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Relativistic Correction to the Equilibrium Statistical Mechanics of a Dense Electron Gas*

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The grand partition function for a degenerate electron gas with relativistic corrections of order $(v/c)^2$ in ring approximation is given. The technique used is the method of Montroll and Ward; in this problem independent field degrees of freedom are not required. The correlation energy, which represents a generalization of the Gell-Mann–Brueckner result, is obtained. Comparison with the field-theoretic results of Akhiezer and Peletminskii is made, with the same results. However, in view of the absence of independent field degrees of freedom in our treatment, renormalization is not required here (to obtain their result Akhiezer and Peletminskii must resort to charge, mass, and vacuum renormalization).

I. INTRODUCTION

There is considerable interest at present in the properties of charged-particle systems in a regime where relativistic effects must be examined. Some of this interest is generated by possible applications to fusion problems and some by astrophysical situations. This paper deals with the equilibrium statistical mechanics of a dense degenerate electron gas; the main application is therefore concerned with problems of the latter type, although there are also implications for hot classical laboratory plasmas (as will be noted).

Apart from the above, there is some bearing on the fundamental problem of a basis for a relativistic statistical mechanics. As is known, relativistic theories of interacting particles, whether based on field theory or not, face serious difficulty of consistency and interpretation. On the one hand, direct interaction theories[1–3] lead to difficulties which have been noted elsewhere.[4,5] On the other hand, field theories are beset by renormalization problems.

Thus, as has been the rationale for previous investigations of the interacting problem, we assume the relativistic $(v/c)^2$ approximation. This work has dealt with classical equilibrium[6–8] and nonequilibrium[9,10] cases, as well as the derivation of a quantum-kinetic equation.[11] To this approximation a Hamiltonian exists[12] and the statistical mechanics of the interacting system is well defined.

The quantum-statistical treatment contained herein has something to add, we feel, toward the presumably more rigorous problems in which exact, rather than approximate, Lorentz invariance is required. Thus part of the paper is devoted to a comparison between a field-theoretic formulation,[13] with its mass, charge, and vacuum renormalizations, and the present renormalization-free procedure.

In regard to astrophysical application, we remark that the $(v/c)^2$ approximation implies that the Fermi momentum p_F of the degenerate gas is less than mc. This restriction confines one to densities less than $O(10^{29})\ \text{cm}^{-3}$. Corresponding temperatures may be of $O(10^8)\ {}^\circ\text{K}$ and still be below the Fermi temperature T_F. Such conditions prevail in white dwarf stars.[14]

The technique used is based on the generalized cluster integral theory of Montroll and Ward[15] (MW); in this relativistic approximation the interactions are similarly treated as instantaneous [in $(v/c)^2$ approximation, effect of noninstanteous interactions is not felt]. Positive energy states only are considered in the interacting problem.

The ring approximation using the Darwin Hamiltonian is made, and the corresponding thermodynamic quantities are found. Included here is the $(v/c)^2$ generalization of the Gell-Mann–Breuckner (GMB) correlation energy.[16] The comparison with the GMB result is made without separate consideration of exchange effects (these will be considered elsewhere).

II. RING CONTRIBUTION TO GRAND PARTITION FUNCTION

A Feynman-type perturbation series is used by MW to obtain the grand partition function. In the present problem the interaction operator H_I, which will be used in the perturbation expansion, is the Darwin interaction given by[17]

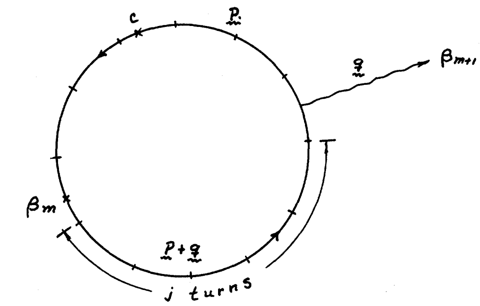

FIG. 1. Typical n-toron diagram.

$$H_I=\sum_{i<j}(e^2/r_{ij})[1-(\overleftrightarrow{\Pi}+\vec{r}_{ij}\vec{r}_{ij}/r_{ij}^2):\vec{p}_i\vec{p}_j/2(mc)^2]. \tag{2.1}$$

Spin-dependent interactions will be neglected in the present theory; these will be generally smaller at the densities considered here.

The $(v/c)^2$ corrections to the ring integral calculations of MW are now given. Figure 1 depicts an n-toron (the windings have been deformed into a circle of circumference $n\beta$). The n-toron is created at c and circulates until it picks up a quantum $\vec{q}$ at reciprocal temperature β_m. It then completes this cycle and moves j complete turns, whence during the next interval, it discharges the quantum at β_{m+1}. Finishing the discharge cycle, the toron returns to point c and is annihilated. Of its total interval of existence $n\beta$, it spends $j\beta+(\beta_{m+1}-\beta_m)$ carrying the quantum $\vec{q}$.

