the problem, we limit ourselves to the simplest physical situation. We shall therefore make the following assumptions: (i) The single-molecule wave functions do not overlap, so it is possible to neglect the problem of the symmetry of the total wave function. (ii) The system is thermally isolated and embedded in a resonant cavity which possesses a single mode resonant with the transition frequency of the molecules. (iii) The system density is so low that the atoms interact with each other only via their coupling with the electromagnetic field.

The Hamiltonian for this simple model is well known to describe the main features of the one-mode laser dynamics for times shorter than the relaxation times.[4] Furthermore the same Hamiltonian, by use of the Schwinger representation of angular momentum in terms of boson creation and annihilation operators,[5] describes trilinear boson scattering processes. This means that our simple model is also able to describe the essential quantum features of parametric processes (such as the coherent Raman and Brillouin scattering) taking into account the nonlinear behavior due to the depletion of the pump field. In addition to the quantum-mechanical approach of Dicke, the previous work on this Hamiltonian model can be divided into two categories: computer calculations and quasiclassical approximations.

The computer calculations have been done for 80^4 and for a few hundred atoms.[6] The results of this analysis exhibit a behavior considerably different from the one expected classically. However such an analysis is hardly applicable to a realistic situation where the number of atoms is macroscopically large. In such a case, we expect the system to evolve in a way closer to the classical theory than predicted by such computer calculations. On the other hand, the semiclassical treatment based on the self-consistent-field approximation is clearly inadequate for at least the following reasons: (a) It does not take into account the quantum spontaneous emission; (b) It cannot give any information about the photon statistics.

Our analysis will modify the usual equations by adding to the source of the field which is responsible for the spontaneous emission, a quantum source, which is always present and becomes essential in the fully excited initial state where the classical source is absent. In general the motion of the system is similar to the classical pendulum motion but with an intrinsic kinetic energy such that there is no unstable equilibrium position.

As a consequence, one gets a periodic series of pulses whose shape is a hyperbolic secant and whose buildup time as well as period is a factor $\ln N$ larger than the duration of the single hyperbolic secant pulses. In particular, the short-time behavior of our solutions is in agreement with that of Dicke's radiation rate.

To our knowledge no previous calculations exist for quantum dispersion of the photon number for these nonlinear and nonstationary processes. In Ref. 7, a closed solution is given only for the linearized parametric amplifier in which the pump field is treated as a given function of time. This approximation is equivalent to neglecting the depletion of the upper atomic level.

We have found two different statistics for the fully excited state $m = r$ and the superradiant state $m = 0$. In the first case, the dispersion is practically that of Bose statistics $\sigma(\bar{n}) \simeq \bar{n}$, and in the second case that of a binomial distribution which assumes Poisson statistics $\sigma(\bar{n}) \simeq (\bar{n})^{1/2}$ when $r \to \infty$. Such different photon statistics can be completely understood in terms of the different physical nature of the fully excited and superradiant states.

In Sec. II we will introduce the model Hamiltonian for the coherent coupling of an N-spin system with a one-mode electromagnetic field and discuss its physical relevance to the trilinear boson scattering processes. The similarity between these two problems[8] is emphasized by introducing the concept of "effective population numbers" of the N-spin system.

In Sec. III, for completeness and for the sake of comparison, we derive the Bloch equations for our problem and discuss the analogy of our system with a rigid pendulum from the Heisenberg equation of motion in the self-consistent-field approximation. For the same reasons the Dicke radiation rate is

briefly derived in the framework of first-order perturbation theory. Furthermore, we generalize the perturbative treatment to obtain the short-time photon statistics by solving the Schrödinger equation when the mean number of photons emitted is much smaller than the effective number of active spins. This solution is essential to construct the all-times solution.

In Sec. IV we discuss in a nonperturbative way the Schrödinger equation, calculating the mean photon number and the quantum dispersions as a function of time. In spite of the analytical complication due to the nonlinearity of the problem, the solution will be expressed in very simple terms.

The approximations made are perfectly justified when the number r is macroscopically large. The reader who does not wish to interrupt the physical discussion can directly go from Sec. III to Sec. V, in which the results of Sec. IV are summarized, extensively discussed, and compared with previous treatments. In carrying out this comparison we give the limit of validity of the self-consistent-field approximation.

II. MODEL HAMILTONIAN

The model Hamiltonian we want to discuss is well known in the literature.[4] Let the energy difference between the two levels be $\hbar\omega$, where ω coincides with one of the cavity eigenfrequencies. In the dipole approximation, the resonant part of the Hamiltonian which describes the entire system is

$$H=\hbar\omega R_3+\hbar\omega a^+a+\hbar k(a^+R^-+aR^+)\,. \tag{1}$$

The operators a^+ and a are the creation and annihilation operators for the electromagnetic mode under consideration and obey the commutation relation

$$[a,\ a^+]=1\,. \tag{2}$$

The operators R^+ and R^-, using Dicke's notations, are the sum over all the spin-flip operators of the N atoms multiplied by the proper phase factor.[1] They obey commutation relations of angular momentum operators:

$$[R^+,\ R^-]=2R_3,\quad [R_3,\ R^\pm]=\pm R^\pm\,. \tag{3}$$

The eigenvalues m of R_3 go from $-\frac{1}{2}N$ to $\frac{1}{2}N$. More precisely, if $n^+(n^-)$ is the number of excited (unexcited) spins, one has

$$m=\tfrac{1}{2}(n_+-n_-),\quad N=n_++n_-\,. \tag{4}$$

The number m represents the total energy of the free spin system. The interaction Hamiltonian describes one-step quantum transitions between two of these collective energy states. In this model, the N spins interact with the electromagnetic field as a single dipole vector with components R^+, R^-, and R_3. In this sense, the model represents a *coherent* radiation process by the N spins. As is well known, this is the fundamental difference between laser and conventional thermal sources, where the single atoms radiate independently of each other.

As already remarked in the Introduction, the Hamiltonian (1) is also relevant for the dynamics of coherent Raman and Brillouin scattering.[9] In fact, the main features of these processes are described by a trilinear boson scattering, in which an incoming photon of frequency ω_2 (pump field) gives rise in a nonlinear medium to a photon of frequency ω_1 and to a scattered phonon of frequency $\omega_2-\omega_1$.

The interaction Hamiltonian for perfect phase matching[9] is given by

$$H_{\rm int}=\hbar k(a_2^+a_1a+a^+a_1^+a_2)\,, \tag{5}$$

where the oscillator operators a_2, a_1, and a describe the incoming photon, the scattered phonon, and the scattered photon, respectively.

