PHYSICAL REVIEW A VOLUME 2, NUMBER 5 NOVEMBER 1970

Fluctuations of Beams of Quantum Particles

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(Received 25 February 1970)

A beam of noninteracting particles, bosons or fermions, is described by the superposition of stochastic wave packets. This description allows in each case (bosons or fermions) the determination of the detection process of the particles. This process is defined by the set of the p-order coincidence probability densities, a general formulation of which is given. In the case of a stationary and weak incoherent beam, these coincidence probability densities are studied thoroughly and several results are obtained. The well-known bunching effect for bosons and the "antibunching effect" for fermions are shown to come from the detection of *indistinguishable* particles. In the boson case, all the well-known results for thermal light are found. In the fermion case, the detection process is, under certain conditions, identical to a renewal process.

I. INTRODUCTION

Photon beam fluctuations have been extensively studied in the last few years from a theoretical as well as from an experimental point of view.[1] These recent developments of quantum optics are primarily due to the use of new sources, such as lasers, and of fast electronic techniques already used in nuclear physics.

Many kinds of experiments can be performed to determine the statistical properties of a photon beam. In particular, the point stochastic process consisting of the detection time instants of photons in an electromagnetic field can be studied by means of coincidence and counting experiments.[2–11] The experimental results are in good agreement with the theoretical calculations,[12] indicating that this process is a compound Poisson process.

These calculations can be obtained by using a quantum-mechanical description of the field and the detectors. In particular, a new description of the old phenomena of coherence has been given on an entirely quantum-mechanical basis by introducing the notion of coherent states of the electromagnetic field.[13–15] The coherent-states formalism depends basically on the commutation relations between boson annihilation and creation operators. Therefore, it can be generalized to any boson beam, and it is possible to predict what one should obtain from coincidence or counting experiments on such beams.

However, if we are interested in the fluctuations of fermion beams, such as electron or neutron beams, none of these new developments in quantum optics are very useful. The coherent-states formalism is not convenient to describe a beam of free fermions, as Ledinegg has shown.[16] Other authors have devised a generalization of the coherent states which deals only with the case of charged particles in an external magnetic field.[17] However, if we consider the developments in electron optics and especially in electron interferences, it seems worthwhile to study fermion beam fluctuations more closely and in particular to know what should be observed by making coincidence or counting experiments on a fermion beam.

The formalism that can be used for this purpose is the wave-packet formalism which was introduced in a series of papers by Goldberger and Watson (GW) to study the second-order intensity correlation function of any beam of particles.[18–20] They associate a stochastic wave packet with every particle and build the wave function of the whole set of particles by symmetrization or antisymmetrization. This wave-packet formalism has the advantage of being convenient for fermions and bosons and then of enabling us to compare the results which should be expected in each case.

In their papers, GW have used a very important approximation of the orthogonality of the wave packets, and they have only considered the second-order correlation function. In this paper, we are starting from the same ideas as the ones introduced by GW but we discuss their approximation and use the wave-packet formalism not only to calculate second-order correlation functions, but to describe completely the statistical properties of boson or fermion beams.

In Sec. II, the wave-packet formalism is described in terms of wave mechanics. A thoroughly general formulation of the wave function of a particles beam is given. In Sec. III, we express the p-order coincidence probabilities which are measured on any beam of quantum particles by an ideal p-order coincidence experiment. These probabilities are introduced because it is possible to show that the whole set of coincidence probabilities defines entirely the statistical properties of the detection process of the particles, including its time dependence.[21] Therefore, in what follows, we will only deal with the coincidence probabilities to describe the fluctuations of the beam. In Sec. IV, we study the stationary incoherent case. It is also the only case that has been studied by GW. We discuss their approximation of the "effective orthogonality" of the wave packets which is shown to be correct for very weak beams. We are giving here the coincidence probabilities for this case. By using these probabilities, we show that the boson detection process is a Poisson compound process, as might be expected, and we specify the properties of the compounding random density. Several features of fermions are pointed out. In particular, the fermion detection process turns out to be a renewal process when the correlation function of every wave packet is exponential.

Finally, we emphasize the fact that the wave-packet formalism shows very clearly the relation between the measured statistical properties and the quantum nature (boson or fermion) of the particles. Hence the well-known Hanbury Brown and Twiss effect observed on thermal light is shown to come only from the stochastic independence of bosons.

II. WAVE FUNCTION OF A BEAM OF PARTICLES

The beam of particles that we are considering here is a set of indistinguishable and *noninteracting* particles. To simplify the calculations, we are not taking into account the spin of the particles. The number of particles in the beam is unknown and is represented by a random variable $N(\omega)$, where ω is a point in a probability space. To simplify, we shall sometimes omit this explicit dependence on ω. We assume that $N(\omega)$ wave packets can be associated with the $N(\omega)$ identical particles and that they are also not known completely. Thus, these wave packets are $N(\omega)$ *identical stochastic* normalized functions $\Phi_i(\vec{r}, t; \omega'_i)$, where $\vec{r}$ is the position and t the time. For every stochastic function $\Phi_i(\vec{r}, t; \omega'_i)$ a given point ω'_i in the probability space Ω'_i determines a given function Φ_i of $\vec{r}$ and t. This function must be normalized in the volume V occupied by the whole set of particles. In general, for given values of ω'_i and ω'_j $(i \neq j)$, the functions $\Phi_i(\vec{r}, t; \omega'_i)$ and $\Phi_j(\vec{r}, t; \omega'_j)$ are *not orthogonal*, and we have have

$$\int_V \Phi_i(\vec{r}, t; \omega'_i)\, \Phi_j^*(\vec{r}, t; \omega'_j)\, d\vec{r} \neq 0. \qquad (2.1)$$

Since the particles are indistinguishable and noninteracting, the wave function of the beam can be deduced from the functions $\Phi_i(\vec{r}, t; \omega'_i)$ by using the appropriate projection operator S on a symmetrized or antisymmetrized space.[22] It can be written

$$\psi(\{\vec{r}_i\}, t; \omega, \omega') = \gamma(\omega, \omega')\, S \prod_{i=1}^{N(\omega)} \Phi\, \omega_{\alpha_i}(\vec{r}_i, t; \omega'_{\alpha_i}), \qquad (2.2)$$

or, in a more detailed formulation,

$$\psi(\{\vec{r}_i\}, t, \omega, \omega') = \gamma(\omega, \omega') \sum_k (\pm)^k \mathrm{P}^k_{N(\omega)} \prod_{i=1}^{N(\omega)} \Phi_{\alpha_i}(\vec{r}_i, t; \omega'_{\alpha_i}) . \qquad (2.3)$$

In this expression, P^k_N stands for the sum of all the different k-order permutations of N elements, and $\{\alpha_i\}$ is any k-order permutation of the ordered set $\{1, 2, \ldots, N(\omega)\}$ which refers to the positions $\{r_1, r_2, \ldots, r_{N(\omega)}\}$.[23] The symbol ω' stands for $\{\omega'_1, \omega'_2, \ldots, \omega'_{N(\omega)}\}$.

