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Energy of Electromagnetic Waves in the Presence of Absorption and Dispersion

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Expressions for the stored energy, energy flow, and power dissipated are derived for electromagnetic waves in terms of the complex permittivity ϵ' and permeability μ' for a frequency-dispersive absorbing medium. This is shown to be possible when ϵ' and μ' are known functions not only of frequency but also of all the loss factors – e.g., collision frequencies, etc. The derivation is not restricted to media with small losses. An oscillating particle system and an electric circuit are used as illustrative examples of the application of the energy expressions derived.

I. INTRODUCTION AND SUMMARY

Stored energy and energy flow are fundamental quantities in the analysis of wave motion. Energy expressions for electromagnetic waves in terms of the fields, and the permittivity ϵ and permeability μ are well known for the nondissipative dispersive medium,[1] but the expressions given for a dissipative medium are based on an assumption of small losses. It has also been questioned if it is possible to determine energy expressions from ϵ and μ in the dissipative medium.[2] This is due to the fact that systems with the same $\epsilon'(\omega)=\epsilon_1(\omega)+j\epsilon_2(\omega)$ and $\mu'(\omega)=\mu_1(\omega)+j\mu_2(\omega)$, where the complex ϵ' and μ' include dissipational terms and where $j=\sqrt{-1}$, may have entirely different energies. In this paper it is shown that it is possible to derive energy expressions from ϵ' and μ' in the dissipative medium when ϵ' and μ' are known functions of the loss factors λ_i such as collision frequency, etc., as well as frequency. This is in general the case when ϵ' and μ' are determined from a model medium. The advantage of deriving the energy expressions directly from ϵ' and μ' is, besides the convenience, that the obtained energy expressions are consistent with the approximations made in ϵ' and μ'. The derivation is not restricted to media with small losses. We have assumed the medium to be linear, homogeneous, and temporally dispersive. Primarily we have considered electromagnetic waves, but the derived expressions are valid for any wave motion provided it is described by one of the two wave amplitudes which are responsible for the energy flow. The expressions can also be used to determine the stored energy in systems of coupled linear oscillators, e.g., a network.

For a nondissipative and temporally dispersive system, it is well known that the average value of the stored energy is determined by

$$\langle W\rangle=\frac{1}{4}\left[\frac{\partial(\omega\epsilon)}{\partial\omega}EE^*+\frac{\partial(\omega\mu)}{\partial\omega}HH^*\right]=\langle W_E\rangle+\langle W_H\rangle\ . \tag{1}$$

A frequently used expression for the time-averaged value of the stored energy in a dissipative system is (1) with ϵ and μ replaced by ϵ_1 and μ_1, respectively. But as pointed out by Ginzburg[2] this may lead to a negative value for the stored energy, for example in a plasma, which apparently is incorrect. Ginzburg derived a conservation relation based on quasimonochromatic waves which includes terms dependent on the manner in which the electric field is established. To avoid these problems Neufeld[3] suggested a reformulation of the dielectric constant in an absorbing medium, such that ϵ is independent of the frictional parameter. The dissipational process is then taken into account by extraneous forces. This formulation is based on a behavior of dispersive media which is physically different from the one customarily used.

In the present work the stored energy, energy flow, and power dissipated are expressed in terms of ϵ' and μ' [or in the equivalent dispersion function $D(\omega, k)$]. It is demonstrated that if the dissipation can be represented by a frequency-independent conductivity we can use (1). For other cases expressions similar to (1) are given by formulating new quantities ϵ_1' and μ_1'.

The analysis will be applied to an assembly of simple oscillators which often serves as a model for a plasma or a molecular medium. The results are also applied to an electric circuit, and the necessity to know the impedence as a function of all dissipational quantities, i.e., all resistances, is demonstrated.

II. ENERGY EXPRESSIONS IN AN ABSORBING MEDIUM

An expression for the stored energy will be derived from the conservation law including power delivered by an external source.[4,5] The stored energy is expressed by means of energy flow and power dissipated. Expressions for these quantities are then obtained from ϵ' and μ'.