A classical solution of this problem can be found in Ref. 9, and can be obtained from the Heisenberg equation of motion for the amplitudes $\langle a_1\rangle$, $\langle a_2\rangle$, and $\langle a\rangle$, assuming *complete factorization*; i.e., expressions like $\langle aa_2^+a_1\rangle$ are assumed to factorize into $\langle a\rangle\langle a_2^+\rangle\langle a_1\rangle$. This hypothesis is evidently equivalent to treating the a's as c numbers.

The formal identity between Hamiltonians (1) and (5) is pointed out in Ref. 8, where a "quasiclassical" solution of Hamiltonian (5) is obtained by assuming a partial factorization; i.e., expressions like $\langle aa_2^+a_1\rangle$ factorize into $\langle a\rangle\ \langle a_2^+a_1\rangle$.

Such an identity follows from the fact that the angular momentum operators R^+ and R^- can be represented by means of two harmonic oscillator operators a_2 and a_1.[5] In fact, we set

$$\begin{aligned} &R^+=a_2^+a_1,\quad |R|^2=R(R+1),\\ &R_3=\tfrac{1}{2}(a_2^+a_2-a_1^+a_1),\quad R=\tfrac{1}{2}(a_2^+a_2+a_1^+a_1)\,. \end{aligned} \tag{6}$$

One can easily verify that this representation is consistent with commutation relations (2) and (3).

Hence Hamiltonians (1) and (5), using identities (6), formally coincide. Furthermore, calling $|r,\ m\rangle$ the eigenstates of $|R|^2$ and R_3 corresponding to the eigenvalues $r(r+1)$ and m, and calling $|n_2,\ n_1\rangle$ the eigenstates of $a_2^+a_2$ and $a_1^+a_1$, one has a one-to-one correspondence between such states, given by the obvious relations

$$m=\tfrac{1}{2}(n_2-n_1),\quad r=\tfrac{1}{2}(n_2+n_1)\,. \tag{7}$$

The process of coherent radiation emission by a system of N atoms already prepared in a collec-

tive state $|r, m\rangle$ is formally equivalent to trilinear scattering processes in which the pumping field a_2 and the phonon field a_1 are prepared in a state $|n_2, n_1\rangle$. The correspondence rule is given by Eqs. (6) and (7).

We note that n_+ and n_- of Eq. (4) and n_2 and n_1 of Eq. (7) have quite different physical meanings. Whereas n_+ and n_- give the number of spins in the two possible levels; n_1 and n_2, once the system has been prepared in the state $|r, m\rangle$, give the effective populations of the two levels which take part in the coherent process. n_2 and n_1 coincide with n_+ and n_- *only* when the correlation among the spins is maximum: $r = \frac{1}{2}N$.

In general the same values of n_+ and n_- can correspond to very different values of n_2 and n_1, which reflect different physical situations. For example, if the system has been incoherently excited to some positive value of $m = \frac{1}{2}(n_+ - n_-)$, the number r has to be chosen equal to m.[1] Hence, in this case one gets $n_2 = n^+ - n^-$ and $n_1 = 0$, which is equivalent to having a fully excited system with a total number of atoms equal to $n^+ - n^-$. On the other hand, if the system has been prepared in a superradiant state with $r \simeq \frac{1}{2}N \gg n^+ - n^-$, one gets $n_2 \simeq n_1 \simeq \frac{1}{2}N$. Once the initial values of the effective populations have been established, the calculations can be carried out equally well with Hamiltonians (1) or (5), where now the oscillator operators a_2 and a_1 operate on the effective energy levels. This correspondence, which is so simple in terms of n states of harmonic oscillators, requires a more careful analysis using other representations, such as the coherent-states representation.[10] The concept of effective populations is by no means of academic interest, but makes it possible to distinguish between two radically different spontaneous radiation processes. Generally, by spontaneous emission one means a radiation process which takes place starting with a zero initial value of the eigenvalue n of a^+a. However, it is easy to see that the spontaneous emission by a superradiant state and by a fully excited state are quite different in nature. In fact, from Hamiltonian (5) we see that the oscillators a_1 and a play a completely symmetrical role. Hence, the time evolution of a superradiant state characterized by $n_2 = n_1 = r$ and $n = 0$, is the same of a state characterized by $n_1 = 0$, $n_2 = r$, and $n = r$. This means that the spontaneous radiation by a superradiant state is substantially equivalent to the stimulated radiation by a fully excited state in which there is already present a number of photons equal to the effective population of the upper level. In this sense one can say that a real spontaneous emission process takes place only if both the effective population of the fundamental level and the initial photon number are zero.

We now proceed to analyze Hamiltonian (1), referring to the N-spin-system problem. Everything can be immediately translated using Eq. (7) to the trilinear scattering processes. We first note that the unperturbed part of Hamiltonian (1) is a constant of motion, i.e.,

$$R_3 + a^+a = M = \text{const.} \tag{8}$$

Hence the Hamiltonian in the interaction picture is time independent and reads

$$H = \hbar k(a^+R^- + R^+a). \tag{9}$$

In order to have a better insight into the dynamical properties of our system, we briefly derive the quasiclassical picture in terms of Bloch equations. This will also allow a closer comparison between the classical and the quantum picture and a better understanding of both.

III. QUASICLASSICAL AND SHORT-TIME SOLUTIONS

We now shortly derive the Bloch equations from the Heisenberg equation of motion by making the self-consistent-field approximation (SCFA).

The Heisenberg equations which follow from Hamiltonian (9) are

$$\dot{a} = -iR^-, \quad \dot{R}^- = 2iaR_3, \quad \dot{R}_3 = i(a^+R^- - aR^+), \tag{10}$$

where time is measured in units of the reciprocal coupling constant $1/k$.

This is not a closed system of equations for the expectation values because expressions like $\langle a^+R^-\rangle$ do not factorize into $\langle a^+\rangle\langle R^-\rangle$. Hence it would be necessary to write equations for a^+R^- which will still depend on the product of atomic and field operators and so on. In other words, to solve exactly the Heisenberg equation of motion requires solving an infinite hierarchy of equations. The SCFA consists in closing this hierarchy at the lowest order, i.e., in factorizing the mean values directly in Eqs. (10). This means that the operators of these equations shall be replaced by their c-number expectation values:

$$\langle R^\pm\rangle = r^\pm, \quad \langle R_3\rangle = r_3, \quad \langle a\rangle = \alpha.$$

In this way the system (10) becomes a closed system of equations for the quantities $r^\pm$, r_3, and α.