We can also write the ket $|\psi(\omega, \omega'))$ corresponding, in the Heisenberg picture, to the wave function $\psi(\{\vec{r}_i\}, t; \omega, \omega')$. It is given by the following expression:

$$|\psi(\omega, \omega')) = \gamma(\omega, \omega') \sum_k (\pm)^k \mathrm{P}^k_{N(\omega)} \prod_{i=1}^{N(\omega)} |\Phi_{\alpha_i}(\omega, \omega'_{\alpha_i})). \qquad (2.3')$$

In this formulation, $\{\alpha_i\}$ is any k-order permutation of the ordered set $\{1, 2, \ldots, N(\omega)\}$ which refers to the order in which the $N(\omega)$ kets $|\ldots)$ are written.

Since the functions $\Phi_i(\vec{r}, t; \omega'_i)$ are stochastic, ψ is a stochastic function too, and its statistical properties are defined by the structure of the probability spaces Ω and Ω'_i. The constant $\gamma(\omega, \omega')$ is determined by the condition

$$(\psi, \psi) = 1. \qquad (2.4)$$

It can be written as

$$|\gamma(\omega, \omega')|^{-2} = \sum_{k', k''} (\pm)^{k'+k''} \mathrm{P}^{k'}_{N(\omega)} \mathrm{P}^{k''}_{N(\omega)} \prod_{j=1}^{N(\omega)} \prod_{h=1}^{N(\omega)} (\Phi_{\beta_j}, \Phi_{\gamma_h}), \qquad (2.5)$$

where $\{\beta_j\}$ and $\{\gamma_h\}$ have the same meaning as $\{\alpha_i\}$. This expression is equivalent to

$$|\gamma(\omega, \omega')|^{-2} = [N(\omega)]!\, D(\omega, \omega'), \qquad (2.6)$$

where

$$D(\omega, \omega') = \sum_k \mathrm{P}^k_{N(\omega)} (\pm)^k \prod_j (\Phi_j, \Phi_{\alpha_j}). \qquad (2.7)$$

In this expression $\{\alpha_j\}$ has the same meaning as in Eq. (2.3′). If the Φ_i were orthogonal, we should have, as usual,

$$D(\omega, \omega') = 1 . \qquad (2.8)$$

However, since this assumption is generally not valid, we shall use in the following discussion the wave function defined by Eqs. (2.3′) and (2.5).

III. GENERAL FORMULATION OF COINCIDENCE PROBABILITY DENSITIES

A. Definitions and Notations

In the particle beam, let us consider p arbitrary infinitesimal volumes $\{\vec{a}_i, d\vec{a}_i\}$ centered in $\vec{a}_i$ and p time intervals $\{t_i, dt_i\}$ centered in t_i. The p-order coincidence event $C_p[\{\vec{a}_i, d\vec{a}_i\}; \{t_i, dt_i\}]$ is realized if we detect any particle of the beam in the volume $\{\vec{a}_1, d\vec{a}_1\}$ during the time interval $\{t_1, dt_1\}$, any other in $\{\vec{a}_2, d\vec{a}_2\}$, $\{t_2, dt_2\}$, ..., and any other in $\{\vec{a}_p, d\vec{a}_p\}$, $\{t_p, dt_p\}$. The probability of this event can be written

$$\Pr[C_p[\{\vec{a}_i, d\vec{a}_i\};\{t_i, dt_i\}]] = P_p[\{\vec{a}_i\};\{t_i\}] \prod_{i=1}^{p} d\vec{a}_i dt_i , \quad (3.1)$$

which defines the p-order coincidence probability density (cpd) $P_p[\{\vec{a}_i\};\{t_i\}]$. Such a quantity $\Pr[C_p\{\vec{a}_i, d\vec{a}_i\};\{t_i; dt_i\}]$ can be measured in a p-order coincidence experiment. This experiment is performed by setting p detectors in the beam, each of them acting in a volume $\{\vec{a}_i, d\vec{a}_i\}$ and during a time interval $\{t_i, dt_i\}$, and absorbing one particle or none in this volume during this time.[24] If the p detectors act during the same time interval, we measure a purely spatial cpd. On the other hand, if we use only one detector acting during different time intervals $\{t_i, dt_i\}$, we measure a purely temporal cpd.

As already mentioned in the Introduction, the detection process of the particles of a beam is defined, in this paper, by the set of the cpd. Since all the statistical properties of a beam can be derived from its wave function, defined by Eqs. (2.3′) and (2.5), it is possible to express the cpd in terms of the wave function, as is being done in the present section. In the calculations, we shall introduce the *a posteriori* cpd calculated for given values of ω and ω'. It has only a mathematical meaning and the cpd which is measured in a coincidence experiment is an *a priori* probability density $P_p[\{\vec{a}_i\};\{t_i\}]$ which is deduced from the *a posteriori* cpd by taking the ensemble average over ω and ω'.

It will also be very useful to introduce in the calculations the probability density (pd) $\xi_p(\{\vec{a}_i\};\{t_i\})$. The quantity $\xi_p(\{\vec{a}_i\};\{t_i\}) \prod d\vec{a}_i \prod dt_i$ is defined mathematically as the probability that a *given* particle will be detected in $\{\vec{a}_1, d\vec{a}_1\}$ during the time interval $\{t_1, dt_1\}$, another *given* particle will be detected in $\{\vec{a}_2, d\vec{a}_2\}$ during the time interval $\{t_2, dt_2\}$..., and another *given* particle detected in $\{\vec{a}_p, d\vec{a}_p\}$ during the time interval $\{t_p, dt_p\}$. As the particles are identical, the pd $\xi_p(\{\vec{a}_i\};\{t_i\})$ is symmetrical with respect to the points $\{\vec{a}_i, t_i\}$. But, since the particles are indistinguishable, it has no physical meaning. The corresponding *a posteriori* pd will be written as $\xi_p[\{\vec{a}_i\};\{t_i\} \mid \omega, \omega']$.

In the calculations below, we shall first compute *a posteriori* cpd and then *a priori* cpd. This will be done in the purely spatial case on one hand, and in the general spatiotemporal case on the other hand.

B. Spatial-Coincidence Probability Densities

Let us consider the *a posteriori* spatial cpd, that is to say, the spatial cpd that should be measured on a system of $N(\omega)=n$ particles, where ω and ω' are given. According to the definition of the cpd and the pd, we can write

$$P_p[\{\vec{a}_i\}; t \mid \omega, \omega'] = [n!/(n-p)!]\, \xi_p[\{\vec{a}_i\}; t \mid \omega, \omega']. \quad (3.2)$$

Since we have

$$\int \xi_p[\{\vec{a}_i\}; t \mid \omega, \omega'] \prod d\vec{a}_i = 1, \quad (3.3)$$

it follows that

$$\int P_p[\{\vec{a}_i\}; t \mid \omega, \omega'] \prod d\vec{a}_i = n(n-1)\cdots(n-p+1) = n^{[p]}. \quad (3.4)$$

In particular,

$$\int P_1[\vec{a}; t]\, d\vec{a} = n , \quad (3.5)$$

$$\int P_n[\{\vec{a}_i\}; t] \prod d\vec{a}_i = n! . \quad (3.6)$$

We can also deduce from the relation

$$\xi_p[\{\vec{a}_i\}; t \mid \omega, \omega'] = \int \xi_{p+1}[\{\vec{a}_i\}; t \mid \omega, \omega']\, d\vec{a}_{p+1} \quad (3.7)$$

and the relation (3.2) that

$$P_p[\{\vec{a}_i\}; t \mid \omega, \omega'] = [1/(n-p)] \int P_{p+1}[\{\vec{a}_i\}; t \mid \omega, \omega']\, d\vec{a}_{p+1} . \quad (3.8)$$

From Eq. (3.8), it follows that

$$P_p[\{\vec{a}_i\}; t \mid \omega, \omega'] = [1/(n-p)!] \int P_n[\{\vec{a}_i\}; t \mid \omega, \omega']\, d\vec{a}_{p+1} \cdots d\vec{a}_n . \quad (3.9)$$

This relation shows that any *a posteriori* spatial cpd can be expressed in terms of the n-order *a posteriori* spatial cpd.