In order to avoid unnecessary complications we will specialize to an isotropic medium. We further assume the medium to be linear, homogeneous, and temporally dispersive. In order to derive energy expressions it is suitable to consider the problem of excitation of the wave system by a source consisting of a current $J^s(t, z)$ and to describe the response by the electrical field $E(t, z)$. The two quantities are chosen such that the product $-E(t, z) \times J^S(t, z)$ is equal to the power delivered (per volume element) by the external source. The relations between the time-space transforms of $E(t, z)$ and of $J^s(t, z)$ can be written [5]

$$D(\omega, k)E(\omega, k) = jJ^s(\omega, k) \quad . \tag{2}$$

We call D the dispersion function. D is an analytic function of frequency, propagation constant, and medium parameters. $D(\omega, k)$ is expressed in ϵ' and μ' by (A4) in Appendix A.

We now consider a wave motion characterized by the angular frequency ω and propagation constant k. $E(t, z)$ is then equal to

$$\mathrm{Re}[E_0 e^{j(\omega t - kz)}] = \mathrm{Re}(E)$$

and $J^s(t, z)$ has the same form. The relation between E and J^s is

(3)

where ω and k may be complex. In the case of free-wave propagation, $J^s = 0$, and the relation between ω and k is determined by the dispersion relation $D(\omega, k) = 0$.

The energy conservation law (Poynting's theorem) can be written

$$\frac{\partial S}{\partial z} + \frac{\partial W}{\partial t} + P_d = P_s \quad , \tag{4}$$

where S is the energy flow, W the stored energy, P_d the power dissipated, and P_s power delivered by the external source. P_s is given by

$$P_s = -\mathrm{Re}[E_0 e^{j(\omega t - kz)}]\,\mathrm{Re}[J_0^s e^{j(\omega t - kz)}]$$

$$= P_{s0} e^{-2(\omega_{\mathrm{Im}} t - k_{\mathrm{Im}} z)} + \mathrm{Re}[P_{s1} e^{j2(\omega t - kz)}] \quad , \tag{5}$$

where ω_{Im} and k_{Im} are the imaginary parts of ω and k, respectively, and

$$P_{s0} = -\tfrac{1}{2}\,\mathrm{Re}(E_0^* J_0^s) \quad , \quad P_{s1} = -\tfrac{1}{2} E_0 J_0^s \quad . \tag{6}$$

When $\omega_{\mathrm{Im}} = 0$, the time average of P_s is given by $P_{s0} e^{2k_{\mathrm{Im}} z}$, while $P_{s1} e^{2k_{\mathrm{Im}} z}$ characterizes that part which oscillates with twice the frequency of the wave; and when the phase of the signal is unknown but uniformly distributed, $P_{s0} e^{-2(\omega_{\mathrm{Im}} t - k_{\mathrm{Im}} z)}$ is the expected value and $(2)^{-1/2} P_{s1} e^{-2(\omega_{\mathrm{Im}} t - k_{\mathrm{Im}} z)}$ is the standard deviation of P_s in the point z at the time t. For S, W, and P_d we use analogous notation. Introducing these expressions into the conservation relation (4) yields

$$2k_{\mathrm{Im}} S_0 - 2\omega_{\mathrm{Im}} W_0 + P_{d0} - P_{s0} + \mathrm{Re}[(-2jkS_1 + 2j\omega W_1 + P_{d1} - P_{s1}) e^{j2(\omega_{\mathrm{Re}} t - k_{\mathrm{Re}} z)}] = 0 \quad . \tag{7}$$

This relation must be fulfilled for all values of t and z, and by expressing P_s by means of (3) and (6) we obtain

$$2k_{\mathrm{Im}} S_0 - 2\omega_{\mathrm{Im}} W_0 + P_{d0} = -\tfrac{1}{2}\,\mathrm{Im} D E_0 E_0^* \quad , \tag{8a}$$

$$2kS_1 - 2\omega W_1 + jPd_1 = -\tfrac{1}{2} D E_0^2 \quad . \tag{8b}$$

After variation of ω_{Im} and k_{Im}, (8a) yields

$$S_0 + k_{\mathrm{Im}} \frac{\partial S_0}{\partial k_{\mathrm{Im}}} - \omega_{\mathrm{Im}} \frac{\partial W_0}{\partial k_{\mathrm{Im}}} + \frac{1}{2} \frac{\partial P_{d0}}{\partial k_{\mathrm{Im}}} = -\frac{1}{4} \mathrm{Re} \frac{\partial D}{\partial k} E_0 E_0^* , \tag{9a}$$

$$W_0 + \omega_{\mathrm{Im}} \frac{\partial W_0}{\partial \omega_{\mathrm{Im}}} - k_{\mathrm{Im}} \frac{\partial S_0}{\partial \omega_{\mathrm{Im}}} - \frac{1}{2} \frac{\partial P_{d0}}{\partial \omega_{\mathrm{Im}}} = \frac{1}{4} \mathrm{Re} \frac{\partial D}{\partial \omega} E_0 E_0^* \quad . \tag{9b}$$