This system is still nonlinear, but can be solved exactly. If one sets $i\alpha \to \alpha$, one obtains the real coefficients system

$$\dot{r}_3 = -2\alpha r_1, \quad \dot{r}_1 = 2\alpha r_3, \quad \dot{\alpha} = r_1, \tag{11}$$

where $r_1 = r^+ = r^-$ is the real transverse component of the macroscopic dipole. This set of equations is well known in the literature as Bloch equations. Using the two constants of motion

$$r_3^2 + r_1^2 = J^2, \quad M = \alpha^2 + r_3, \tag{12}$$

the system (11) can be reduced to a unique nonlinear equation for the field amplitude

$$(\dot{\alpha})^2 = r_1^2 = (J^2 - r_3^2) = J^2 - (M - \alpha^2)^2. \tag{13}$$

This equation will be useful in order to have a direct comparison with the quantum solution.

To have the analogy of our system with a rigid pendulum, it is convenient to write Eq. (11) in a different form. The dipole variables r_1 and r_2 can be considered as the components of two-dimensional vector of constant length which we shall refer to as the Bloch vector. The time evolution of this vector is simply a rotation with angular velocity proportional to the field amplitude α. In fact if one sets

$$r_1 = J \sin\varphi(t), \quad r_3 = J \cos\varphi(t), \tag{14}$$

the first two equations (11) are equivalent to the equation

$$\dot{\varphi} = 2\alpha. \tag{15}$$

By using the third equation (11), the system reduces to a unique equation for the Bloch angle

$$\ddot{\varphi} = 2J \sin\varphi. \tag{16}$$

Hence, the Bloch vector rotates as a rigid pendulum whose stable equilibrium point is $\varphi = \pi$. The radiation from this pendulum is obtained by the simple relation (15).

This simple pendulum analogy is well known to describe undistorted pulse propagation in a laser amplifier[11] as well as self-induced transparency phenomena.[12]

This formalism is quite simple and clear, but does not take into account the quantum spontaneous emission. This point deserves a few comments. From Eq. (13), we see that the initial time derivative of the field amplitude, if $\alpha(0) = 0$, is given by $\dot{\alpha}(0) = r_1(0) = J^2 - M^2$, where $M = R_3(0)$ is the initial value of the inversion. If $M \ll J$, the intensity of the field increases proportionally to J^2. This is the superradiant spontaneous emission due to the presence of a macroscopic dipole r_1. Hence this spontaneous emission is correctly accounted for by a classical theory.

We suppose $J = M$. In this case no radiation is emitted by the system. This corresponds to the fact that the initial position of the pendulum is the unstable equilibrium point with zero velocity $\dot{\varphi} = 2\alpha = 0$. We stress that this is the physical initial situation for an incoherently excited system. Clearly this result is in principle wrong because the quantum spontaneous emission by the excited spins, which is missing here, has to set the system into motion. In other words, we do not expect our system to have an unstable equilibrium point. The trick one uses to bypass this difficulty is to give an arbitrary small initial value to $\dot{\varphi}$ or φ. The solution for $\alpha(t)$ is generally a periodic elliptic function. However, if one lets $\varphi(0)$ and $\dot{\varphi}(0)$ approach zero, the period of the pulses goes to infinity and the solution becomes a single spike, which is the well-known hyperbolic secant pulse[11,12]:

$$\alpha^2 = (\tfrac{1}{2}\dot{\varphi})^2 = 2J \operatorname{sech}^2[(2J)^{1/2}t]. \tag{17}$$

One can directly verify that this is the singular solution of the pendulum equation (16) which starts from the unstable equilibrium point at $t = -\infty$. Any information about the buildup time and the period of the pulses is evidently lost.

We now turn to discuss Hamiltonian (9) in the Schrödinger picture. We use the representation in which a^+a, R_3, and $|R|^2$ are diagonal with respective eigenvalues n, m, and $r(r+1)$, which label the common eigenstates $|n, m, r\rangle$.

We suppose the initial state to be a state with zero photons and a fixed value of r and m of $|0, M, r\rangle$. As we have already pointed out, there are two constants of motion, R^2 and $R_3 + a^+a$. Hence, the system will evolve in the subspace of constant r undergoing transitions to states $|n, m, r\rangle$ such that $n + m = M$. Therefore we can label the states with only the quantum number n, and $|n\rangle$ is a shorthand for

$$|n\rangle \equiv |n, M - n, r\rangle. \tag{18}$$

The quantum problem is completely solved once we know the probability amplitude of having n photons at the time t:

$$p(n, t) \equiv \langle n | e^{-iHt/\hbar} | 0\rangle. \tag{19}$$

In order to construct the solution for all times, it will turn out to be particularly useful to derive the short-time solutions. For very short times, we can calculate $p(n, t)$ by means of the first-order perturbation theory:

$$p(n, t) = -i\delta_{n,1}[(r + M)(r - M + 1)]^{1/2}t,$$

where we have used the well-known properties

$$\begin{aligned} &a|n\rangle = (\sqrt{n})|n-1\rangle \quad a^+|n\rangle = (n+1)^{1/2}|n+1\rangle, \\ &R^{\pm}|r, m\rangle = [(r \pm m + 1)(r \mp m)]^{1/2}|r, m \pm 1\rangle. \end{aligned} \tag{20}$$

The mean number of photons at time t is given by

$$\bar{n}(t) = (r^2 - M^2 + r + M)t^2. \tag{21}$$

This result substantially gives Dicke's radiation rate and should be compared with the classical behavior of the intensity given by Eq. (13). By solving Eq. (13) to first order in time, with the initial conditions $\alpha(0) = 0$, $r_3(0) = M$, we get

$$\alpha^2 = (J^2 - M^2)t^2 . \tag{22}$$

Comparing Eqs. (21) and (22) we see that, provided the cooperation number r is identified with the total length of the Bloch vector J, the quantum and the classical expressions differ by a term $(r+M)t^2$.

Therefore, in the quantum equation (21) we may distinquish two sources of the field: (i) that which goes as $r^2 - M^2$, is of classical origin and can be identified with the classical polarization r_1^2 of the Bloch equations; (ii) that which goes as $r+M$, is the quantum spontaneous emission term which is absent in the classical equation. This term is irrelevant for the superradiant states in which $r \gg M$, but becomes crucial for the ordinary spontaneous emission in which $r = M$. The expression for $\bar{n}(t)$ at any time, which we give below, will coincide with Eq. (21) if specialized for very short times.