The momentum-space propagator corresponding to the n-toron and emitted quantum may be written as

$$(2\pi\hbar)^{-3/2}\exp\{-\epsilon_p[(n-j)\beta-\alpha_m]\}\hbar^{-3/2}u(q_m,p_m,p_{m+1})\times(2\pi\hbar)^{-3/2}\exp[-\epsilon_{p+q}(j\beta+\alpha_m)], \tag{2.2}$$

where $\alpha_m\equiv|\beta_{m+1}-\beta_m|$. The exponential terms occurring in (2.2) represent the momentum-space representation of the free-particle propagators, and ϵ_p and ϵ_{p+q} are the relativistic kinetic energies of the particle with momentum p and $p+q$, respectively.

$u(q_m,p_m,p_{m+1})$ is the Fourier transform of the interaction energy and is expressed as

$$\hbar^{-3/2}u(q_m,p_m,p_{m+1})=(2/\pi)^{1/2}(e^2\hbar^2/q_m^2)\times[1-\vec{p}_m\vec{p}_{m+1}:(\overleftrightarrow{\Pi}-\vec{q}_m\vec{q}_m/q_m^2)]. \tag{2.3}$$

If θ and ϕ represent the polar and azimuthal momentum angles, (2.3) may also be written as

$$\hbar^{-3/2}u=(2/\pi)^{1/2}(e^2\hbar^2/q_m^2)[1-p_mp_{m+1}\sin\theta_m\sin\theta_{m+1}\times\cos(\phi_m-\phi_{m+1})]. \tag{2.4}$$

The momentum-space propagator of torons of all orders is computed in the same manner as in the original MW paper. Taking the inverse Fourier transform of the result and integrating over p then gives the propagator in configuration space of all possible diagrams resulting from Fig. 1:

$$F(R_{m+1}-R_m;\ \beta_{m+1}-\beta_m)=(2\pi\hbar^2)^{-3/2}\sum_{n=1}^{\infty}(-1)^{n+1}z^n\sum_{j=0}^{n-1}\int d^3p_m\exp\{-\epsilon_p[(n-j)\beta-\alpha_m]\}u(q_m,p_m,p_{m+1})$$
$$\times\exp[-\epsilon_{p+q}(j\beta+\alpha_m)]\exp[-iq\cdot(R_{m+1}-R_m)/\hbar]d^3q_m. \quad (2.5)$$

A ring of N n-torons is constructed from the basic diagram shown in Fig. 1. A cut exists between the Nth and first torons, and after the calculations are complete, one sets $N+1=1$ to close the chain. The total propagator for the ring is

$$F_N(R_{N+1}-R_1;\beta_{N+1}-\beta_1)=\int_0^{\beta}\cdots\int_0^{\beta}\prod_{m=1}^{N}F(R_{m+1}-R_m,\ \beta_{m+1}-\beta_m)\,d\beta_2\cdots d\beta_N\,d^3R_2\cdots d^3R_N. \quad (2.6)$$

Substituting (2.5) into (2.6) and integrating over the R variables and then the variables q_1 to q_{N-1}, one finds that

$$q_1=q_2=\cdots q_N=q.$$

The product of the interaction transforms which results from substituting (2.5) into (2.6) may be simplified by employing the azimuthal ϕ_p integrations. One finds that

$$\int d^3p_1\cdots d^3p_N\,u(p_1p_2q)\cdots u(p_Np_1q)=\int d^3p_1\cdots d^3p_N\{1-2(-1)^N[p_1^2\sin^2\theta_1/2(mc)^2]\times\cdots\times[p_N^2\sin^2\theta_N/2(mc)^2]\}. \quad (2.7)$$

The β integrations in (2.6) are facilitated by defining the functions

$$G_s(\alpha_m)=(2\pi\hbar^2)^{-3/2}\sum_{n=1}^{\infty}(-1)^{n+1}z^n\sum_{j=0}^{n-1}\int d^3p_m\ \exp\{-\epsilon_p[(n-j)\beta-\alpha_m]\}\exp[-\epsilon_{p+q}(j\beta+\alpha_m)], \quad (2.8)$$

$$G_R(\alpha_m)=G_s(\alpha_m)p_m^2\sin^2\theta_m/2(mc)^2. \quad (2.9)$$

Using (2.7)-(2.9), one finds that (2.6) reduces to

$$F_N(R_{N+1}-R_1;\ \beta_{N+1}-\beta_1)=(2\pi\hbar)^{-3}\int d^3q\,(A/q^2)^N\exp[-iq\cdot(R_{N+1}-R_1)\hbar^{-1}]\int_0^{\beta}\cdots\int_0^{\beta}G_S(\alpha_1)G_S(\alpha_2)\times\cdots$$
$$\times G_S(\alpha_{N-1})G_S(\alpha_N)\,d\beta_2\cdots d\beta_N+2(-1)^N(2\pi\hbar)^{-3}\int d^3q(A/q^2)^N\exp[-iq\cdot(R_{N+1}-R_1)\hbar^{-1}]$$
$$\times\int_0^{\beta}\cdots\int_0^{\beta}G_R(\alpha_1)G_R(\alpha_2)\times\cdots\times G_R(\alpha_{N-1})G_R(\alpha_N)\,d\beta_2\cdots d\beta_N, \quad (2.10)$$