Equation (3.9) can also be written in the form

$$P_p[\{\vec{a}_i\}; t \mid \omega, \omega'] = [1/(n-p)!] \int P_n[\{\vec{r}_i\}; t \mid \omega, \omega'] \times \delta(\vec{a}_1 - \vec{r}_1) \cdots \delta(\vec{a}_p - \vec{r}_p)\, d\vec{r}_1 \ldots d\vec{r}_n . \quad (3.10)$$

The expressions appearing in Eqs. (3.9) and (3.10) are not symmetrical with respect to the space coordinates $\vec{a}_i$, but since $P_n[\{\vec{r}_i\}; t \mid \omega, \omega']$ is symmetrical with respect to the coordinates $\vec{r}_i$, Eq. (3.10) can be rewritten in a symmetrical way:

$$P_p[\{\vec{a}_i\}; t \mid \omega, \omega'] = [1/(n-p)!]\, P_n[\{\vec{r}_i\}; t \mid \omega, \omega'] \times (1/n^{[p]}) \sum_{\beta_1 \neq \cdots \neq \beta_p} \delta(\vec{a}_1 - \vec{r}_{\beta_1}) \cdots \delta(\vec{a}_p - \vec{r}_{\beta_p})\, d\vec{r}_1 \cdots d\vec{r}_n . \quad (3.11')$$

The set of integers $\{\beta_1 \cdots \beta_p\}$ characterizes any permutation of p elements taken in n elements. The symbol $\sum_{\beta_1 \neq \cdots \neq \beta_p}$ indicates a sum over the $n^{[p]}$ permutations $\{\beta_1 \cdots \beta_p\}$ taken in n elements.

Thus, the *a posteriori* cpd can be given by the relation

$$P_p[\{\vec{a}_i\};t\mid\omega,\omega']=(1/n!)\,P_n[\{\vec{r}_i\};t\mid\omega,\omega']\times\sum_{\beta_1\neq\cdots\neq\beta_p}\prod_{i=1}^{p}\delta(\vec{a}_i-\vec{r}_{\beta_i})\,d\vec{r}_1\cdots d\vec{r}_n\,. \tag{3.11'}$$

Let us now consider the wave function $\psi(\{\vec{a}_i\},t;\omega,\omega')$ defined by Eq. (2.4), where ω and ω' are given. According to Eq. (2.3), $|\psi(\vec{a}_1,\cdots,\vec{a}_n;t;\omega,\omega')|^2$ is generally referred to as the "presence probability density" of the n particles in $\vec{a}_1,\ldots,\vec{a}_n$, at time t.[25] By considering Eqs. (2.4) and (3.6) we see that the presence probability density $|\psi(\{\vec{a}_i\},t;\omega,\omega')|^2$ is related to the cpd by

$$|\psi(\{\vec{a}_i\},t;\omega,\omega')|^2=[n!]^{-1}P_n[\{\vec{a}_i\};t\mid\omega,\omega']\,. \tag{3.12}$$

From Eq. (3.12) it follows that the function $|\psi(\{\vec{a}_i\},t;\omega,\omega')|^2$ may be considered as the probability density, defined only on a mathematical point of view, that a given particle is at $\vec{a}_1,\ldots,$ a given particle at $\vec{a}_n$, for given values of ω and ω'. This function is identical to the probability $\xi_n[\vec{a}_1,\ldots,\vec{a}_n;t\mid\omega,\omega']$ that we have already introduced.

From Eqs. (3.11′) and (3.12), we can derive the expression of the spatial *a posteriori* cpd in terms of the wave function

$$P_p[\{\vec{a}_i\};t\mid\omega,\omega']=\int|\psi(\{\vec{r}_i\},t;\omega,\omega')|^2\times\sum_{\beta_1\neq\cdots\neq\beta_p}\prod_{i=1}^{p}\delta(\vec{a}_i-\vec{r}_{\beta_i})\,d\vec{r}_1\cdots d\vec{r}_n\,. \tag{3.13}$$

According to the symmetry of ψ with respect to the space coordinates, this expression is equivalent to

$$P_p[\{\vec{a}_i\};t\mid\omega,\omega']=n^{[p]}\,|\psi(\{\vec{r}_i\},t;\omega,\omega')|^2\times\prod_{i=1}^{p}\delta(\vec{a}_i-\vec{r}_i)\prod_{j=1}^{n}d\vec{r}_j\,. \tag{3.14}$$

This expression could have been directly derived from Eqs. (3.10) and (3.12).

The *a priori* spatial cpd is the only one that can be measured on a beam where $N(\omega)$ and $\Phi_i(\omega_i')$ are random. We have

$$P_p[\{\vec{a}_i\};t]=\langle P_p[\{\vec{a}_i\};t\mid\omega,\omega']\rangle\,, \tag{3.15}$$

where the symbol $\langle\cdots\rangle$ refers to an ensemble average over ω and ω'.

From Eqs. (3.13) and (3.15), we derive

$$P_p[\{\vec{a}_i\};t]=\int|\psi(\{\vec{r}_i\},t;\omega,\omega')|^2\times\sum_{\beta_1\neq\cdots\neq\beta_p}\prod_{i=1}^{p}\delta(\vec{a}_i-\vec{r}_{\beta_i})\,d\vec{r}_1\cdots d\vec{r}_{N(\omega)}\,. \tag{3.16}$$

For the *a priori* cpd there is no simple relation between P_{p+1} and P_p, but from Eqs. (3.4) and (3.15) we deduce that

$$P_p[\{\vec{r}_i\},t)]\,d\vec{r}_1\cdots d\vec{r}_p=\langle N(\omega)^{[p]}\rangle_\omega\,, \tag{3.17}$$

where $\langle N(\omega)^{[p]}\rangle_\omega$ is the factorial moment of $N(\omega)$. (The symbol $\langle\cdots\rangle_\omega$ refers to an ensemble average over ω only.)

Equation (3.16) gives the most general formulation of a spatial cpd. However, to define completely the statistical properties of the detection process of the particles, we have to know a more general cpd, that is to say, the spatiotemporal cpd.