In order to have a suitable expression for the short-time transition probability, we now solve the Schrödinger equation with an approximation less crude than the first-order one. Differentiating both sides of Eq. (19) and using Eqs. (20), we get

$$\begin{aligned} i\dot{p}(n,t) = {} & [n(r-M+n)(r+M-n+1)]^{1/2} p(n-1,t) \\ & + [(n+1)(r-M+n+1)(r+M-n)]^{1/2} \\ & \times p(n+1,\ t) . \end{aligned} \tag{23}$$

We solve this equation with the initial condition $p(n,\ 0) = \delta_{n,0}$ in a range of time so short that the probability amplitude for having n photons is negligible unless $n \ll r$. This range of time will be later specified. We consider the two interesting particular cases: the superradiant case $M = 0$ and the fully excited state $M = r$. In both cases r is assumed to be a very large number of the order of $\frac{1}{2}N$.

For $M = 0$, neglecting n as compared with r, Eq. (23) reduces to

$$i\dot{p}(n,\ t) \simeq r[(\sqrt{n})p(n-1,\ t) + (n+1)^{1/2} p(n+1,\ t)] . \tag{24}$$

This is the equation of a simple harmonic oscillator driven by a constant classical force. This classical force is here proportional to the length of the macroscopic dipole r.

One can easily verify that the exact solution of this equation is given by

$$|p(n,\ t)|^2 = \left| (-i)^n e^{-r^2 t^2/2} \frac{(rt)^n}{\sqrt{n!}} \right|^2 = e^{-\bar{n}} \frac{\bar{n}^n}{n!} , \tag{25}$$

where the mean number of photons $\bar{n}(t)$ is given by

$$\bar{n}(t) = r^2 t^2 . \tag{26}$$

This formula practically coincides with Eq. (21), specialized for $M = 0$. The statistics given by Eq. (25) are Poissonian, and the dispersion is

$$\sigma(\bar{n})^2 = (n^2)_{\text{av}} - \bar{n}^2 = \bar{n} .$$

The region of time in which this solution holds is clearly given by $t \ll t_s \simeq r^{-1/2}$, so that $\bar{n} \ll r$. Hence, if the system is prepared in a superradiant state, the short-time photon statistics is that of a coherent radiation field.[13]

If $M = r$, in the same approximation as before, Eq. (23) reduces to

$$i\dot{p}(n,\ t) = 2r^{1/2}[np(n-1,\ t) + (n+1)p(n+1,\ t)] . \tag{27}$$

The exact solution of this equation is

$$\begin{aligned} |p(n,\ t)|^2 = {} & |(-i)^n \tanh^n (2r)^{1/2} t \\ & \times \operatorname{sech}(2r)^{1/2} t|^2 = \bar{n}^n/(1+\bar{n})^{n+1} , \end{aligned} \tag{28}$$

where

$$\bar{n} = \sinh^2 (2r)^{1/2} t . \tag{29}$$

This expression for $t \to 0$ is in agreement with Eq. (21).

The photon statistics for $\bar{n} \ll 2r$, i.e., for $t \ll t_s = \frac{1}{2} \ln 2r/(2r)^{1/2}$, are Bose statistics which are characteristic of an incoherent thermal field.[13] The dispersion associated with $\bar{n}$ is

$$\sigma(\bar{n})^2 = \bar{n}(\bar{n}+1) ,$$

which is larger by a factor $(\bar{n}+1)^{1/2}$ than the previous one. Furthermore, the time t_s is here longer by a factor $\ln 2r$ than the time t_s of a superradiant state.

We remark that Eqs. (28) and (23) are known to describe the parametric amplifier if in the Hamiltonian (9) the pump field a_2 is replaced by the constant number $(2r)^{1/2}$.[7]

The markedly different behavior of the photon statistics is clearly understandable if one thinks of the different sources of the field in the two cases. In a superradiant state the source of the field is a big macroscopic polarization proportional to r^2, which, being a classical source, drives the field into a coherent state. In the second case, the only source is the quantum spontaneous emission, so that it is no surprise to find Bose statistics characteristic of a chaotic field.

As we shall see, the photon statistics at later times basically retain their short-time features, except for values of $\bar{n}$ near their maximum values.

IV. SOLVING THE SCHRÖDINGER EQUATION

We are now in a position to attack the dynamical problem based on Hamiltonian (9) and on the related Schrödinger equation (23).

This equation is a quite complicated difference-differential equation which cannot be solved exactly. However, quite reasonable approximations can

be made when the cooperation number r is macroscopically large.

First of all we eliminate the imaginary factor in Eq. (23) by means of the simple transformation

$$p(n,t) \to (-i)^n p(n,t).$$

In this way Eq. (23) becomes

$$\dot{p}(n,t) = [n(r-M+n)(r+M-n+1)]^{1/2} p(n-1,t) - [(n+1)(r-M+n+1)(r+M-n)]^{1/2} \times p(n+1,t). \quad (30)$$

Now the equation has real coefficients, so the amplitude $\dot{p}(n,t)$ is real; for $t=0$, $p(n,0)=\delta_{n,0}$.

The easiest way of finding a solution of a difference-differential equation is to approximate it with a partial differential equation. However, in order to have a better control of the validity of the approximation, it is convenient to first make some transformations in Eq. (30). These transformations shall be performed according to the following two steps. (i) Transform $p(n)$ into a new variable $A(n)$ such that the coefficient of the terms $A(n-1)$ and $A(n+1)$ are equal. (ii) Change the variable n into a new variable which varies in an almost continuous way when r is very large.

We proceed with the first step. We define a function $g(h)$ as

$$g(h) = \sqrt{2}\Gamma(1+\tfrac{1}{2}h)/\tfrac{1}{2}\Gamma(1+h), \quad h=0,\ 1,\ 2,\ \ldots. \quad (31)$$

Using the well-known property $\Gamma(1+z)=z\Gamma(z)$ of Γ functions, one can verify that $g(h)$ satisfies the recurrence equation

$$g(h+1)g(h) = h+1, \quad \text{with } g(0)=(2/\pi)^{1/2}. \quad (32)$$

Equation (30) can then be written

$$\dot{p}(n,t) = [g(n)g(n-1)g(r-M+n)g(r-M+n-1) \times g(r+M-n+1)g(r+M-n)]^{1/2} p(n-1,t) - [g(n+1)g(n)g(r-M+n+1)g(r-M+n) \times g(r+M-n)g(r+M-n-1)]^{1/2} p(n+1,t),$$

and setting

$$G(n) = g(n)g(r-M+n)g(r+M-n), \quad (33)$$

we get

$$\dot{p}(n,t) = [G(n)]^{1/2}\{[G(n-1)]^{1/2} p(n-1,t) - [G(n+1)]^{1/2} p(n+1,t)\}.$$

Finally, if we define $A(n)$ as

$$A(n,t) = [G(n)]^{1/2} p(n,t),$$

we obtain

$$\dot{A}(n,t) = G(n)[A(n-1,t) - A(n+1,t)]. \quad (34)$$

Hence we have reduced Eq. (30) to a more symmetrical form.