where

$$A\equiv e^2\hbar^2(2/\pi)^{1/2}.$$

The β integrations of the iterated kernels in (2.10) are done in an analogous manner to that of the MW paper, i.e., by means of an integral equation technique. Choosing a set of characteristic functions $\{\psi_j^{(S)}\}$, $\{\psi_j^{(R)}\}$ and associated eignevalues $\{\lambda_j^{(S)}\}$, $\{\lambda_j^{(R)}\}$, one then finds, after setting $R_{N+1}=R_1$ and $\beta_{N+1}=\beta_1$, that

$$\int_0^{\beta}F_N(0,0)\,d^3R_1\,d\beta_1=V(2\pi\hbar)^{-3}\sum_j\int d^3q$$
$$\times[(A\lambda_j^{(S)}/q^2)^N+2(-1)^N(A\lambda_j^{(R)}/q^2)^N]. \quad (2.11)$$

The statistical weight necessary in (2.11) is the same as that used in the nonrelativistic problem, i.e.,

$$\tfrac{1}{2}2^N(-1)^N(N-1)!/N!\ .$$

Combining these statistical factors and summing over all N gives the contribution to the free-particle grand partition function from the ring integrals:

$$\ln Z_G=\ln Z_G^{(0)}+V(2\pi\hbar)^{-3}2^{-1}\sum_j\int d^3q$$
$$\times[2A\lambda_j^{(S)}/q^2-\ln(1+2A\lambda_j^{(S)}/q^2)]$$
$$-V(2\pi\hbar)^{-3}\sum_j\int d^3q\,[2A\lambda_j^{(R)}/q^2$$
$$+\ln(1-2A\lambda_j^{(R)}/q^2)]. \quad (2.12)$$

The $N=1$ term gives rise to self-interaction (exchange) effects which are not considered in the present paper. The direct-interaction term for $N=1$ vanishes for an assumed electrically neutral system. In the nonrelativistic limit, G_R (and hence $\lambda_j^{(R)}$) is 0 and (2.12) reduces to the expression obtained by MW.

III. CALCULATION OF $\lambda_j^{(S)}$ AND $\lambda_j^{(R)}$

The periodicity requirements of G_S and G_R lead, as in the nonrelativistic problem, to the following expression for $\lambda_j^{(S)}$ and $\lambda_j^{(R)}$:

$$\lambda_j^{(S,R)}=\int_0^\beta G_{S,R}(\alpha)\,e^{2\pi ij\alpha/\beta}\,d\alpha. \tag{3.1}$$

The notation S, R means that $\lambda_j^{(S)}$ ($\lambda_j^{(R)}$) results when G_S (G_R) is chosen.

Defining $\Delta\epsilon=\epsilon_p-\epsilon_{p+q}$, one may rewrite (2.8) and (2.9) in the following way:

$$G_S=(2\pi\hbar^2)^{-3/2}\sum_{n=1}^{\infty}(-1)^{n+1}z^n\sum_{j=0}^{n-1}\int d^3p\,e^{-\epsilon_p n\beta}\,e^{j\beta\Delta\epsilon}\times e^{\alpha\Delta\epsilon}, \tag{3.2}$$

$$G_R=G_S(p^2\sin^2\theta)/2(mc)^2. \tag{3.3}$$

Summing directly over j and then over n in (3.2) and (3.3) leaves

$$G_S=(2\pi\hbar^2)^{-3/2}(n_{p+q}-n_p)/(e^{\beta\Delta\epsilon}-1), \tag{3.4}$$

where $n_p=1/(1+z^{-1}e^{\beta\epsilon_p})$ is the occupation number of electrons with momentum p. Performing the α integration in (2.1) yields

$$\lambda_j^{(S)}=(2\pi\hbar^2)^{-3/2}\int d^3p(n_{p+q}-n_p)\Delta\epsilon/[(\Delta\epsilon)^2+(2\pi j/\beta)^2], \tag{3.5}$$

$$\lambda_j^{(R)}=(2\pi\hbar^2)^{-3/2}\int d^3p(p^2\sin^2\theta/2m^2c^2)(n_{p+q}-n_p)\times\Delta\epsilon/[(\Delta\epsilon)^2+(2\pi j/\beta)^2]. \tag{3.6}$$

In the degenerate limit, the nonzero contribution from $\lambda_j^{(S)}$ and $\lambda_j^{(R)}$ occurs if $n_{p+q}=0$ and $n_p=1$. By writing the energies out explicitly in the expression for n_p and n_{p+q}, the nonzero contributions to (3.5) and (3.6) are found to be satisfied if

$$|\vec{p}+\vec{q}|>p_0,\quad p<p_0 \tag{3.7}$$

where p_0 is defined through the fugacity as

$$z=\exp[\beta(p_0^2c^2+m^2c^4)^{1/2}].$$

Further reductions on the integrand in (3.5) and (3.6) are found by considering the small-q limit. Under these circumstances an expansion of ϵ_{p+q} in a Taylor series about small q leads to

$$\Delta\epsilon\cong-\frac{\vec{p}\cdot\vec{q}}{m[1+(p/mc)^2]^{1/2}}+O(q^2). \tag{3.8}$$