C. Spatiotemporal Conincidence Probabilities

Let us consider the expression of $P_p[\{\vec{a}_i\};t\mid\omega,\omega']$ given by Eq. (3.13). If we introduce in this expression the ket $|\psi)$, which has been defined in Sec. II, [Eq. (2.3′)], we can write Eq. (3.13′) in the following way:

$$P_p[\{\vec{a}_i\};t\mid\omega,\omega']=|\gamma(\omega,\omega')|^2\sum_k(\pm)^k P_n^k\prod_{i=1}^{n}(\Phi_{\alpha_i}|\,n^{[p]}\times\prod_{i=1}^{p}|a_i(t))(a_i(t)|\,I_{n-p}\sum_{k'}(\pm)^{k^1}P_{n_i}^{k'}\prod_{i=1}^{n}|\Phi_{\gamma_i})\,, \tag{3.18}$$

or

$$P_p[\{\vec{a}_i\};t\mid\omega,\omega']=(\psi|\,\mathcal{P}_p^n[\{\vec{a}_i\},t]\,|\psi)\,, \tag{3.18'}$$

where

$$\mathcal{P}_p^n[\{\vec{a}_i\},t]=n^{[p]}\prod_{i=1}^{p}|a_i(t))(a_i(t)|\,I_{n-p}\,. \tag{3.19}$$

The set of integers $\{\alpha_i\}$ and $\{\gamma_i\}$ are permutations of the ordered set $\{i\}$ which refers to the order of the n kets $|\ldots)$. The operator I_{n-p} is the identity operator in the states space of $n-p$ identical particles. The operator $|a_i(t))(a_i(t)|$ is a Heisenberg operator. It is equal to $U^\dagger(t,t_0)\,|a_i)(a_i|\,U(t,t_0)$, where $U(t,t_0)=\exp[-iH(t-t_0)/\hbar]$. The operator H is the Hamiltonian of the n particles. The operator $\mathcal{P}_p^n[\{a_i\},t]$ is a Heisenberg operator in the states space of the n identical particles.

We can assume that, in Eq. (3.18), every ket i is associated with a given particle i.[25] Moreover, as the particles are independent, we have

$$U(t,t_0)=\prod_{j=1}^{n}u_i(t,t_0).$$

The operator $u_i(t,t_0)$ is equal to $\exp[-ih_i(t-t_0)/\hbar]$, and the operator h_i is the Hamiltonian of particle i. Hence we can write

$$|a_i(t))(a_i(t)|=u_i^\dagger(t,t_0)\,|a_i)(a_i|\,u_i(t,t_0). \tag{3.19'}$$

Thus, every operator $|a_i(t))(a_i(t)|$ can be considered as a projection operator acting on particle

i. It localizes particle i at point $\vec{a}_i$ at time t, and the p-order coincidence measurement is described by the action of these p projection operators. It follows that any p-order coincidence measurement performed at different points $\vec{a}_i$ and *different* times t_i can be described by the action of the operator

$$\mathcal{P}_p^n[\{\vec{a}_i\};\{t_i\}] = n^{[p]}\prod_{i=1}^{p} |a_i(t_i))(a_i(t_i)| I_{n-p} , \quad (3.20)$$

where

$$|a_i(t_i))(a_i(t_i)| = u_i^\dagger(t_i,t_0)\,|a_i)(a_i|\,u_i(t_i,t_0). \quad (3.21)$$

Thus, we have

$$\mathrm{P}_p[\{\vec{a}_i\};\{t_i\}\,|\,\omega,\omega'] = |\gamma(\omega,\omega')|^2 \sum_k (\pm)^k \mathrm{P}_n^k \prod_{i=1}^{n} (\Phi_{\alpha_i}| n^{[p]} \prod_{i=1}^{p} |a_i(t_i))(a_i(t_i)| I_{n-p} \sum_{k'} (\pm)^{k'} \mathrm{P}_n^{k'} \prod_{i=1}^{n} |\Phi_{\gamma_i}). \quad (3.22)$$

The *a priori* cpd is given by

$$\mathrm{P}_p[\{\vec{a}_i\};\{t_i\}] = \langle |\gamma(\omega,\omega')|^2 \sum_k (\pm)^k \mathrm{P}_n^k \prod_{i=1}^{n} (\Phi_{\alpha_i}|\mathcal{P}_p^n[\{\vec{a}_i\};\{t_i\}] \sum_{k'} (\pm)^{k'} \mathrm{P}_n^{k'} \prod_{i=1}^{n} |\Phi_{\gamma_i})\rangle. \quad (3.23)$$

In this expression, the sum of permutations $\{\alpha_i\}$ can be replaced by $n!$ times the zero-order permutation, because of the identity of the stochastic functions $\Phi_i(\vec{r}, t, \omega_i')$. Furthermore, using Eq. (2.6), we can write

$$\mathrm{P}_p[\{\vec{a}_i\};\{t_i\}] = \langle D(\omega,\omega')^{-1} (\prod_{i=1}^{n} \Phi_i | n^{[p]} \prod_{i=1}^{p} |a_i(t_i)) \times (a_i(t_i)| \sum_{k'} (\pm)^{k'} \mathrm{P}_n^{k'} |\Phi_{\gamma_i})\rangle. \quad (3.24)$$

Finally, the detailed formulation of $\mathrm{P}_p[\{a_i\};\{t_i\}]$ is

$$\mathrm{P}_p[\{\vec{a}_i\};\{t_i\}] = \left\langle \frac{n^{[p]} \sum_k (\pm)^k \mathrm{P}_n^k \prod_{i=1}^{p} \Phi_i(\vec{a}_i,t_i;\omega_i') \Phi^*_{\alpha_i}(\vec{a}_i,t_i;\omega'_{\alpha_i}) \prod_{j=p+1}^{n} (\Phi_j,\Phi_{\alpha_j})}{\sum_{k'} (\pm)^{k'} \mathrm{P}_n^{k'} \prod_{i=1}^{n} (\Phi_i,\Phi_{\beta_i})} \right\rangle . \quad (3.25)$$

For any quantum system of particles described by random wave packets, the number of which is equally random, the detection process is determined in theory by Eq. (3.25). However, the expression of the cpd given by this equation is so complicated that no interesting general consideration can be made about the properties of the cpd. To obtain some simple results, we shall have to limit ourselves to certain types of beams, namely, to stationary and weak incoherent beams.

However, it is worth noting that when the functions Φ_i and Φ_j are orthogonal for any ω_i' and ω_j' $(i \neq j)$, Eq. (3.25) reduces to

$$\mathrm{P}_p[\{\vec{a}_i\};\{t_i\}] = \langle n^{[p]} \sum_k (\pm)^k \mathrm{P}_p^k \prod_{i=1}^{p} \Phi_i(\vec{a}_i,t_i;\omega_i') \times \Phi^*_{\alpha_i}(\vec{a}_i,t_i;\omega'_{\alpha_i})\rangle. \quad (3.25')$$

IV. STATIONARY AND WEAK INCOHERENT BEAMS

A. Definition

The beams we are studying in this section are stationary, weak, and incoherent. Moreover, they are assumed to be quasimonochromatic. They are defined by the following hypotheses:

(i) The wave packets $\Phi_i(\vec{r}, t; \omega_i')$ associated with the particles are stochastic functions with the same probability distribution. We assume that $\Phi_i(\vec{r}, t; \omega_i')$ can be written as $\Phi(\vec{r}, t-t_i)$. The point t_i can be considered as the "emission time" of the packet Φ_i. The point process consisting of the time instants t_i is without memory, and the number $N(\omega)$ of emission times t_i in a time interval T is a Poisson variable. Therefore, as it can easily be shown, the stochastic functions $\Phi_i(\vec{r}, t; \omega'_i)$ are *independent*.