We now find a simpler expression of the coefficient $G(n)$. The Γ functions which define the g functions in Eq. (30) are conveniently expressed by the Stirling formula[14]

$$\Gamma(z) \sim (2\pi)^{1/2} e^{-z} z^{z-1/2}\left(1+\frac{1}{12z}+\frac{1}{288z^2}+\cdots\right), \quad \mathrm{Re}\, z > 0.$$

To lowest order in $1/z$, using definition (31) we get

$$g(h) \simeq (1+h)^{1/2}. \quad (35)$$

This approximation also gives good results for small values of h, as the reader can easily check by calculating $g(h)$ via Eq. (31) and comparing the result with $(1+h)^{1/2}$. For $h \gg 1$, Eq. (35) becomes $g(h) \sim \sqrt{h}$, which can be written directly using the Wallis formula (14). Finally, using Eqs. (33) and (35), $G(n)$ can be written as

$$G(n) \cong [(1+n)(r-M+n+1)(r+M-n+1)]^{1/2}. \quad (36)$$

The second step of our procedure is to define the angle θ_n as

$$n = (r+M)\sin^2\theta_n, \qquad \theta_n = \arcsin[n/(r+M)]^{1/2}. \quad (37)$$

We note that as m goes from the initial value M to the minimum value $-r$, the photon number varies from zero to its maximum value $r+M$. This can be easily seen by remembering that, by definition, $n+m=M$. Correspondingly, the angle θ_n goes from zero to $\frac{1}{2}\pi$.

Using Eqs. (14) and (37), one can easily see that the angle θ is related to the angle described by the Bloch vector by the relation

$$\cos\tfrac{1}{2}\varphi = [(M+r)^{1/2}/2r]\cos\theta,$$

which reduces to $\theta = \frac{1}{2}\varphi$ for the fully excited state $M=r$. Clearly, if one assumes that r is very large and M is a positive quantity, one sees that the variation of θ when n goes from $n-1$ to $n+1$ is very small. A quite accurate expression for this variation is given by

$$\Delta\theta_n = \theta_{n+1} - \theta_{n-1} = [(n+1)(r+M-n+1)]^{-1/2}. \quad (38)$$

The biggest value of the jump of the angle θ is at the extreme values of n: $n \simeq 0$ and $n \cong r+M$. In this region

$$\Delta\theta_n \simeq (r+M)^{-1/2}.$$

We remark that this value of $\Delta\theta$ is infinitesimally small compared with the total variation $\frac{1}{2}\pi$,

when r is macroscopically large. Hence we will assume that the angle θ has continuous variation. Therefore it should be clear that the validity of our calculation rests heavily on the fact that r is very large, as is physically the case.

Let us now go back to Eq. (35). Thinking of n as a function of the new independent variable θ_n, we get

$$\dot{A}(\theta_n, t)+G(\theta_n)[A(\theta_{n+1}, t)-A(\theta_{n-1}, t)]=0. \quad (39)$$

Using the previous remark, we pass to the continuum in θ and get

$$\frac{\partial A(\theta, t)}{\partial t}+G(\theta)\left[\frac{\partial A}{\partial \theta}\Delta(\theta)+\tfrac{1}{2}\frac{\partial^2 A}{\partial \theta^2}(\Delta(\theta))^2+\cdots\right]=0, \quad (40)$$

where $G(\theta)$ and $\Delta(\theta)$ are continuous functions of the variable θ obtained from Eqs. (36) and (38). Explicitly,

$$\Delta(\theta)=\{[1+(r+M)\sin^2\theta][1+(r+M)\cos^2\theta]\}^{-1/2},$$
$$G(\theta)=\{[1+(r+M)\sin^2\theta][1+(r+M)\cos^2\theta]$$
$$\times[r-M+1+(r+M)\cos^2\theta]\}^{1/2}. \quad (41)$$

If $\partial A/\partial\theta$ does not vary too rapidly in the small interval $\Delta(\theta)$, we can drop the higher-order derivatives in Eq. (40).

The assumption of slow variation of $\partial A/\partial\theta$ is certainly not true in the neighborhood of $t=0$. In fact, at $t\cong 0$ the probability of having n photons is $\delta_{n,0}$ and consequently $A(\theta)$ is a very sharply peaked function around $\theta=0$.

Hence the differential equation we are deriving will not describe the short-time behavior but, in the sense we shall make precise in the following, can be used to construct the wanted solution out of the short-time solutions which have already been derived in Sec. III. Therefore, we will use the differential equation (27) for times $t>t^*$, where t^* is a time such that the amplitude distribution $A(\theta, t)$ is such a sufficiently smooth function that the higher-order derivatives of Eq. (40) can be neglected.

Afterwards t^* will be consistently determined from the short-time solutions by the requirement $1\ll\bar{n}(t^*)\ll N$ [for practical purposes $\bar{n}(t^*)$ can be chosen to be of the order of a few hundred].

Thus, for sufficiently smoothly varying $A(\theta)$, Eq. (40) becomes

$$\frac{\partial A}{\partial t}+v(\theta)\frac{\partial A}{\partial \theta}=0, \quad (42)$$

where

$$v(\theta)=[(r-M+1)+(r+M)\sin^2\theta]^{1/2}=G(\theta)\Delta(\theta).$$

This equation has to be solved starting at the initial time t^* with an initial condition fixed by the short-time solution.

We observe that $A(\theta, t)$ is a probability amplitude which propagates in the θ space with a velocity which depends on the "point" θ. This means that the θ space behaves like an inhomogeneous medium that clearly will deform the initial shape of A. We can already see that this effect is much smaller in the superradiant state $M=0$ than in the fully excited state $r=M$. In fact, if $M=0$, the "velocity" $v(\theta)$ varies from $(r+1)^{1/2}$ for $\theta=0$ to $(2r+1)^{1/2}$ for $\theta=\frac{1}{2}\pi$. On the other hand, when $M=r$, this variation is by a factor $(2r)^{1/2}$.