If one substitutes (3.8) into (3.6) and (3.5), and defines the quantities

$$\mathcal{P}\equiv|\vec{p}|/p_0,\quad \mathcal{Q}\equiv|\vec{q}|/p_0,$$

$$x\equiv\cos(\mathcal{P},\mathcal{Q})=\cos\theta,$$

$$u\equiv 2\pi j/\beta p_0 q=2\pi j/\beta p_0^2\mathcal{Q},\quad a\equiv p_0/mc,$$

then to order $a^2=p_0^2/m^2c^2$, $\lambda_j^{(S)}$ and $\lambda_j^{(R)}$ may be written as[18]

$$\lambda_j^{(S)}\cong(2\pi\hbar^2)^{-3/2}\,2\pi mp_0\int_0^1dx\int_{1-qx}^1 d\mathcal{P}\,\mathcal{P}^2\times\{\mathcal{P}x/\mathcal{Q}(\mathcal{P}^2x^2+u^2)[1+a^2\mathcal{P}^2(\tfrac12-u^2/\mathcal{P}^2x^2+u^2)]\}, \tag{3.9}$$

$$\lambda_j^{(R)}\cong(2\pi\hbar^2)^{-3/2}2\pi mp_0a^2\int_0^1dx\,(1-x^2)\times\int_{1-qx}^1 d\mathcal{P}\,\mathcal{P}^4[\mathcal{P}x/\mathcal{Q}(\mathcal{P}^2x^2+u^2)]. \tag{3.10}$$

The limits on the $\mathcal{P}$ integration follow directly from (3.7) in the small-$\mathcal{Q}$ limit.[16] The indefinite $\mathcal{P}$ integrations are tabulated in Ref. 19 so, after evaluating the result at the limits, taking care only to include terms of $O(\mathcal{Q})$, and integrating over x, one finds

$$\lambda_j^{(S)}=2(2\pi)^{-1/2}\hbar^{-3}mp_0\{R(u)+a^2[R(u)-1/2(1+u^2)]\}, \tag{3.11}$$

$$\lambda_j^{(R)}=2(2\pi)^{-1/2}\hbar^{-3}mp_0a^2[(1+u^2)R(u)-\tfrac13], \tag{3.12}$$

where $R(u)=1-u\tan^{-1}u^{-1}$.

It is to be noted, also, that in the limit of $\beta\to\infty$ the parameter $u=2\pi j\,m/\beta\,\mathcal{Q}p_0^2$ becomes a continuous variable with $du=(2\pi m/\beta\,\mathcal{Q}p_0^2)dj$, so that the sum over j appearing in (2.12) may be replaced by $(\beta\,\mathcal{Q}p_0^2/2\pi m)\times\int_{-\infty}^{\infty}du$.

IV. EVALUATION OF GRAND PARTITION FUNCTION

Substituting (3.11) and (3.12) into (2.12) and defining

$$\alpha_S\equiv(4me^2/\pi\hbar p_0)\{R(u)+a^2[R(u)-1/2(1+u^2)]\}, \tag{4.1}$$

$$\alpha_R\equiv(4me^2a^2/\pi\hbar p_0)[R(u)(1+u^2)-\tfrac13], \tag{4.2}$$

one finds that

$$\ln Z_G-\ln Z_G^{(0)}=V(2\pi\hbar)^{-3}\beta m^{-1}p_0^5\int_{-\infty}^{\infty}du\int_0^{\infty}d\mathcal{Q}\,\mathcal{Q}^3\times[\alpha_S\mathcal{Q}^{-2}-\ln(1+\alpha_S\mathcal{Q}^{-2})]-2V(2\pi\hbar)^{-3}\beta m^{-1}p_0^5\times\int_{-\infty}^{\infty}du\int_0^{\infty}d\mathcal{Q}\,\mathcal{Q}^3[\alpha_R\mathcal{Q}^{-2}+\ln(1-\alpha_R\mathcal{Q}^{-2})]. \tag{4.3}$$

The quantities α_S and α_R are greater than 0 for all values of u provided that $0<\tan^{-1}u^{-1}<\pi$. The $\mathcal{Q}$ integrations diverge at the upper limit; but a cutoff at $\mathcal{Q}=1$ is justified.[16]

Evaluation of the $\mathcal{Q}$ integrals in (4.3) will be done in the high-density limit ($\rho\to\infty$). MW have shown that in the ground state ($\beta\to\infty$) the appropriate dimensionless quantity describing thermodynamic properties is $me^2/\hbar^2\rho^{1/2}$, so that expansions performed in e^2 are equivalent to expansions in inverse powers of density. Following the GMB theory, the integrals in (4.3) are evaluated to order e^4 (i.e., $1/\rho$) which, in view of definitions (4.1) and (4.2), is equivalent to $O(\alpha^2_{S,R})$.