(ii) Any wave packet $\Phi_i(\vec{r}, t; \omega'_i)$ can be written as

$$\Phi_i(\vec{r}, t; \omega_i') = \Phi(\vec{r}, t-t_i) = e^{i\hbar^{-1}[p_0 r - E_0(t-t_0)]} \times \int dp\, a(p) e^{i\hbar^{-1}[pr - E(p)(t-t_i)]} . \quad (4.1)$$

Let us call Δp the width of the function $a(p)$ and let ΔE be corresponding variation of $E(p)$. We assume that the beam is quasimonochromatic, or that

$$\Delta E \ll E_0 . \quad (4.2)$$

By definition, the coherence time is $\tau = h(\Delta E)^{-1}$, and the coherence length is $l = \tau v$, where v is the mean velocity of the particles. The coherence area σ is defined by the width of the wave packets in the directions y and z normal to the propagation direction x. In the following, to simplify the calculations, we shall consider only measurements on a beam whose section is of the order of the coherence area; we are not interested in the variations of Φ_i with y and z and we write Φ_i as $\Phi_i(x, t; \omega'_i)$.

(iii) The random functions $\Phi_i(x, t; \omega_i')$ and the random variable $N(\omega)$ are stationary in space and time.[26] Since a wave packet can only be stationary during a limited time interval T and on a limited length interval $L = vT$, where the spreading of the wave packet may be neglected, this implies that the dimensions of the stationary physical system S being considered are limited by T and L. As can easily be shown,[27] the time interval T is related to the coherence time τ by

$$T \ll (E_0/h)\,\tau^2 . \tag{4.3}$$

(iv) The number of particles that can be detected in the time interval T is equal to the number $N(\omega)$ of time instants t_i in this interval. This hypothesis means that for every wave packet emitted during the time interval T a particle corresponds which can be detected during the time interval T. This implies that T is much larger than τ, the time-width of the wave packets, and that the probability of a particle being emitted during a time τ at the beginning or the end of the time interval T is very small. This last condition is fulfilled if δ, the mean number of particles emitted during a time τ, is such that

$$\delta \ll 1 . \tag{4.4}$$

This quantity δ is the degeneracy parameter. Condition (4.4) is fulfilled for thermal light ($\delta \approx 10^{-3}$) and for the most powerful and monocinetic electron beams ($\delta \approx 10^{-5}$), but it is not fulfilled in the laser case and, as a consequence, in the experiments performed with "pseudothermal light"[8] (δ may be larger than 10^3). Finally, according to hypotheses (iii) and (iv), T must verify the two conditions

$$\tau \ll T \ll (E_0/h)\,\tau^2. \tag{4.5}$$

Both these conditions can be fulfilled because, according to Eq. (4.2), we have

$$\tau \ll (E_0/h)\,\tau^2.$$

B. Approximation

The preceding hypotheses bring very few simplifications to expression (3.25) of the cpd, but the problem can be considerably simplified by assuming, as GW did, that the functions Φ_i are "effectively orthogonal".[20] This assumption of "effective orthogonality" merely signifies that for any value of $N(\omega)$, the $N(\omega)$ functions Φ_i are orthogonal for any set of points ω_i', as can be verified by performing the calculations which lead to Eqs. (2.1b) and (2.26) of Ref. 20. Thus the assumption of GW is equivalent to condition (2.8) of the present paper. If this condition is fulfilled, the cpd's are given by Eq. (3.25′). So, it would be very convenient to make the same assumption as GW. However, before doing so, we must determine for what conditions such an assumption is valid and what its meaning is. In this subsection, we show that this assumption is necessarily an approximation. In fact, the functions Φ_i, being identical and independent, cannot be orthogonal for every set $\{\omega_i'\}$. In Sec. IV C, we show that the assumption of GW is nevertheless a good approximation if a certain condition limiting the density of the particles is fulfilled.

It can easily be shown that the Φ_i, verifying hypothesis (i), cannot be orthogonal for any value of $\{\omega_i'\}$. For this purpose, let us consider the random function $G(t_1, t_2;\ \omega_i',\ \omega_j')$, which is the simplest term in $D(\omega, \omega')$ identified with zero by GW:

$$G(t_1, t_2;\ \omega_i',\ \omega_j') = \int \Phi_i(x_1, t_1;\ \omega_i')\, \Phi_i^*(x_2,\ t_2;\ \omega_i') \times \Phi_j(x_2, t_2;\ \omega_j')\, \Phi_j^*(x_1, t_1; \omega_j')\,dx_1 dx_2. \tag{4.6}$$

This function is equal to zero if, and only if, the functions Φ_i and Φ_j are orthogonal ($i \neq j$). However, taking into account the fact that Φ_i and Φ_j are identical and independent and knowing that the order of the integration in space and of the mean value can be interchanged, we can write

$$G(t_1, t_2; \omega_i', \omega_j') = \int \left|\langle \Phi_i(x_1, t_1; \omega_i')\, \Phi_i^*(x_2, t_2; \omega_i')\rangle\right|^2 dx_1 dx_2. \tag{4.7}$$

The function $\langle G(t_1, t_2; \omega_i', \omega_j')\rangle$ is strictly positive. It follows that, when Φ_i and Φ_j are identical and independent, $G(t_1, t_2; \omega_i', \omega_j')$ is not identical to zero and that the functions Φ_i are not orthogonal with one another for every value of $\{\omega_i'\}$. Thus the random variable $D(\omega, \omega')$ is never equal to 1, and the assumption of GW is always an approximation.

C. Condition of Validity

We shall now establish the condition which allows us to consider the random variable $D(\omega, \omega')$ as nearly equal to 1. In what follows, the proposition "$D(\omega, \omega')$ is nearly equal to 1" is strictly equivalent to the relations

$$\langle D(\omega, \omega')\rangle = 1 \pm \epsilon,\ \ (\epsilon \ll 1), \tag{4.8}$$

$$\sigma_D^2 = \langle D^2(\omega, \omega')\rangle - \langle D(\omega, \omega')\rangle^2 = \epsilon' \quad (\epsilon \ll 1). \tag{4.9}$$

No condition is imposed upon higher-order moments of $D(\omega, \omega')$. The values of the quantities ϵ and ϵ' determine the range of error which is introduced by admitting condition (2.8).

We shall first determine the necessary and sufficient condition for $\langle D(\omega, \omega')\rangle$ to satisfy Eq. (4.8). Next we shall show that this condition is sufficient for σ_D^2 to fulfill Eq. (4.9)(Appendix B).

From Eq. (2.7), it follows that

$$\langle D(\omega, \omega')\rangle = 1 + \left\langle \sum_{k=1}^{N(\omega)-1} \mathrm{P}^k_{N(\omega)}(\pm)^k\right.$$

$$\times \int \prod_{i=1}^{N(\omega)} \Phi_i(x_i, t_i; \omega_i') \Phi^*_{\alpha_i}(x_i, t_i; \omega_i') dx_1 \cdots dx_{N(\omega)} \Big\rangle . \tag{4.10}$$