It is quite natural to study Eq. (42) using the variable $u(\theta)$ defined by

$$u(\theta)=\int_0^\theta d\theta'/v(\theta'). \quad (43)$$

Using the fact that, by definition, $du/d\theta=1/v(\theta)$, Eq. (42) is transformed to

$$\frac{\partial A(u, t)}{\partial t}+\frac{\partial A(u, t)}{\partial u}=0. \quad (44)$$

The general integral of this equation is

$$A(u, t)=A_0(u-t),$$

where A_0 is an arbitrary function which has to be determined by the initial condition. The relation between θ and u is specified by the integral in Eq. (43), which turns out to be an elliptic integral of the first kind with a negative parameter; i.e., we have

$$\sin\theta=sn[(r-M+1)^{1/2}u\,|-(r+M)/(r-M+1)],$$

where sn is a Jacobi elliptic function.[14]

Using standard elliptic transformations in order to have a positive parameter we get

$$\sin\theta=[(r-M+1)^{1/2}/(2r+1)]sd[(2r+1)^{1/2}u\,|k]$$
$$=cn[(2r+1)^{1/2}u-K\,|k], \quad (45)$$

where the parameter k of the elliptic function is

$$k=(r+M)/(2r+1),$$

and K is a quarter of the real period of the elliptic function. It is bounded by the relation[15]

$$\ln 4\le[K+\tfrac{1}{2}\lg(1-k)]\le\tfrac{1}{2}\pi. \quad (46)$$

In this relation the upper limit is approached if $k\to 0$ and the lower limit is approached if $k\to 1$.

We write

$$K(k)=a+\tfrac{1}{2}\ln[(2r+1)/(r-M+1)],$$

where $\ln 4\le a\le\frac{1}{2}\pi$. The previously introduced quantities $G(\theta)$, $v(\theta)$, $\Delta(\theta)$, can now be expressed in terms of u. In particular, from Eqs. (37) and (45) we get

$$n=(r-M)cn^2[(2r+1)^{1/2}u_n-K\,|k], \quad (47)$$

which relates a discrete set of values u_n to the

discrete values of n from 0 to $r+M$. It is now important to calculate the variation of u_n for each quantum step $n \to n+1$. To a very good approximation we get

$$\frac{\Delta u}{\Delta n}=\frac{\Delta u}{\Delta\theta}\frac{\Delta\theta}{\Delta n}\simeq\frac{\Delta(\theta)}{2v}=\frac{1}{2G}\,, \tag{48}$$

where Eqs. (42) and (34) have been used.

Hence, the G function approximately gives the metrics which connect the u space to the n space. We notice that, due to the smallness of $1/G(n)$, we can consider u as well as θ to be practically continuous variables.

We can obtain the physical meaning of $|A(u, t)|^2$ from Eq. (48). In fact using Eq. (48) and the definition (34), the normalization condition can be written

$$\sum_n |p(n, t)|^2 \simeq \int |A_0(u-t)|^2 du = 1$$

for all times t. Hence $|A|^2$ is the probability distribution in the u space.

From the previous discussion, it is clear that a compact way to calculate the kth moment of the photon distribution at the time t is given by

$$\langle n^k(t)\rangle = \sum_{n=0}^{r+M} n^k |p(n,t)|^2 \simeq \int du\, n^k(u)\,|A_0(u-t)|^2\,, \tag{49}$$

where $n(u)$ is given by Eq. (47). The function $|A_0(u-t)|^2$ is determined by the initial condition

$$|A_0(u-t^*)|^2 = G[n(u)]\,|p[n(u), t^*]|^2\,, \tag{50}$$

where $p(n, t^*)^2$ is the short-time solution of Eqs. (25) and (28).

A detailed use of Eqs. (49) and (50) is computationally quite cumbersome and is best done by computer. However, taking into account the fact that $|A_0|^2$ is quite a sharply peaked function, we can give a simple approximate evaluation both of the mean photon number $\bar{n}(t)$ and of the dispersion $\sigma(\bar{n}, t)$. In order to evaluate $\bar{n}(t)$, we notice that since $|A_0(u-t)|^2$ is sharply peaked about $u=t$, the mean value $\bar{n}(t)$ can be approximated by the most probable value $\bar{\bar{n}}(t)$, i.e., the value of n obtained by setting $u=t$ in Eq. (47).

Thus we have

$$\bar{n}(t)\simeq\bar{\bar{n}}(t)=(r+M)cn^2[(2r+1)^{1/2} \times t - K|(r+M)/(2r+1)]\,. \tag{51}$$

We next proceed to calculate the dispersion of the photon number around the value $\bar{n}(t)$. We know that $A(u, t)$, after some time t^*, obeys the differential equation (44). This means that the dispersion $\sigma(\bar{u}, t)$ of this curve around the value $\bar{u}=t$ is a function of time which starts from zero at $t=0$ and after the time t^* reaches some stationary value σ^*.

We now calculate this stationary value. From Eq. (48) we may write, approximately,

$$\sigma(\bar{n}, t)=2G(\bar{n})\sigma(\bar{u}, t) = 2[(1+\bar{n})(r-M+\bar{n}+1)(r+M-\bar{n}+1)]^{1/2}\sigma(\bar{u}, t)\,. \tag{52}$$

For $M=0$, relation (52) becomes

$$\sigma(\bar{n}, t)=2\{(1+\bar{n})[(r+1)^2-\bar{n}^2]^2\}^{1/2}\sigma(\bar{u}, t)\,. \tag{53}$$

On the other hand, we already know from Eq. (25) that for times such that $\bar{n}\ll r$, we have $\sigma(\bar{n}, t) = [\bar{n}(t)]^{1/2}$.

Specializing Eq. (53) in this region, we get

$$\sigma(\bar{u}, t)=\frac{1}{2r'}\left(\frac{\bar{n}}{1+\bar{n}}\right)^{1/2}.$$

Hence, $\sigma(\bar{u}, t)$ starts from zero at $t=0$, and after a very short time t^*, such that, say, $\bar{n}(t^*)$ ~100, reaches the stationary value

$$\sigma^*\simeq 1/2r\,. \tag{54}$$

After this time t^*, $A(u, t)$ is consistently undeformed so that $\sigma(\bar{u}, t)=\sigma^*$. Hence, by Eq. (53), we get at any time

$$\sigma(\bar{n}, t)\simeq\{\bar{n}(t)[1-\bar{n}(t)^2/r^2]\}^{1/2}\,. \tag{55}$$

We remark on the consistency of our calculation. First of all, in order to use the short-time dispersion $\sigma(\bar{n})=\sqrt{\bar{n}}$ to evaluate the stationary dispersion of n, it has been implicitly assumed that the time t^* after which $\sigma(\bar{u}, t)$ is stationary is smaller than the time t_s after which the short-time solution ceases to hold so the time regions of validity of the two solutions *do* overlap. This assumption is perfectly consistent with our results since we had defined t^* and t_s by $\bar{n}(t^*)\gg 1$ and $\bar{n}(t_s)\ll r$.