In the lossless case we obtain with $\omega_{\mathrm{Im}} = 0$ and $k_{\mathrm{Im}} = 0$ the time- or space-averaged expressions for the energy flow and stored energy (see Refs. 4 and 5):

$$S_0 = -\frac{1}{4} \frac{\partial D}{\partial k} E_0 E_0^* \quad , \quad W_0 = \frac{1}{4} \frac{\partial D}{\partial \omega} E_0 E_0^* \quad . \tag{10}$$

When $\omega_{\mathrm{Im}} = 0$ and $k_{\mathrm{Im}} \neq 0$ we obtain for the time-averaged expressions

$$\langle S \rangle = S_0 e^{2k_{\mathrm{Im}} z}, \quad \langle W \rangle = W_0 e^{2k_{\mathrm{Im}} z} \quad , \quad \langle P_d \rangle = P_{d0} e^{2k_{\mathrm{Im}} z} \quad , \tag{11}$$

where S_0 and W_0 are given by (8a) and (9b):

$$S_0 = -(\mathrm{Im} D E_0 E_0^* + 2P_{d0})/4k_{\mathrm{Im}} \quad ,$$

$$W_0 = \frac{1}{4} \mathrm{Re} \frac{\partial D}{\partial \omega} E_0 E_0^* + \frac{1}{2} \frac{\partial P_{d0}}{\partial \omega_{\mathrm{Im}}} + k_{\mathrm{Im}} \frac{\partial S_0}{\partial \omega_{\mathrm{Im}}} \quad . \tag{12}$$

If we for the moment assume P_d to be known, we conclude that it is necessary to know S_0 as a function of ω_{Im} in order to determine W_0 when $\omega_{\mathrm{Im}} = 0$. The situation is still more complicated when ω_{Im} as well as k_{Im} differ from 0. The energy-conservation relation only yields a relation between S, W, and P_d in this general case. In the Sec. III we consider some cases when it is possible to derive expressions for S and P_d which together with (8) yield an expression for W.

III. DERIVATION OF DISSIPATIVE POWER AND ENERGY FLOW

The main problem in an absorbing medium is to separate the energy components. Primarily, we need an expression for the power dissipated, and it will be shown that it is possible to derive such an expression from the dispersion function if this is an explicit function of the dissipational parameters λ_i, i.e., $D=D(\omega, k, \lambda_1)$. We assume that the power dissipated is given by

$$P_d=\sum_i \lambda_i(\mathrm{Re}v_i)^2 \quad , \tag{13}$$

where v_i is such an amplitude that λ_i is frequency and wavelength independent and describes the "real" loss process. (If not, energy-storing processes will be "hidden." For the meaning of λ_i see Sec. VI.) We now introduce the amplitude ratio $A_i=v_i/E$, where A_i is in general a function of frequency as well as wavelength. It will be shown that A_i can be obtained from the so-called dispersion function $D(\omega, k, \lambda_i)$ by varying the loss factor λ_i. We assume linear relations between E, v_i, and J^s. After elimination of the irrelevent amplitudes in the set of equations describing our system, we can describe the relation between E and v_i by (cf. Ref. 5)

$$D^i_{aa}E+D^i_{ab}v_i=jJ^s \quad , \tag{14a}$$

$$D^i_{ba}E+D^i_{bb}v_i=-jF_i \quad . \tag{14b}$$

For our objectives it is not necessary to know the exact form of the coefficients. We introduce a fictitious force F_i, and the signs of D^i_{ba} and D^i_{bb} are then defined by the fact that the power delivered to the system is $\mathrm{Re}F_i\,\mathrm{Re}v_i$. Elimination of v_i finally yields ($F_i=0$)

$$DE=(D^i_{aa}-D^i_{ab}D^i_{ba}/D^i_{bb})E=jJ^s \quad . \tag{15}$$

According to (13) the dissipation is described by a damping force $\lambda_i v_i$, which means that λ_i is included in D^i_{bb} only as $-j\lambda_i$. According to (14b) $A_i=-D^i_{ba}/D^i_{bb}$, which means that

$$\frac{\partial D}{\partial\lambda_i}=\frac{D^i_{ab}D^i_{ba}}{(D^i_{bb})^2}\frac{\partial D^i_{bb}}{\partial\lambda_i}=-jA_i^2\frac{D^i_{ab}}{D^i_{ba}} \quad . \tag{16}$$

