Quantum Statistics of One-Photon Interaction of Light with Matter

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The time development of the coherence properties of a beam of light interacting linearly with a gas of two-level atoms maintained at fixed population levels is determined. For initially coherent light, the results obtained differ from those of a previous theory.

In a recent letter,[1] Chandra and Prakash considered the time development of the density matrix for a beam of light in resonant interaction with a collection of identical two-level atoms. They derive the time dependence of the second moment of the photon number distribution, and use the result to draw conclusions about the changes in statistical and coherence properties of the light. Their theory provides the constant and linear terms in the time development.

In the present paper we treat essentially the same problem as Chandra and Prakash, but we obtain solutions correct to all orders in the time t. The more complete solutions lead to entirely different conclusions about the effect of one-photon interactions on the statistical properties of an initially coherent beam.

The Hamiltonian for the system of photons in interaction with a two-level atom is

$$H=\hbar\omega(a^\dagger a+\sigma^\dagger\sigma)+\hbar g(\sigma^\dagger a+a^\dagger\sigma), \tag{1}$$

where ω is the common frequency of the light and the atomic transition, g is the usual atom-radiation interaction, $a^\dagger$ and a are photon creation and destruction operators, and $\sigma^\dagger$ and σ are operators which excite and deexcite the atom.

We work in the number representation, using states $|n\rangle$ for the photon field. Let ρ_n be the corresponding diagonal element of the photon-density matrix; only the diagonal elements are required to evaluate the moments of the photon distribution. The equation of motion for ρ_n is[2,3]

$$d\rho_n/dt = -N_2G(n+1)\rho_n + N_2Gn\rho_{n-1} - N_1Gn\rho_n + N_1G(n+1)\rho_{n+1}\ , \tag{2}$$

where $G=4g^2/\Gamma$. (3)

Here N_1 and N_2 are the populations of the ground and excited states of the atom and Γ is the ordinary linewidth of the atomic transition. The equation of motion can be derived by standard density-matrix techniques[2,3]; it has a very simple significance, the terms on the right representing the effect of transitions on the occupancy of the state $|n\rangle$ of the photon field. We assume that N_1 and N_2 are fixed quantities.

The first and second moments of the photon distribution are defined by

$$\langle n\rangle=\textstyle\sum_n n\rho_n\ , \tag{4}$$

$$\langle n^2\rangle=\textstyle\sum_n n^2\rho_n\ . \tag{5}$$

Simple differential equations for $\langle n\rangle$ or $\langle n^2\rangle$ can be obtained by multiplying both sides of Eq. (2) by n or n^2 and summing over n. The solution for $\langle n\rangle$ is

$$\langle n\rangle=\{[\langle n\rangle_0(N_2-N_1)+N_2]e^{(N_2-N_1)Gt} - N_2\}/(N_2-N_1)\ , \tag{6}$$

where the zero subscript refers to the value at $t=0$. It is convenient to give the second-moment result in terms of the combination

$$\langle n^2\rangle-2\langle n\rangle^2-\langle n\rangle = (\langle n^2\rangle_0-2\langle n\rangle_0^2-\langle n\rangle_0)e^{2(N_2-N_1)Gt}\ . \tag{7}$$

Results equivalent to these were first derived many years ago.[4]

The first-order coherence properties of light can be inferred from a knowledge of the first and second moments of the photon distribution.[1] For chaotic, or incoherent, light

$$\langle n^2\rangle-2\langle n\rangle^2-\langle n\rangle=0, \tag{8}$$

while for coherent light,

$$\langle n^2\rangle-\langle n\rangle^2-\langle n\rangle=0. \tag{9}$$

If the photon distribution is initially incoherent, or if there are no photons present initially, Eq. (7) shows that the distribution is incoherent at all subsequent times. The terms linear in t derived from Eqs. (6) and (7) agree with the results of Chandra and Prakash (note a missing minus sign in the final exponent of Eq. (12) of Ref. 1).

Now consider an initially coherent photon distribution. It is convenient to define the quantities

$$\langle n\rangle_c=\langle n\rangle_0e^{(N_2-N_1)Gt}\ , \tag{10}$$

$$\langle n\rangle_i=N_2[e^{(N_2-N_1)Gt}-1]/(N_2-N_1), \tag{11}$$

in terms of which Eqs. (6) and (7) can be rewritten

$$\langle n\rangle = \langle n\rangle_c + \langle n\rangle_i \,, \tag{12}$$

$$\langle n^2\rangle - \langle n\rangle = \langle n\rangle_c^2 + 4\langle n\rangle_c\langle n\rangle_i + 2\langle n\rangle_i^2 \,, \tag{13}$$

where the condition (9) has been assumed to hold at $t = 0$. Results (12) and (13) are those which hold[5,6] for the statistical properties of a mixture of coherent light of mean photon number $\langle n\rangle_c$ and incoherent light of mean photon number $\langle n\rangle_i$. Thus the initially coherent light is amplified or attenuated in accordance with Eq. (10), but remains coherent; the amplification process generates incoherent light, of strength specified by Eq. (11), which is added to the coherent light.[7]

For the case of absorption, where $N_1 > N_2$, the coherent light is ultimately removed and only an incoherent contribution remains at sufficiently large times t. For atomic population inversion, where $N_2 > N_1$, both $\langle n\rangle_c$ and $\langle n\rangle_i$ grow exponentially with time. The statistics of the light at large times are seen from Eqs. (10) and (11) to depend on the relative magnitudes of $\langle n\rangle_0 (N_2 - N_1)$ and N_2. For a feeble initial excitation of coherent light, where $\langle n\rangle_0 (N_2 - N_1) \ll N_2$, the spontaneous emission of photons dominates the net result of the absorption and stimulated emission processes and leads at large times to a predominance of incoherent light. On the other hand, for a strong initial excitation of coherent light, where $\langle n\rangle_0 (N_2 - N_1) \gg N_2$, the stimulated emission process dominates and leads to predominantly coherent light at large times.

This result is not contained in the small-t behavior given by Chandra and Prakash; their predictions of the statistical properties of amplified light differ from ours. The conclusions of the present work confirm the accepted view[8] that stimulated emission maintains the coherence properties of the stimulating light, whereas spontaneous emission generates incoherent light.

The above theory does not explain the operation of a laser oscillator, since coherent light can be obtained only if a sufficiently large amount of coherent light is initially supplied to the system. The full quantum-mechanical theory of the laser[9] requires a consideration of saturation effects in the rate of photon emission achievable for a given atomic pumping rate, and of the loss of photons from the laser cavity. These factors produce a more complicated version of Eq. (2) which is not easily amenable to the type of exact solution carried out here for the simpler system.

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