Solving the α_S integral in (4.3), to order e^4, yields

$$\tfrac{1}{4}\alpha_S^2(\tfrac{1}{2} - \ln\alpha_S).$$

The α_R integral can be computed by an appropriate integration by parts and then by employing the Cauchy principal part on the result (see Appendix A). One finds that

$$\int_0^1 d\,\mathcal{Q}\,\mathcal{Q}^3[\alpha_R\,\mathcal{Q}^{-2} + \ln(1-\alpha_R\mathcal{Q}^{-2})] \cong \tfrac{1}{4}\alpha_R^2(\tfrac{1}{2} - \ln\alpha_R).$$

From (4.2) it is observed that $\alpha_R \propto (p_0/mc)^2$. Therefore, in the high-density limit, the correction to $\ln Z_G$ from the Darwin interaction is of order $(v/c)^4$ and may be neglected in the $(v/c)^2$ approximation. This is consistent with the classical result which can also be interpreted as contributing terms of $O(v/c)^4$ from the Darwin interaction.[20]

Before evaluating the u integrals the quantity $\frac{1}{4}\alpha_S^2(\frac{1}{2} - \ln\alpha_S)$ is expanded to order a^2. Using the definition of α_S leads to

$$\tfrac{1}{4}\alpha_S^2(\tfrac{1}{2} - \ln\alpha_S) \cong [\tfrac{1}{4}\gamma^2R^2(1+2a^2)](\tfrac{1}{2} - \ln\gamma R) + \tfrac{1}{4}a^2\gamma^2[R\ln\gamma R/(1+u^2) - R^2], \qquad (4.4)$$

where

$$\gamma \cong 4me^2/p_0\pi\hbar. \qquad (4.5)$$

All the quantities of $O(a^2)$ are not grouped together because, in its present form, the integration of the first expression on the right-hand side of (4.4) has been evaluated already by GMB. Using their result and evaluating the remaining terms in (4.4) (see Appendix B for details of the u integrations), one finds that the grand partition function becomes

$$\Delta\ln Z_G = -V(2\pi\hbar)^{-3}\beta m^{-1}p_0^5[\gamma^2K(\ln\gamma + C - \tfrac{1}{2}) + \gamma^2Ka^2(\tfrac{1}{2}\ln\gamma - 0.026)], \qquad (4.6)$$

where

$$K \equiv \pi\tfrac{1}{6}(1-\ln 2), \quad C \equiv -0.551.$$

Equation (4.6) is valid for low-momentum transfer $(q\to 0)$ at ground state $(\beta\to\infty)$ and high-density conditions $(\rho\to\infty)$ and in the limit as $a^2 \ll 0$.

V. CORRELATION ENERGY AND ΔP_{ing}

The procedure to find the relativistic correction to the correlation energy for a system of fermions proceeds in a manner similar to the nonrelativistic calculations of MW. The energy and density of the system may be found from the grand partition function from the following equations:

$$E = \frac{-\partial(\ln Z_G)}{\partial\beta}, \quad \rho = \frac{\partial(\ln Z_G)}{V\,\partial\alpha}, \qquad (5.1)$$

where

$$\mu = \text{chemical potential} = (p_0^2c^2 + m^2c^4)^{1/2}, \qquad (5.2)$$

$$\alpha = \ln(ze^{-\beta mc^2}) = \beta[(p_0^2c^2+m^2c^4)^{1/2} - mc^2] \equiv \beta\mu'. \qquad (5.3)$$

To order $(v/c)^2$, the quantity p_0 may be expressed in terms of α:

$$p_0 \cong (2m\alpha/\beta)^{1/2}[1 + (2m\alpha/\beta)(8m^2c^2)^{-1}]. \qquad (5.4)$$

Substituting (5.4) into (4.6), and using only orders of $(v/c)^2$, one finds that

$$\Delta\ln Z_G = \ln Z_G - \ln Z_G^{(0)} = V(2\pi\hbar)^{-3}m^{-1}\beta^{-3/2}(2m\alpha)^{5/2} \times[1 + (5/8m^2c^2)(2m\alpha/\beta)]I_T, \qquad (5.5)$$

where

$$I_T = -[\gamma^2K(\ln\gamma - 1.05) + \gamma^2K(p_0/mc)^2(\tfrac{1}{2}\ln\gamma - 0.026)]. \qquad (5.6)$$