Since the system is reciprocal we have $D^i_{ba}=\pm D^i_{ab}$, yielding

$$A_i^2=\pm j\frac{\partial D}{\partial\lambda_i} \quad . \tag{17}$$

The sign of A_i^2 is determined from an analysis for large ω and k, when we know that A_i^2 takes the form $(j\omega)^{2n}(-jk)^{2m}C^2$, where C is real. As long as we need $A_iA_i^*$ the sign is unimportant. The expressions wanted now are

$$P_{d0}=\sum_i \tfrac{1}{2}\lambda_i\left|\left(j\frac{\partial D}{\partial\lambda_i}\right)^{1/2}\right|^2 |E_0|^2 \quad ,$$
$$P_{d1}=\sum_i \tfrac{1}{2}\lambda_iA_i^2E_0^2=\sum_i\left(\pm\tfrac{1}{2}j\lambda_i\frac{\partial D}{\partial\lambda_i}\right)E_0^2 \quad . \tag{18}$$

Equations (18) demonstrate that it is not necessary to know λ_i, but only that part of λ_i which directly characterizes the dissipational process, say ν_i. The assumption that P_d can be written in the form given by (13) is fulfilled when D is a rational function of the loss factors.

In order to obtain expressions for the energy flow we restrict ourselves to media without spatial dispersion. For such media the dispersion function has the form (see Appendix A)

$$D(\omega,k)=\omega\epsilon'(\omega)-k^2/\omega\,\mu'(\omega) \quad . \tag{19}$$

From Appendices A and B we have $S=\mathrm{Re}(E)\mathrm{Re}(H)$ where $H=-\frac{1}{2}(\partial D/\partial k)E$, and we obtain

$$S_0=-\tfrac{1}{4}\mathrm{Re}\frac{\partial D}{\partial k}E_0E_0^* \quad ,$$
$$S_1=-\frac{1}{4}\frac{\partial D}{\partial k}E_0^2 \quad . \tag{20}$$

These expressions can be generalized to other wave motions characterized by $D(\omega, k)$ on the form given by (19). This implies that we use one of the propagating field amplitudes (e.g., analogous to E or H) to describe the medium response and that the quantities corresponding to $\omega\epsilon'$ and $\omega\mu'$ are independent of the propagation constant.

IV. EXPRESSIONS FOR STORED ENERGY

From (8), (18), and (20) we obtain

$$W_0=\frac{1}{4\omega_{\mathrm{Im}}}\mathrm{Im}\left[D-jk_{\mathrm{Im}}\frac{\partial D}{\partial k}+j\lambda_i\left|\left(j\frac{\partial D}{\partial\lambda_i}\right)^{1/2}\right|^2\right]E_0E_0^* \quad ,$$
$$W_1=\frac{1}{4\omega}\left(D-k\frac{\partial D}{\partial k}\mp\lambda_i\frac{\partial D}{\partial\lambda_i}\right)E_0^2 \quad , \tag{21}$$

where the sign is determined according to the text after (17). Particularly when $\omega_{\mathrm{Im}}=0$ we obtain from (12), (18), and (20)

$$W_0=\tfrac{1}{4}\mathrm{Re}\left[\frac{\partial D}{\partial\omega}-jk_{\mathrm{Im}}\frac{\partial^2D}{\partial\omega\,\partial k}+j2\lambda_i\left(j\frac{\partial D}{\partial\lambda_i}\right)^{1/2}\times\frac{\partial}{\partial\omega}\left(j\frac{\partial D}{\partial\lambda_i}\right)^{1/2}\right]E_0E_0^* \quad . \tag{22}$$

These expressions are valid in a medium without spatial dispersion. D is included for convenience in (21) although $D=0$ for free waves. By means of (19) the dispersion function D can be expressed in ϵ' and μ'.