The quantity $P^k_{N(\omega)}$ is the sum of all the k-order permutations of $N(\omega)$ elements. Any k-order permutation $\{\alpha_i\}$ taken in $N(\omega)$ elements $[(N(\omega) > k]$ can be written as a product of l circular permutations of q_i elements ($2 \le q_i \le k$) and 1 circular permutation of one element, the sum of the orders of all these permutations being equal to k.[28] Since the order of a circular permutation of q elements is equal to $q-1$, for any positive integer q, we have

$$\sum_{i=1}^{l} (q_i - 1) = k$$

or

$$\sum_{i=1}^{l} q_i = k + l \quad . \tag{4.11}$$

The integer l depends on the permutation $\{\alpha_i\}$ under consideration and may vary from 1 to k. When $l = 1$, the permutation $\{\alpha_i\}$ is a k-order circular permutation. When $l = k$, the permutation $\{\alpha_i\}$ is a product of k circular permutations of two elements. The integers l and q_i being given, different k-order permutations can be written as a product of l circular permutations of q_i elements. We shall call them equivalent permutations. All these equivalent permutations give an equal contribution to $\langle D(\omega, \omega')\rangle$. This contribution is

$$\prod_{i=1}^{l} F_{q_i} = \prod_{i=1}^{l} \int \chi_{1,2}\, \chi_{2,3} \cdots \chi_{q_{i-1}, q_i}\, \chi_{q_i, 1}\, dx_1 dx_2 \cdots dx_{q_i} , \tag{4.12}$$

where

$$\chi_{1,2} = \langle \Phi_i(x_1, t; \omega_i') \Phi^*_i(x_2, t; \omega_i')\rangle . \tag{4.13}$$

As shown in Appendix A the function F_{q_i} is real and positive. The number of equivalent permutations is equal to the number of different sets of l circular permutations of q_i ($1 \le i \le l$) different elements that can be chosen in $N(\omega)$ elements. In other words, it is equal to the number of sets of l nonordered group of q_i different elements that can be taken in $N(\omega)$ elements, times the product of the numbers of the different circular permutations that can be respectively built with $q_1, q_2, \ldots, q_l$ elements. Thus the number of equivalent k-order permutations is

$$Q[k, \{q_i\}] = \binom{N(\omega)}{q_1} \binom{N(\omega) - q_1}{q_2} \cdots \times \binom{N(\omega) - q_1 - q_2 \cdots - q_{l-1}}{q_l} \prod_{i=1}^{l} A^{q_i - 1}_{q_i} \left[\prod_{a=2}^{k+l} (n_a !)\right]^{-1} . \tag{4.14}$$

The quantity $A^{q_i-1}_{q_i}$ is the number of circular permutations of q_i elements. We have

$$A^{q_i - 1}_{q_i} = (q_i - 1)! \; . \tag{4.15}$$

The integer n_a is the number of q_i equal to a. (The number a is a positive integer varying from 2 to $k + l$).

According to Eq. (4.14), we can write $\langle D(\omega, \omega')\rangle$ in the form

$$\langle D(\omega, \omega')\rangle = 1 + \Big\langle \sum_{k=1}^{N(\omega)-1} (\pm)^k \sum_{l=1}^{k} \sum_{\{q_i\}} \binom{N(\omega)}{q_1} \binom{N(\omega) - q_1}{q_2} \cdots \times \binom{N(\omega) - q_1 - q_2 \cdots - q_{l-1}}{q_l} \left[\prod_{a=2}^{k+l} (n_a !)\right]^{-1} \prod_{i=1}^{l} A^{q_i - 1}_{q_i} F_{q_i} \Big\rangle . \tag{4.16}$$

The sum $\sum_{\{q_i\}}$ is obtained over all the different sets of l nonordered positive integers larger than 1, the sum of which is $\sum_{i=1}^{l} q_i = k + l$.

From Eqs. (4.15) and (4.16) it follows that

$$\langle D(\omega, \omega')\rangle = 1 + \Big\langle \sum_{k=1}^{N(\omega)-1} (\pm 1)^k \sum_{l=1}^{k} \sum_{\{q_i\}} N(\omega)[N(\omega) - 1] \cdots \times [N(\omega) - k - l + 1] \left[\prod_{a=2}^{k+l} (n_a !) \prod_{i=1}^{l} q_i\right]^{-1} \prod_{i=l}^{l} F_{q_i} \Big\rangle . \tag{4.17}$$

Let us call $H(k, l)$ the positive function of k and l which is written as

$$H(k, l) = \sum_{\{q_i\}} \left[\prod_{a=2}^{k+l} (n_a !) \prod_{i=1}^{l} q_i\right]^{-1} \prod_{i=1}^{l} F_{q_i} , \tag{4.18}$$

and let us call $p(n)$ the probability that $N(\omega)$ will be equal to n. We can write

$$\langle D(\omega, \omega')\rangle = 1 + \sum_{n=0}^{\infty} p(n) \sum_{k=1}^{n-1} (\pm)^k \sum_{l=1}^{k} n(n-1) \cdots \times (n - k - l + 1)\, H(k, l). \tag{4.19}$$

After some calculations, we find that this expression is equivalent to

$$\langle D(\omega, \omega')\rangle = 1 + \sum_{k=1}^{\infty} (\pm)^k \sum_{l=1}^{k} H(k, l) \times \sum_{n=k+l}^{\infty} p(n)\, n(n-1) \cdots (n - k - l + 1) . \tag{4.20}$$

As $N(\omega)$ is a Poisson variable, we have

$$\langle D(\omega, \omega')\rangle = 1 + \sum_{k=1}^{\infty} (\pm)^k \sum_{l=1}^{k} H(k, l) \langle N(\omega)\rangle^{k+l} . \tag{4.21}$$

From Eqs. (4.8) and (4.21), it follows that the condition we are looking for is the condition necessary and sufficient (ns condition) for having

$$\sum_{k=1}^{\infty}(\pm)^k\sum_{l=1}^{k}H(k,l)\langle N(\omega)\rangle^{k+p}\ll 1. \qquad (4.22)$$

The ns condition for Eq. (4.22) to be satisfied for bosons (+sign) is a sufficient condition for Eq. (4.22) to be satisfied for fermions (−sign). So, the ns condition we are looking for is that required to get

$$\langle\Delta D\rangle=\sum_{k=1}^{\infty}(\pm)^k\sum_{l=1}^{k}H(k,l)\langle N(\omega)\rangle^{k+l}\ll 1. \qquad (4.23)$$

Let us study the function $H(k,l)$. It is shown in Appendix A that, if $q\geq 2$,

$$(\beta\tau/T)^{q-1}\leq F_q\leq(m\tau/T)^{q-1}\quad . \qquad (4.24)$$

In this expression m is a positive quantity of the order of 2, and the constant β is such that $0<\beta\leq m$. It is given (Appendix A) by

$$\int|\chi_{12}|^2dx_1dx_2=\beta(\tau/T).$$

From Eq. (4.24), it follows that

$$(\beta\tau/T)^k\leq\prod_{i=1}^{l}F_{q_i}\leq(m\tau/T)^k\quad , \qquad (4.25)$$

and from Eqs. (4.18) and (4.25), it follows that

$$\left(\frac{\beta\tau}{T}\right)^k\sum_{\{q_i\}}\left[\prod_{a=2}^{k+l}(n_a!)\prod_{i=1}^{l}q_i\right]^{-1}\leq H(k,l)\leq\left(\frac{m\tau}{T}\right)^k\sum_{\{q_i\}}\left[\prod_{a=2}^{k+l}(n_a!)\prod_{i=1}^{l}q_i\right]^{-1}. \qquad (4.26)$$

Moreover, we have

$$1/(k+l)<\sum_{\{q_i\}}\left[\prod_{a=2}^{k+l}(n_a!)\prod_{i=1}^{l}q_i\right]^{-1}<1\ . \qquad (4.27)$$

The first inequality is derived from the fact that the quantity $(k+l)^{-1}$ is one of the terms of the sum of positive terms

$$S(k,l)=\sum_{\{q_i\}}\left[\prod_{a=2}^{k+l}(n_a!)\prod_{i=1}^{l}q_i\right]^{-1}.$$