Performing the differentiations in (5.1) and using the ground-state energy and density[14] to order $(v/c)^2$ leads to

$$E/N = [2\rho m(2\pi\hbar)^3]^{-1}\left[\tfrac{8}{5}\pi(1 - 5p_0^2/28m^2c^2) + 3p_0^5I_T + 2p_0^5\left(\beta\frac{\partial I_T}{\partial\beta} + \beta I_T\frac{\partial M}{\partial\beta}\right)\right], \qquad (5.7)$$

$$\rho = [2(2\pi\hbar)^3]^{-1}\left[(\tfrac{16}{3}\pi p_0^3) + 10p_0^3I_T(1 + p_0^2/4m^2c^2) - 2(1 + p_0^2/4m^2c^2)\left(\beta\frac{\partial I_T}{\partial\beta} + \beta I_T\frac{\partial M}{\partial\beta}\right)\right], \qquad (5.8)$$

where

$$M \equiv [1 + 5p_0^2/8m^2c^2].$$

Now, p_0 must be eliminated from both (5.7) and (5.8) and E expressed in terms of ρ. Assume that

$$p_0 = p_F + \delta p_0, \qquad (5.9)$$

where p_F is the Fermi momentum of a set of noninteracting particles and δp_0 is a small correction due to interactions and relativistic effects. From (5.8) and (5.9) it follows that

$$\delta p_0/p_F = -\tfrac{1}{8}\pi\Big[5I_T(1 + p_F^2/4m^2c^2) - 2(1 + p_F^2/4m^2c^2)$$

$$\times\left(\beta\frac{\partial I_T}{\partial\beta}+\beta I_T\frac{\partial M}{\partial\beta}\right)\bigg]. \tag{5.10}$$

The correlation energy $\epsilon_{\rm corr}$ is defined as

$$\epsilon_{\rm corr}=\Delta(E/N)=(E-E_{\rm ideal})/N.$$

Substituting $p_0=p_F\,(1+\delta p_0/p_F)$ into (5.7) and using (5.10), one finds that

$$\epsilon_{\rm corr}=-(3p_F^2/8\pi m)I_T\ ,$$

the terms involving $[\beta\,\partial I_T/\partial\beta+\beta I_T\,\partial M/\partial\beta]$ cancelling out. Defining the dimensionless quantity

$$r_s=r_0/r_B, \tag{5.11}$$

where r_B = Bohr radius and r_0 is related to the density by $\rho^{-1}=\frac{4}{3}\pi r_0^3$ and using the explicit expression for I_T, $\epsilon_{\rm corr}$ may be written in units of rydbergs/electron as

$$\epsilon_{\rm corr}=(0.062\ln r_s-0.091)+(p_F/mc)^2$$
$$\times(0.031\ln r_s\ -0.021). \tag{5.12}$$

When the density is expressed in electrons/cm^3, it follows that

$$(p_F/mc)^2=1.93\times10^{-4}r_s^{-2}\ .$$

In the nonrelativistic limit $\epsilon_{\rm corr}$ corresponds to the GMB value.[21]

Nozières and Pines[22] have shown that the GMB result is valid if $r_s\lesssim1$. This corresponds to a lower limit on the density of about 10^{24} cm^{-3}. Up to densities of 10^{28} cm^{-3} the $(v/c)^2$ approximation is quite good since, in these regions, $(p_F/mc)^2$ is reasonably small [for example, at $\rho=10^{28}$ cm^{-3}, $(p_F/mc)^2=0.065$]. One may extend the range of ρ to about 10^{29} cm^{-3} [$(p_F/mc)^2=0.3$ and $r_s=0.025$], but above this density $(p_F/mc)^2$ grows quite rapidly and the approximation breaks down.

The next thermodynamic property to be calculated from (4.6) is the $(v/c)^2$ ring contribution to the equation of state for a dense electron gas. Using the definition of I_T in (5.6) it follows that

$$\ln Z_G=\ln Z_G^{(0)}+V(2\pi\hbar)^{-3}\beta m^{-1}p_0^5I_T\ . \tag{5.13}$$

The contribution from $\ln Z_G^{(0)}$ is found by performing a $(v/c)^2$ reduction on the result of Chandrasekhar.[14]

By assuming that $p_0=p_F\,(1+\delta p_0/p_F)$ and then substituting this form of p_0 in (5.13), dropping orders of $I_T\,\delta p_0$, one finds

$$\Delta P=P-P_{\rm ideal}=[8\pi p_F^5/15m(2\pi\hbar)^3]\{5\delta p_0/p_F$$
$$\times[1-(p_F/mc)^2/2]+15I_T/8\pi\}, \tag{5.14}$$

where the ideal-gas contribution is given by[14]

$$P_{\rm ideal}=8\pi p_F^5/15(2\pi\hbar)^3m(1-5p_F^2/14m^2c^2). \tag{5.15}$$

Performing the differentiations in (5.10) and collecting quantities of like order in $1/m^2c^2$, one finds that

$$\delta p_0/p_F=(1/8\pi)\,[3\gamma^2K(\ln\gamma-1.05)-\gamma^2K$$
$$+\gamma^2K(p_F^2/m^2c^2)\,(4\ln\gamma-2.70)]\ . \tag{5.16}$$

Putting (5.16) in (5.14) ultimately leads to the ring contribution to the pressure in units of dynes/cm^2:

$$\Delta P_{\rm ring}=-2.18\times10^{12}r_s^{-3}$$
$$\times\{\tfrac{1}{3}+(p_F/mc)^2[(-\tfrac{1}{3})\ln r_s+0.324]\}, \tag{5.17}$$

where r_s is given in (5.11). Note that the nonrelativistic contribution to the pressure in (5.17) is independent of terms proportional to $e^4\ln e^2$. This point has been observed by other investigators[23]; however, in Ref. 23, if one performs all the differentiations of the density, the factor of $\frac{1}{3}$ shown here in (5.19) will also occur (proportional to e^4).