The analysis above assumes the power dissipated and the stored energy in the medium to be located where the medium interacts with the electric field.

A crucial point in the analysis of an abosrbing medium seems to be the definition of one quantity responsible for the energy storage and one for the dissipation. By definition we have $\epsilon'=\epsilon_1+j\epsilon_2=\epsilon-j(\sigma/\omega)$. Ginzburg used $\epsilon=\epsilon_1$ and $\epsilon_2=-\sigma/\omega$ (real frequencies) and found that σ as well as ϵ contributes to the stored energy. Neufeld reformulated the definition of ϵ such that it was independent of the dissipation. In this paper we have determined the stored energy and the dissipation directly from expressions for D or ϵ' and μ'. However, introducing the definitions

$$\epsilon_2' = -\sum_i \frac{\lambda_i}{\omega}\left|\left(j\frac{\partial(\omega\epsilon')}{\partial\lambda_i}\right)^{1/2}\right|^2 ,$$

$$\mu_2' = -\sum_j \frac{\lambda_j}{\omega}\left|\left(j\frac{\partial(\omega\mu')}{\partial\lambda_j}\right)^{1/2}\right|^2 , \qquad (23)$$

$$\epsilon_1' = \epsilon' - j\epsilon_2' , \quad \mu_1' = \mu - j\mu_2' ,$$

we obtain from (18) when ω is real,

$$\langle P_d \rangle = -\tfrac{1}{2}(\omega\epsilon_2'|E|^2 + \omega\mu_2'|H|^2) ,$$
$$\langle W \rangle = \tfrac{1}{4}\,\mathrm{Re}\left[\frac{\partial(\omega\epsilon_1')}{\partial(j\omega_{\mathrm{Im}})}|E|^2 + \frac{\partial(\omega\mu_1')}{\partial(j\omega_{\mathrm{Im}})}|H|^2\right]_{\omega_{\mathrm{Im}}=0} . \qquad (24)$$

ϵ_1' and μ_1' are equal to ϵ_1 and μ_1 for real ω. However, ϵ_1' and μ_1' do not determine the oscillating part of the stored energy and the dissipation as long as $\omega\epsilon_2'$ and $\omega\mu_2'$ are dependent of ω. This means that we cannot in the general case give definitions of the ϵ_1' and μ_1' which can be used to determine the instantaneous value of the stored energy.

Finally we obtain for the stored energy when ω is real

$$\langle W \rangle = \frac{1}{4}\left[\frac{\partial(\omega\epsilon_1)}{\partial\omega} - \frac{\partial(\omega\epsilon_2')}{\partial\omega_{\mathrm{Im}}}\right]E_0E_0^* + \frac{1}{4}\left[\frac{\partial(\omega\mu_1)}{\partial\omega} - \frac{\partial(\omega\mu_2')}{\partial\omega_{\mathrm{Im}}}\right]H_0H_0^* \quad (\omega_{\mathrm{Im}}=0) . \qquad (25)$$

This expression is identical to the frequently used expression

$$\langle W \rangle = \frac{1}{4}\frac{\partial(\omega\epsilon_1)}{\partial\omega}E_0E_0^* + \frac{1}{4}\frac{\partial(\omega\mu_1)}{\partial\omega}H_0H_0^* , \qquad (26)$$

when $\omega\epsilon_2'$ and $\omega\mu_2'$ are frequency independent, which is equivalent to $\omega\epsilon_2$ and $\omega\mu_2$ being frequency independent. However, if the correction term is small, (26) can be an acceptable approximation although it does not yield a physically correct expression.

V. ENERGY EXPRESSIONS FOR ARBITRARY SIGNALS

The derived expressions can as well be used for arbitrary signals. We have, according to (3), (13), and (20), expressions for $S(\omega, k)$, $P_d(\omega, k)$, and $P_s(\omega, k)$ in terms of convolution integrals (marked by $*$),

$$S(\omega, k) = -\frac{1}{2}\frac{\partial D}{\partial k}E(\omega, k) * E(\omega, k) ,$$

$$P_d(\omega, k) = \sum \lambda_i A_i(\omega, k)E(\omega, k) * A_i(\omega, k)E(\omega, k) , \qquad (27)$$

$$P_s(\omega, k) = jD(\omega, k)E(\omega, k) * E(\omega, k) .$$

$W(\omega, k)$ is obtained from the transformed energy-conservation relation

$$W(\omega, k) = (1/j\omega)[P_s(\omega, k) - P_d(\omega, k) + jkS(\omega, k)] . \qquad (28)$$

The time-space expressions are obtained by inversion.