To justify the use of the quasidifferential relation (52) between the dispersions of n and u, it is clearly necessary that n, as a function of u [as given by Eq. (47)], does not vary appreciably in an interval σ^*. This is well verified by Eq. (47), which shows that n as a function of u varies on a scale $1/\sqrt{r}$, which is bigger than $\sigma^*=1/2r$ by a factor $2\sqrt{r}$.

We now turn to the fully excited state $r=M$; here Eq. (52) becomes

$$\sigma(\bar{n}, t)=2(1+\bar{n})(2r+1-\bar{n})^{1/2}\sigma(\bar{u}, t)\,. \tag{56}$$

For $\bar{n}\ll 2r$, i.e., for times much shorter than the period, we know that $\sigma(\bar{n})=[\bar{n}(\bar{n}+1)]^{1/2}$. Hence, Eq. (56) becomes

$$[\bar{n}(\bar{n}+1)]^{1/2}=2(1+\bar{n})(2r+1)^{1/2}\sigma(\bar{u}, t),$$

from which one derives that as soon as n is of the order of a few hundred we have

$$\sigma(u, t)=\sigma^*\simeq 1/2(2r)^{1/2}\,. \tag{57}$$

This stationary dispersion is now bigger than the dispersion σ^* of the superradiant state by a factor $(\frac{1}{2}r)^{1/2}$. More precisely, from Eq. (47) one sees that when n is near to the maximum value $2r$, i.e., for $u \simeq K/(2r+1)^{1/2}$, the variation of n corresponding to a variation σ^* of u is of the order of $2r$.

From all this we conclude that $\sigma(\bar{n}, t)$ is always of the order $\bar{n}$; in other words, that the chaotic nature of the short-time distribution is preserved as the time elapses.

V. DISCUSSION AND CONCLUSIONS

We discuss here the results of the previous sections. We have found that the mean photon number $\bar{n}(t)$ is approximately given by the Jacobi elliptic function cn or sd as

$$\bar{n}(t) = \frac{(r+M)(r-M+1)}{2r+1}\, sd^2[(2r+1)^{1/2}t \mid k]$$

$$= (r+M)cn^2[(2r+1)^{1/2}(t-T) \mid k], \qquad (58)$$

where the elliptic parameter k is given as a function of the initial value of the inversion M and of the cooperation number r by the relation

$$k = (r+M)/(2r+1). \qquad (59)$$

The quarter of period T[15] is given by

$$T = (2r+1)^{-1/2}[a - \tfrac{1}{2}\ln(1-k)], \qquad (60)$$

where $\ln 4 \leq a \leq \frac{1}{2}\pi$.

The number a is equal to $\frac{1}{2}\pi$ in the trigonometric limit $k \to 0$ and is equal to ln4 in the hyperbolic limit $k \to 1$. The time T is the build up time of the pulse, i.e., the time in which the mean number of photons increases from zero to the maximum value $r+M$. We first compare our formula with the short-time solution (21). To first order in time, the sd function of Eq. (58) is $(2r+1)^{1/2}t$. Hence we have

$$\bar{n}(t) \simeq (r+M)(r-M+1)t^2, \qquad (61)$$

which coincides with Eq. (21).

The behavior of $\bar{n}(t)$ as given by Eq. (58) depends strongly on the relative magnitudes of M and r; if $M = r$, it increases very slowly and reaches the maximum value after a time

$$T \simeq \ln 2r/(2r)^{1/2}. \qquad (62)$$

The time distance between two subsequent peaks is $2T$.

One may ask what relation exists between these and the classical hyperbolic secant pulses. We observe that if $M = r$, the elliptic parameter k as given by Eq. (60) is $k = 2r/(2r+1)$, which is extremely near, but not exactly equal to, 1. Now, when $k \to 1$, it is well known that the elliptic function cn approaches the hyperbolic function sech.

So we conclude that if $M = r$, we get a periodic train of pulses separated by a distance $2T$ whose shape is very close to

$$\bar{n}(t) \cong 2r\,\mathrm{sech}^2(2r)^{1/2}[t - (2m+1)T],$$

$$m = 0, 1, \cdots.$$

The shape of each pulse coincides with the classical one. The great difference is that the buildup time T is not infinite, but is $\ln(2r)$ larger than the time of duration of the single spike $(2r)^{-1/2}$. In the case $M = r$, the system radiates a series of well-separated hyperbolic secant pulses. On the other hand, in the superradiant case $M = 0$, $k \cong \frac{1}{2}$, and we get a train of pulses whose separation time is of the same order as the duration time, and the shape resembles very much the square of a sine function. The quarter period T is given by

$$T \simeq (2r+1)^{-1/2}(\ln a + \tfrac{1}{2}\ln 2).$$

Finally, when only a few spins are excited, i.e., when $M \simeq -r$, one gets elliptic functions with parameter $k \to 0$, i.e., pure trigonometric functions. That is what we expect for the small oscillations of a pendulum near the stable equilibrium point.

In order to elucidate the contribution of the quantum spontaneous emission as well as the pendulum analogy, we find the equation satisfied by the quantity $\alpha(t) = [\bar{n}(t)]^{1/2}$. This quantity is clearly the analog of the classical amplitude of the field. Using Eq. (58) and differentiating with respect to time, we have

$$(\dot{\alpha})^2 = (r - M + 1 + \alpha^2)(r + M - \alpha^2). \qquad (63)$$

We know by definition that $\alpha^2 + \bar{R}_3 = M$. Hence Eq. (63) can be written

$$(\dot{\alpha})^2 = (r^2 - \bar{R}_3^2) + (r + \bar{R}_3). \qquad (64)$$

We compare this equation with the classical equation (13). If one identifies r_3 with $\bar{R}_3$ and the cooperation number r with the total length of the Bloch vector J, the first term of Eq. (64) coincides with the second member of Eq. (13); therefore the quadratic terms of Eq. (64), which give rise to the superradiant emission, are of classical origin.

However, the second term in Eq. (64) has no counterpart in the classical equation (13). This term, which coincides with the effective population of the upper level $\bar{n}_2$, is the quantum spontaneous emission source. This source is essential to set the system into motion when initially no transverse polarization is present, i.e., $\bar{R}_3(0) = M = r$, as is the case when the system has been excited incoherently.