VI. GRAND PARTITION FUNCTION DERIVED FROM QUANTUM FIELD THEORY

The purpose of this section is to give a comparison, to order $(v/c)^2$, of the grand partition function derived by the cluster integral method and the grand partition function calculated from quantum field theory. The latter procedure was used by Akhiezer and Peletminskii[13] to determine the thermodynamic properties of a gas of electrons, positrons, and photons with an accuracy proportional to $e^4\ln e^2$. The log of the grand partition function is defined as $-\beta\Delta\Omega_c$, and naturalized units are used ($\hbar=c=1$).

For extremely low temperatures ($\beta\ll\mu-m$) AP (Ref. 13) find that the correlation contribution to the thermodynamic potential is given by

$$-\beta\,\Delta\Omega_c=V(2\pi)^{-3}\int_{-\infty}^{\infty}dk_4\int_0^{\infty}d\omega\,\omega^2$$
$$\times\{\ln[1+4\pi m^2e^2\,\Lambda_1(k)/k^2]$$
$$+2\ln[1+4\pi m^2e^2(\Lambda(k)-\Lambda_1(k))/2k^2]$$
$$-4\pi m^2e^2\Lambda(k)/k^2\}. \tag{6.1}$$

As pointed out in Ref. 13, the main contribution in (6.1) is derived from the region of small ω and k_4 so, in analogy with the cluster integral method, the upper limit in the ω integral is chosen to be the quantity $p_0=(\mu^2-m^2)^{1/2}$.

With the change of variables $\omega=k\sin\phi$, $k_4=k\cos\phi$, (6.1) becomes[24]

$$-\beta\Delta\Omega_c=-V\beta(2\pi)^{-3}\int_0^{\pi}d\phi\,\sin^2\phi\int_0^{p_0/\sin\phi}dk\,k^3$$
$$\times\{\ln[1+4\pi m^2e^2\,\Lambda_1(0,\phi)/k^2]$$
$$+2\ln[1+4\pi m^2e^2(\Lambda(0,\phi)$$

$$- \Lambda_1(0,\phi)/2k^2)] - 4\pi m^2 e^2 \Lambda(0,\phi)/2k^2\} \quad . \qquad (6.2)$$

If one defines $k = p_0\kappa$ in (6.2) and

$$\alpha \equiv 4\pi m^2 e^2/p_0^2, \quad \Delta\Lambda \equiv \Lambda(0,\phi) - \Lambda_1(0,\phi), \qquad (6.3)$$

then the k integrations of (6.2) lead to

$$\tfrac{1}{8}(\alpha\Delta\Lambda)^2 \ln(\tfrac{1}{2}\alpha\Delta\Lambda \sin^2\phi) + \tfrac{1}{4}(\alpha\Delta\Lambda)^2 \ln(\alpha\Lambda_1 \sin^2\phi)$$

$$-\tfrac{1}{8}[\tfrac{1}{2}(\alpha\Delta\Lambda)^2 + (\alpha\Lambda_1)^2] \; . \qquad (6.4)$$

In order to put (6.2) and (6.4) in a form more suggestive of the integrations involved in the cluster integral method, the variables are changed once again. With $a = p_0/mc$, let

$$u \equiv (1+a^2)^{1/2} \cot\phi/a; \qquad (6.5)$$

$\Lambda(0,\phi)$ and $\Lambda_1(0,\phi)$ may now be written as

$$\Lambda(u) = a[R(u) + a^2]/\pi^2(1+a^2)^{1/2} \; , \qquad (6.6)$$

$$\Lambda_1(u) = aR(u)[1 + a^2(1+u^2)]/\pi^2(1+a^2)^{1/2}, \qquad (6.7)$$

where $R(u) = 1 - u\tan^{-1}u^{-1}$. Also, it is found that

$$\int_0^\pi d\phi \sin^2\phi f(\phi) \rightarrow a(1+a^2)^{3/2} \int_{-\infty}^\infty du[1 + a^2(1+u^2)]^{-2} f(u). \qquad (6.8)$$

Using relations (6.3)–(6.8) in (6.2) and using the small-a^2 approximation, one finds that

$$-\beta\Delta\Omega_c \cong - V(2\pi)^{-3}\beta m^{-1} p_0^5 \int_{-\infty}^\infty du \{\tfrac{1}{4}\gamma^2 R^2[\ln(\gamma R) - \tfrac{1}{2}] + \tfrac{1}{4}a^2\gamma^2 R^2[\tfrac{1}{2}(\ln\gamma R) + \tfrac{1}{4}]\} \; , \qquad (6.9)$$

where $\gamma \equiv \alpha a^2/\pi = 4me^2/\pi p_0$. Performing the u integrations in (6.9) leaves

$$-\beta\Delta\Omega_c = - V(2\pi)^{-3}\beta p_0^5 m^{-1} \times [\gamma^2 K(\ln\gamma + C - \tfrac{1}{2}) + a^2\gamma^2 K(\tfrac{1}{2}\ln\gamma - 0.026)] \; , \qquad (6.10)$$

which is identical to the result given in (4.6).