VI. APPLICATIONS

Energy Density of an Assembly of Oscillators

We will now apply our expressions to an idealized structure of an atomic medium described by an assembly of independent oscillators. The motion of the electrons can be expressed by

$$m\left(\frac{d^2r_i}{dt^2} + \nu_i\frac{dr_i}{dt} + \omega_i^2 r_i\right) = eE , \qquad (29)$$

where $m\nu_i(dr_i/dt)$ is the frictional force and $m\omega_i^2 r_i$ the restoring force on the bound electron. Assuming that the mean velocity of the electrons is 0, we obtain after linearization

$$\epsilon' = \epsilon_0\left(1 - \sum_i \frac{\omega_{pi}^2}{\omega^2 - \omega_i^2 - j\omega\nu_i}\right) , \qquad (30)$$

where $\omega_{pi}^2 = N_i e^2/m\epsilon_0$ and N_i is the density of the electrons characterized by ω_i and ν_i. From (19) we now obtain

$$D = \omega\epsilon' - \frac{k^2}{\omega\mu'} = \epsilon_0\left(\omega - \sum_i \frac{\omega_{pi}^2}{\omega - \omega_i^2/\omega - j\nu_i}\right) - \frac{k^2}{\omega\mu_0} . \qquad (31)$$

We will now determine the stored energy from the knowledge of the dispersion function. We first notice that $D(\omega, k)$ is not real for ω and k real. This means that the medium is lossy. It is inferred that the loss factors are ν_i. The power dissipated is given by (18) with $\lambda_i = \nu_i$:

$$P_d = \sum_i \tfrac{1}{2}\epsilon_0\nu_i\frac{\omega_{pi}^2}{|\omega - \omega_i^2/\omega - j\nu_i|^2}|E|^2 - \mathrm{Re}\left(\sum_i \tfrac{1}{2}\epsilon_0\nu_i\frac{\omega_{pi}^2}{(\omega - \omega_i^2/\omega - j\nu_i)^2}E^2\right), \qquad (32)$$

which is identical to

$$\sum_i N_i m\nu_i\left[\mathrm{Re}\left(\frac{dr_i}{dt}\right)\right]^2$$

as expected. From Eq. (21) we obtain

$$W = \tfrac{1}{4}\left(\epsilon_0 + \sum_i \epsilon_0\frac{\omega_{pi}^2(1 + \omega_i^2/|\omega|^2)}{|\omega - \omega_i^2/\omega - j\nu_i|^2} + \frac{k^2}{|\omega|^2\mu_0}\right)|E|^2$$

$$+\frac{1}{4}\,\mathrm{Re}\left[\left(\epsilon_0-\sum_i \epsilon_0\frac{\omega_{pi}^2(1-\omega_i^2/\omega^2)}{(\omega-\omega_i^2/\omega-j\nu_i)^2}+\frac{k^2}{\omega^2\mu_0}\right)E^2\right], \tag{33}$$

which can be written

$$W=\frac{1}{2}\left\{\epsilon_0\left[\mathrm{Re}(E)\right]^2+\sum_i N_i m\left[\mathrm{Re}\left(\frac{dr_i}{dt}\right)\right]^2 +\sum_i N_i m\cdot\omega_i^2[\mathrm{Re}(r_i)]^2+\mu_0[\mathrm{Re}(H)]^2\right\}, \tag{34}$$

i.e., the same as would be expected by physical arguments whether or not the dissipation process takes place.

For $\omega_{\mathrm{Im}}=0$, the time average of the stored energy is given by the first term in (33). The same expression can be obtained from (22).

Electrical Circuits

In electrical circuits the equation corresponding to (3) is[4] $DI=jV$, and D is given by $D=-jZ$. We have no energy flow, and Eq. (21) is reduced to

$$W_0=-\frac{\mathrm{Re}(Z-\sum R_i|A_i|^2)}{4\omega_{\mathrm{Im}}}|I_0|^2,$$
$$W_1=-\frac{jZ-\sum jR_iA_i^2}{4\omega}I_0^2, \tag{35}$$

where

$$A_i^2=\frac{\partial Z}{\partial R_i},\quad I_i=A_iI_0. \tag{36}$$

A_i relates the current in a certain branch of the network to the input current. If the resistance R_i is represented by a conductance G_i in the expression (35), the other sign for A_i^2 must be used, and we obtain

$$A_i^2=-\frac{\partial Z}{\partial G_i},\quad V_i=A_iI_0. \tag{37}$$

V_i is the voltage across G_i. These relations may be useful in other connections too.