On the other hand, the quantum term $r + \bar{R}_3$ becomes completely unessential when the system is already prepared in the superradiant region $\bar{R}_3 \ll r$

or when the system, starting from $\bar{R}_3 = r$, reaches this region by spontaneous radiation. Mathematically speaking, the structure of Eq. (64) is essentially different from Eq. (13) because the only equilibrium point of Eq. (64) is the stable equilibrium point $\bar{R}_3 = -r$.

Our result can be immediately translated to the trilinear boson interacting using the effective population notation given by Eq. (7) and the simple constant-of-motion relations

$$\bar{n}_1 - \bar{n} = \bar{n}_1(0), \quad \bar{n}_2 + \bar{n} = \bar{n}_2(0).$$

In particular Eq. (64) takes the simple form

$$\left(\frac{d\sqrt{\bar{n}}}{dt}\right)^2 = \bar{n}_1\bar{n}_2 + \bar{n}_2. \tag{65}$$

We see that the ratio between the classical source $\bar{n}_1\bar{n}_2$ and the quantum source $\bar{n}_2$ is equal to $\bar{n}_1$. This means that the classical theory is able to describe the parametric effect only if a substantial number of excitations are already present in the mode 1.

One may ask whether the quantum equation (64) does or does not preserve the pendulum analogy. To answer such a question, we define in analogy with Eq. (37) the mean angle $\bar{\theta}$ as

$$\bar{n} = (r + M)\sin^2\bar{\theta}(t). \tag{66}$$

Using Eq. (64) we get

$$\left(\frac{d\bar{\theta}(t)}{dt}\right)^2 = (r - M + 1) + (r + M)\sin^2\bar{\theta}. \tag{67}$$

By differentiating both terms with respect to time, we obtain

$$2\ddot{\bar{\theta}} = (r + M)\sin 2\bar{\theta}. \tag{68}$$

Hence, the system is equivalent to a rigid pendulum whose displacement from the unstable equilibrium point is $2\bar{\theta}$.

However, from Eq. (67) we see that this pendulum has a minimum kinetic energy $r - M + 1$, and because of this intrinsic energy, the quantum motion does not have an unstable equilibrium position. We note that the pendulum angle $\bar{\theta}$ defined by Eq. (66) is generally different from the angle φ described by the Bloch vector (14). In fact, using the relation $\bar{R}_3 + \bar{m} = M$ one sees that φ and $\bar{\theta}$ are related by the equation

$$\cos\tfrac{1}{2}\varphi = [(M + r)/2r]^{1/2}\cos\bar{\theta}. \tag{69}$$

This relation reduces simply to $2\bar{\theta} = \varphi$ if $r = M$. In general, the equation for φ, using Eqs. (68) and (69), is quite complicated:

$$(\dot{\varphi})^2 = 4r\left(\frac{M}{r} - \cos\varphi\right)\left(1 + \frac{1}{2r\sin^2\tfrac{1}{2}\varphi}\right). \tag{70}$$

We now compare this equation with the first-order equation that one can easily derive, integrating Eq. (16) with $\varphi(0) = 0$. Observing that $\tfrac{1}{2}M$ is the initial value of $\cos\varphi$, we see that the first term in large parentheses is the classical term, whereas the second term is purely quantum mechanical. The system behaves classically when the Bloch angle is such that

$$2r\sin^2\tfrac{1}{2}\varphi \equiv \bar{n}_1 \gg 1. \tag{71}$$

Hence the range of φ for which the quantum spontaneous emission is important is very small if r is macroscopically large, but it is not at all small if r is of the order of a few hundred. This may serve as an explanation of the disagreement of our solution (macroscopic r) with the various computer computations.

Despite the fact that the quantum spontaneous emission affects the mean photon number only at the onset of the pulses when $r = M$ and becomes practically unessential as soon as φ satisfies Eq. (71), its effects on the photon statistics are much more dramatic.

In fact, in the case $M \cong r$ we found a dispersion law which at any time is practically of a Bose type: $\sigma(\bar{n}) \simeq \bar{n}$, whereas for the superradiant states $M \ll r$, we have found

$$\sigma(\bar{n}) = [\bar{n}(1 - \bar{n}^2/r^2)]^{1/2}. \tag{72}$$

This dispersion within a factor $[1 + \bar{n}/r]^{1/2}$ coincides with that of binomial distribution. However, for times t not too close to $(2m + 1)T$, we have $\sigma(\bar{n}) \simeq \sqrt{\bar{n}}$, which is the characteristic dispersion of a coherent field.[13]

We conclude by emphasizing that the chaotic and coherent character of the short-time photon distributions is preserved by the dynamics of our system and therefore that the different photon statistics in the case $M = r$ and $M \ll r$ are due to the different nature of the initial state.

The state $M = r$ is simply a state in which some energy has been stored in the system. No phase relation exists between the dipoles of each atom, so the transverse macroscopic dipole $r_1^2 = r^2 - M^2$ is zero. This system starts radiating incoherently by a purely quantum-mechanical spontaneous emission.

In the case $M \ll r$, the system has been prepared in a correlated state with a large transverse polarization. This polarization acts as a classical source of spontaneous emission much larger than the quantum source $r + M$ and therefore the field emitted has coherent or quasiclassical properties, as we expect for a field generated by a "classical current."

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Radiative Decay of Coupled Atomic States

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The radiative decay of an atom with two excited states coupled by an external perturbation is investigated. The differential equations of motion are Fourier transformed and the probability amplitudes are obtained by contour integration. The real parts of the poles in the complex plane are the perturbed energies of the excited states, and the imaginary parts yield the decay characteristics. The decay probabilities of the excited states contain three different decay terms; two exponential decays and one modulated exponential decay. The probabilities of the final states give the frequency distribution of the emitted photons as a function of time. In an Appendix, the Heitler-Ma formalism is used to eliminate the final states of the system, and the resulting equations which contain damping terms are compared with the phenomenological method.

I. INTRODUCTION

When Weisskopf and Wigner[1] considered the radiative decay of multilevel coupled atomic systems, they showed that for a certain class of decays it is possible to simplify the set of differential equations of motion for the system by eliminating the final states. This procedure yields equations for the decaying states only, with coupling to the final state accounted for by the inclusion of damping terms. Other authors[2–5] have extended this method to cases of two or more excited states which are coupled by external perturbations, and which decay via several channels to a common ground state despite the fact that the original derivation excluded such situations.[6]

A formalism developed by Heitler and Ma[7,8] can