To establish the second inequality, it is sufficient to notice that the number of k-order permutations of q elements taken in $N(\omega)$ elements is smaller than the number of all the permutations of q elements taken in $N(\omega)$ elements. Thus we have

$$\sum_{\{q_i\}}Q[k,\{q_i\}]\,N(\omega)(N(\omega)-1)\cdots(N(\omega)-q+1), \qquad (4.28)$$

which gives

$$\sum_{\{q_i\}}\left[\prod_{a=2}^{k+l}(n_a!)\prod_{i=1}^{l}q_i\right]^{-1}<1.$$

From Eqs. (4.23), (4.26), and (4.27), it follows that

$$\sum_{k=1}^{\infty}\left(\frac{\beta\tau}{T}\right)^k\sum_{l=1}^{k}\frac{\langle N(\omega)\rangle^{l+k}}{l+k}<\langle\Delta D\rangle<\sum_{k=1}^{\infty}\left(\frac{3m\tau}{T}\right)^k\sum_{l=1}^{k}\langle N(\omega)\rangle^{l+k}. \qquad (4.29)$$

We see in this formula that the ns condition to have $\langle\Delta D\rangle\ll 1$ is

$$[\langle N(\omega)\rangle^2\tau/T]\ll 1\ , \qquad (4.30)$$

which can also be written as

$$\langle N(\omega)\rangle\,\delta\ll 1\ . \qquad (4.31)$$

Relation (4.31) is the ns condition in order that the approximation of GW might be used. It limits the intensity of the beam. It is fulfilled for weak beams only, such as thermal light beams $[\langle N(\omega)\rangle\delta\lesssim 10^{-2}]$ and electron beams $[\langle N(\omega)\rangle\delta\lesssim 10^{-6}]$. However, there are cases where the approximation of GW cannot be used because $\langle N(\omega)\rangle\,\delta$ is much larger than 1. Such is the case for pseudothermal light.[8] In this paper, we shall deal only with weak beams which respect condition (4.31).

Condition (4.31) allows us to write Eq. (3.25) in the following way:

$$P_p[\{a_i\};\{t_i\}]=\langle n^{[p]}\sum_k(\pm)^kP_n^k\prod_{i=1}^{p}\Phi_i(a_i,t_i;\omega_i')\times\Phi^*_{\alpha_i}(a_i,t_i;\omega'_{\alpha_i})\prod_{j=p+1}(\Phi_j,\Phi_{\alpha_j})\rangle. \qquad (4.32)$$

It can readily be shown that it follows from condition (4.31) that the sum of all the terms in which permutations occur within the scalar products can be neglected. As a consequence, Eq. (4.32) gives

$$P_p[\{a_i\};\{t_i\}]=\langle N(\omega)\rangle_\omega^p\times\left\langle\prod_{i=1}^{p}\Phi_i(a_i,t_i;\omega_i')\sum_k(\pm)^kP_p^k\Phi^*_{\alpha_i}(a_i,t_i;\omega'_{\alpha_i})\right\rangle_{\omega'}\ , \qquad (4.33)$$

a result which is identical to what we should obtain if the wave packets were orthogonal.

If we set

$$\eta_i(a,t;\omega_i')=\Phi_i(a,t;\omega_i')L^{1/2},$$

we can write $P_p[\{a_i\};\{t_i\}]$ in the following way:

$$P_p[\{a_i\};\{t_i\}]=\langle\rho\rangle^p\left\langle\prod_{i=1}^{p}\eta_i(a_i,t_i;\omega_i')\times\sum_kP_p^k(\pm)^k\eta^*_{\alpha_i}(a_i,t_i;\omega'_{\alpha_i})\right\rangle_{\omega'}\ , \qquad (4.34)$$

where $\langle\rho\rangle$ is the mean number of particles by unit of length

$$[\langle\rho\rangle=\langle N(\omega)\rangle L^{-1}].$$

Equation (4.34) can also be written as

$$P_p[\{a_i\};\{t_i\}]=\langle\rho\rangle^p\sum_kP_p^k(\pm 1)^k\times\prod_{i=1}^{p}\langle\eta_i(a_i,t_i;\omega_i')\eta_i^*(a_{\beta_i},t_i;\omega_i')\rangle_{\omega_i'}\ , \qquad (4.34')$$

or, by introducing the correlation function,

$$\gamma_{12} = \gamma_{21}^* = \langle \eta_i (a_1, t_1, \omega_i') \eta_i^* (a_2, t_2; \omega_i') \rangle, \qquad (4.35)$$

$$P_p[\{a_i\}; \{t_i\}] = \langle \rho \rangle^p \sum_k P_p^k (\pm 1)^k \prod_{i=1}^{p} \gamma_{i_{\beta_i}} . \qquad (4.34'')$$

This equation shows that in the case of a weak incoherent beam the cpd can be expressed in terms of the second-order correlation function of the wave packets. Moreover, in the case of fermions, we see that the cpd can be considered as the determinant of a matrice Γ_p:

$$\Gamma_p = \langle \rho \rangle^p \begin{pmatrix} \gamma_{11}\ \gamma_{12} \cdots & \gamma_{1j} & \cdots \gamma_{1p} \\ \gamma_{21}\ \gamma_{22} \cdots & \gamma_{2j} & \cdots \gamma_{2p} \\ \gamma_{j1} \cdots & \gamma_{jj} & \cdots \gamma_{jp} \\ \gamma_{p1} \cdots & \gamma_{pj} & \cdots \gamma_{pp} \end{pmatrix} . \qquad (4.36)$$

The cpd given by Eq. (4.34) or by Eq. (4.34″) defines the detection process of any beam of particles characterized by the hypotheses of Sec. IV and verifying condition (4.31).

It remains for us to study the properties of the point processes defined by these cpd. To begin with, in the present paper, we shall only point out some particular features of these processes.

V. DISCUSSION OF SOME PARTICULAR RESULTS

Several comments can be made about the expression (4.34) of $P_p[\{a_i\}; \{t_i\}]$.

(a) If we write Eq. (4.34) for $p=2$, we obtain

$$P_2[a_1 - a_2,\ t_1 - t_2] = \langle \rho \rangle^2 [1 + |\gamma_{12}|^2] , \qquad (5.1)$$

where γ_{12} is given by Eq. (4.35).

The purely temporal cpd $P_2[0, (t_1 - t_2 = t)]$ is given in Fig. 1 in the particular case where $\gamma(0, t_1 - t_2) = \exp[-|t_1 - t_2|/\tau]$. We have

$$P_2[0,\ t] = P_2(t/\tau) = \langle \rho \rangle^2 [1 - \exp(-2|t|/\tau)]. \qquad (5.2)$$

Equation (5.1) expresses the well-known bunching effect for bosons (+ sign), and it shows, in the fermion case (− sign), another effect that may be called the "antibunching effect." In the latter case, the *a posteriori* cpd $\langle \rho \rangle [1 - |\gamma_{12}|^2]$, to detect any particle at a given point and at a given time, when another particle has already been detected in a neighboring point and time (distance in space and time of the order of the coherence length and time), is smaller than the *a priori* cpd $\langle \rho \rangle$ to detect a particle at that point and at that time. This second-order effect is well known; it is the so-called Fermi-hole effect, and the principal interest of our results lies in the knowledge of *higher-order* fermion cpd.