When we have forced oscillations with $\omega_{\mathrm{Im}}=0$, and $Z=R+jX$, the stored energy is according to (12) given by

$$W_0=\frac{1}{4}\frac{\partial X}{\partial\omega_{\mathrm{Re}}}|I_0|^2+\frac{1}{2}\frac{\partial P_{d0}}{\partial\omega_{\mathrm{Im}}}, \tag{38}$$

in agreement with Ref. 6. From (18) we obtain that

$$\frac{1}{2}\frac{\partial P_{d0}}{\partial\omega_{\mathrm{Im}}}=-\frac{1}{2}\sum\mathrm{Im}\left(R_i\frac{\partial A_i}{\partial\omega}A_i^*\right)|I_0|^2, \tag{39}$$

where A_i is given by (36).

It was remarked above that it is necessary to know how the dispersion function (input impedance) varies with all the loss parameters. If we choose certain values of the parameters of the circuit illustrated in Fig. 1 (see p. 485 of Ref. 2), namely, $R_1=R_2=R$ and $L=R^2C$, we obtain $Z\equiv R$. Thus, it is not possible to determine the stored energy from Z. This is due to the fact that Z is not a function of the "real" loss factors R_1 and R_2.

VII. CONCLUSIONS

The essential feature of this analysis is to demonstrate how energy expressions can be derived directly from the expressions for the complex permittivity ϵ' and permeability μ', or rather the dispersion function D. It is shown that the power dissipated P_d can be derived when ϵ' and μ' or D are known rational functions of all the loss factors. Furthermore, an expression for the energy flow S is given for frequency-dispersive media. The expression for the stored energy is obtained from the energy-conservation relation by means of P_d and S.

We have considered primarily the energy expressions of plane electromagnetic waves in isotropic media. However, the results of the present work can be applied to arbitrary plane waves in lossy frequency-dispersive isotropic media, and generalized to anisotropic media. (Then it may be convenient to introduce separate loss factors in each coordinate direction.)

The present analysis may also be of value in connection with a discussion of the signal velocity. The same method of analysis can, furthermore, be applied to the momentum-conservation relation.

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APPENDIX A

Maxwell's equations for the macroscopic electric and magnetic fields in the presence of material media may be written as

$$\nabla\times\vec{H}=\epsilon_0\frac{\partial\vec{E}}{\partial t}+\vec{J}_e+\vec{J}_e^s,$$
$$\nabla\times\vec{E}=-\mu_0\frac{\partial\vec{H}}{\partial t}-\vec{J}_m-\vec{J}_m^s, \tag{A1}$$

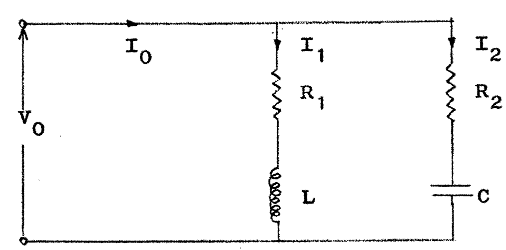

FIG. 1. Electrical circuit illustrating the necessity to know how the dispersion function varies with all the loss parameters. If $R_1=R_2=R$, and $L=R^2C$, we obtain $Z\equiv R$, and Z does not keep information of the stored energy.

where $\vec{J}_e$ and $\vec{J}_m$ are linearized functions of $\vec{E}$ and $\vec{H}$ describing the medium, while $\vec{J}_e^s$ and $\vec{J}_m^s$ represent external sources exciting the system. The power delivered by the external source is

$$P_s = -\vec{E}\cdot\vec{J}_e^s - \vec{H}\cdot\vec{J}_m^s . \qquad \text{(A2)}$$

After time-space transformation we can write (A1) in an isotropic case without spatial dispersion (ϵ' and μ' independent of k):

$$\begin{aligned} &\omega\epsilon'(\omega)E(\omega,k) - kH(\omega,k) = jJ_e^s(\omega,k), \\ &-kE(\omega,k) + \omega\mu'(\omega)H(\omega,k) = jJ_m^s(\omega,k). \end{aligned} \qquad \text{(A3)}$$