VII. CONCLUSIONS

The correlation energy calculation for an approximately relativistic electron gas generalized the GMB nonrelativistic calculation. The comparison with the $(v/c)^2$ approximation of the field-theoretic treatment of AP (Ref. 13) has been made, with similar results. However, it is important to emphasize that no renormalization is necessary in our treatment, since independent field degrees of freedom are not introduced.

The contribution of order $(v/c)^2$ given in (5.12) has resulted entirely from terms involving the ring contributions from the relativistic free-particle propagators and the Coulomb interaction. The momentum-dependent interaction, in agreement with classical results,[20] gives effectively no contribution to this order.

The agreement between the quantum-electrodynamic result and our own would appear to augment the contention that (to this order) a screening result for the transverse interaction such as that obtained in classical argument by Trubnikov and Kosachev[8,10] may not be valid. That is to say, if their procedures and approximations are equivalent to a ring approximation procedure, then the results to order $(v/c)^2$ for the thermodynamic quantities should be the same.

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APPENDIX A

The integral in (4.3) is readily solved by integration by parts. With $\alpha^{1/2}x \equiv \mathcal{Q}$ one finds that

$$\alpha_R^2 \int_0^{\alpha_R^{-1/2}} dx\, x^3[x^{-2} + \ln(1-x^{-2})]$$

$$= \alpha_R^2 [\tfrac{1}{2}x^2 + \tfrac{1}{4}x^2 \ln(1-x^{-2})]_0^{\alpha_R^{-1/2}}$$

$$+ \tfrac{1}{2}\alpha_R^2 \int_0^{\alpha_R^{-1/2}} [x^3/(x^2-1)]\, dx.$$

Solving the integral by means of the Cauchy principal part and evaluating the remaining term to $O(\alpha_R^2)$, one finds that the result is $(\tfrac{1}{4}\alpha_R^2)(\tfrac{1}{2} - \ln\alpha_R)$.

APPENDIX B

The u integrals in (4.4) have the following values:

$$\int_{-\infty}^\infty R(u) \ln\gamma\, du/(1+u^2) = \pi(1 - \ln 2)\ln\gamma \; ,$$

$$\int_{-\infty}^\infty [R(u)]^2 du = -\tfrac{1}{3}2\pi(1-\ln 2) \; , \qquad (B1)$$

$$\int_{-\infty}^\infty R(u) \ln R(u)\, du/(1+u^2) = -0.694 \; .$$

The last integral was integrated numerically, first by substituting $x = \tan^{-1}u^{-1}$ and then evaluating

$$2\int_0^{\pi/2} (1 - x\cot x)\ln(1 - x\cot x)\, dx \; .$$

Substituting the expressions in (B1) into (4.4) and using the GMB value leads to (4.6).

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General Theory of the van der Waals Interaction: A Model-Independent Approach*

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We study the van der Waals interaction $V_{2\gamma}^{AB}(R)$ arising from two-photon exchange between neutral spinless systems A and B. By using the analytic properties of the two-photon contribution to the scattering amplitude for $A+B\to A+B$ and of the full amplitudes for $\gamma+A\to\gamma+A$ and $\gamma+B\to\gamma+B$, we show that it is possible to express $V_{2\gamma}^{AB}(R)$ entirely in terms of measurable quantities, the elastic scattering amplitudes for photons of various frequencies ω. This approach includes relativistic corrections, higher multipoles, and retardation effects from the outset and thus avoids any v/c expansion or any direct reference to the detailed structure of the systems involved. We obtain a generalized form of the Casimir-Polder potential, which includes both electric and magnetic effects, and, correspondingly, a generalized asympotic form $V_{2\gamma}^{AB}(r)\sim -D/R^7$, where $D=[23(\alpha_E^A\alpha_E^B+\alpha_M^A\alpha_M^B)-7(\alpha_E^A\alpha_M^B+\alpha_M^A\alpha_E^B)]/4\pi$ and the α's denote static polarizabilities. In addition, we show that the potential may be written as a single integral over ω, involving products of the dynamical polarizabilities $\alpha_X(\omega)$ evaluated at *real* frequencies, in contrast to the familiar integral over imaginary frequencies; for the case of interacting atoms, the domain of applicability of the various formulas is clarified, and the problem of evaluating $V_{2\gamma}^{AB}(R)$ from present experimental information is discussed. Some simple interpolation formulas are presented, which may accurately describe $V_{2\gamma}^{AB}(R)$ in terms of a few constants.

I. INTRODUCTION

In this paper we present a theoretical description of the van der Waals interaction between two neutral spinless systems. We show that this interaction may be expressed in terms of measurable quantities that describe the interaction of the individual systems with real photons, i.e., the elastic photon scattering amplitudes. We are thereby able to avoid any reference to the detailed structure of the system, such as is involved in the conventional