If the wave motion is excited by an electric source ($J_e^s = J^s \neq 0$, $J_m^s = 0$) the medium response is determined by

$$\begin{aligned} D(\omega,k)E(\omega,k) &= [\omega\epsilon'(\omega) - k^2/\omega\mu'(\omega)]E(\omega,k) \\ &= jJ^s(\omega,k), \\ H(\omega,k) &= \frac{k}{\omega\mu'(\omega)}E(\omega,k) = -\frac{1}{2}\frac{\partial D}{\partial k}E. \end{aligned} \qquad \text{(A4)}$$

$D(\omega,k)$ is closely related to the transform of the Green's function.[5]

APPENDIX B

From Appendix A we obtain for a non-spatial-dispersive medium

$$\omega\epsilon'(\omega)E - kH = jJ_e^s, \quad -kE + \omega\mu'(\omega)H = jJ_m^s. \qquad \text{(B1)}$$

If we now change k by δk this change can be compensated by the currents $\delta J_e^s = jH\cdot\delta k$ and $\delta J_m^s = jE\cdot\delta k$ in such a way that the system for the rest is unchanged. This means that S, W, and P_d is unchanged and that the input power tends to

$$P_s - \mathrm{Re}(jH\cdot\delta k)\ \mathrm{Re}(E) - \mathrm{Re}(jE\cdot\delta k)\ \mathrm{Re}(H).$$

From Eq. (7) we then obtain

$$S_0 = \tfrac{1}{2}\mathrm{Re}(H_0\cdot E_0^*), \quad S_1 = \tfrac{1}{2}H_0\cdot E_0, \qquad \text{(B2)}$$

or

$$S = \mathrm{Re}(H)\ \mathrm{Re}(E). \qquad \text{(B3)}$$

[1]W. P. Allis, S. J. Buchsbaum, and A. Bers, *Waves in Anisotropic Plasmas* (MIT Press, Cambridge, Mass., 1962), Sec. 8.5; P. R. McIsaac, RLE Chalmers University of Technology, Research Report No. 57, 1966 (unpublished); T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill, New York, 1962), Sec. 3.

[2]V. L. Ginzburg, *The Propagation of Electromagnetic Waves in Plasmas* (Pergamon, Oxford, 1964), Sec. 22, Appendix B.

[3]J. Neufeld, Phys. Rev. 152, 708 (1966); Intern. J. Electron. 27, 301 (1969); Nuovo Cimento 65B, 33 (1970); 66B, 51 (1970).

[4]J. Askne and O. Nilsson, Proc. IEEE 57, 1423 1969.

[5]J. Askne, RLE Chalmers University of Technology, Research Report No. 91, 1968 (unpublished).

[6]I. Lundström and O. Nilsson, Proc. IEEE 58, 796 1970.

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Excitation of Highly Ionized Atoms by Electron Impact*

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We use the Coulomb-Born approximation to calculate the total cross section for the electron-impact excitation of ionized atoms which have one electron outside their closed shells. When the incident energy is high and the atom is highly ionized, we can use the eikonal and classical approximations in conjunction with the Coulomb-Born approximation. Then, the scattering amplitude reduces to a one-dimensional path integral. The path integral, in turn, can be reduced, to any desired accuracy, to a sum over integrals which can be evaluated analytically. Using this procedure, we have calculated the total cross sections for the heavily ionized atoms Fe^{15+}, Co^{16+}, Ni^{17+}, and Cu^{18+}.

I. INTRODUCTION

Electron impact excitation of various types of atomic ions has been examined in different approximations.[1] Of particular interest have been the Coulomb-Born (CB) and the close-coupling approximations. By use of the Coulomb-Born approximation, electron-impact excitation cross sections have been calculated[2] for hydrogenic ions by Tully and Burgess, for He^+ by Tully, for lithiumlike ions (Be^+, C^{3+}, O^{5+}, and Mg^{9+}) by Belly, Tully, and Van Regemorter, for sodiumlike ions (Mg^+, Si^{3+}, and Fe^{15+}) by Belly, Tully, and Van Regemorter; also by Kreuger and Czyzak, and for potassiumlike ions