(b) Equation (4.34) shows that the bunching effect, as well as the antibunching effect, arising

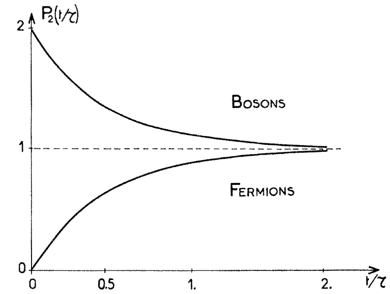

FIG. 1. Variation with respect to t/τ of the purely temporal second-order cpd $P_2(t/\tau)$ for bosons and fermions.

in a weak incoherent beam, comes from the terms corresponding to nonzero-order permutations. If the particles were distinguishable, these terms would not exist and we should have

$$P_p[\{a_i\}; \{t_i\}] = \langle \rho \rangle^p .$$

The detection process would be a Poisson process. Thus *it is because of their indistinguishability that the particles, which are assumed to be stochastically independent when emitted, are detected as bunched or antibunched.*

(c) The space and time variations of the p-order cpd given by Eq. (4.34) come from the indistinguishability of p particles. The existence of other particles appears only in the coefficient $\langle \rho \rangle^p$. Therefore, the actual system on which a p-order coincidence experiment is performed can be considered to result from the interaction of a system of p indistinguishable particles on which the measurement is performed, with a reservoir consisting of the $[N(\omega) - p]$ other particles.

(d) Equation (4.34) enables us to find all the well-known results for thermal photon fields.[29] The cpd given by Eq. (4.34) written with the + sign, are the cpd of a compound Poisson process. In fact, we know that if the set of the cpd of a point process is identical to the set of the moments of a stochastic positive function $\rho(x, t)$, this process is a compound Poisson process, the density of which is $\rho(x, t)$. It can be easily seen in Eq. (4.34) that the cpd are identical to the moments of

$$\rho(x, t) = \alpha(x, t)\, \alpha^*(x, t) , \qquad (5.3)$$

where $\alpha(x, t)$ is a Gaussian analytic signal[29] whose correlation function is

$$\langle \alpha(a_1, t_1)\, \alpha^*(a_2, t_2) \rangle = \langle \rho \rangle \gamma_{12} . \qquad (5.4)$$

For this purpose, let us consider the p-order moment $\langle \rho(a_1, t_1)\, \rho(a_2, t_2) \cdots \rho(a_p, t_p) \rangle$ which is equal to the $2p$-order moment $\langle \alpha^{2p} \rangle = \langle \alpha(a_1, t_1) \times \alpha^*(a_1, t_1)\, \alpha(a_2, t_2)\, \alpha^*(a_2, t_2) \cdots \alpha(a_p, t_p)\, \alpha^*(a_p, t_p) \rangle$ of the Gaussian function $\alpha(x, t)$. This $2p$-order moment of a Gaussian function is the sum of all the different *nonordered* nonzero products of p second-order moments of $\alpha(x, t)$. Since $\alpha(x, t)$ is an analytic signal, we know that[30]

$$\langle \alpha(a_1, t_1)\alpha(a_2, t_2)\rangle = 0, \quad \langle \alpha^*(a_1, t_1)\alpha^*(a_2, t_2)\rangle = 0 . \tag{5.5}$$

It follows that the sum of all the different nonordered nonzero products of p second-order moments is obtained by associating any $\alpha^*(a_i, t_i)$ with any $\alpha(a_j, t_j)$. We have

$$\langle \alpha^{2p}\rangle = \left\langle \prod_{i=1}^{p} \alpha_i(a_i, t_i) \sum_{k=0}^{p-1} \mathrm{P}_p^k \alpha^*_{\beta_i}(a_i, t_i)\right\rangle , \tag{5.6}$$

where P_p^k and $\{\beta_i\}$ have the meaning already defined in Sec. III.

From Eqs. (5.4) and (5.6), it follows that

$$\langle \alpha^{2p}\rangle = \langle\rho\rangle^p \prod_{i=1}^{p} \eta_i(a_i, t_i, \omega_i') \sum_{k=0}^{p-1} \mathrm{P}_p^k \eta^*_{\beta_i}(a_i, t_i, \omega'_{\beta_i}). \tag{5.7}$$

From Eqs. (5.3) and (5.7) we deduce that the cpd given by Eq. (4.34) are identical to the p-order moments of the stochastic function $\rho(x, t)$.

We must note that this stochastic function $\rho(x, t)$ has a physical interpretation only when the beam is strong enough. In the case of very weak beams, it appears to have only a mathematical meaning. The quantity $\rho(x, t)v\,dt$, according to the definition of $\rho(x, t)$, is equal to the number of particles passing through the section of the beam at time t during the time interval dt. This number can be measured only if the detector is fast enough to follow the fluctuations of $\rho(x, t)$, but another condition must be fulfilled for $\rho(x, t)$ to be mensurable and thus to have a physical interpretation. In fact, because of the quantum nature of the interaction, a detector would measure the density $\rho(x, t)$ only if the number of particles arriving to the detector during its resolution time is much larger than 1. If this condition is not satisfied, that is to say if the beam is very weak, it is only possible to count particles and impossible to measure any intensity.

(e) For fermions, several interesting features may be pointed out. Let us consider the one-dimensional stochastic process, a function of time only, that is obtained when all the coincidence measurements are made at the same point in space:

(i) We see in Eq. (4.34) that, for fermions as well as for bosons, if the coherence time is very small compared with all the times introduced by these measurements, the detection process is a Poisson process. It is the only case that has been attained experimentally. In fact, the best fermion sources that could be used for coincidence experiments are point-cathode electron sources, because they are monocinetic and powerful.[31] Even these sources, which give a coherence area large enough to allow good measurements, give a coherence time of the order of 10^{-13} sec. We know that the electronic detection devices we might use are generally integrating over a time of the order of 10^{-9} sec. Consequently, they would smooth out the effects that we would like to observe; we meet the same problem with coincidence experiments on white light.

(ii) In what follows, we assume that the measurements introduce times of the order of τ; we obtain theoretical results that cannot be verified with the present experimental devices.

We assume that the time correlation function $\gamma_{12} = \langle\eta(a, t_1)\eta^*(a, t_2)\rangle$ is such that $\gamma_{13} = \gamma_{12}\gamma_{23}$, if $t_1 \leq t_2 \leq t_3$. Thus it can be written as

$$\gamma_{12} = \exp[-|t_1 - t_2|/\tau] . \tag{5.8}$$

In the expression $\mathrm{P}_p(a, t_1; \cdots ; a, t_p)$ we can always assume that $t_1 \leq t_2 \leq \cdots \leq t_p$. Then after some calculations, it can be shown[32] that, when Eq. (5.8) is satisfied, we have

$$\mathrm{P}_p(a, t_1, \cdots a, t_p) = \langle\rho\rangle^p \prod_{i=1}^{p-1} (1 - e^{-2\Delta t_i/\tau}), \tag{5.9}$$

where $\Delta t_i = t_{i+1} - t_i$.

The quantity Δt_i is the waiting time between two successive detections.

The relation (5.9) is a sufficient condition for the stochastic process to be a renewal process, as has been shown by Macchi.[21] A renewal process may be defined as follows: It is a stochastic process in which the successive waiting times are mutually independent random variables.[33] In our case, this means that the waiting time Δt_i between two successive detections does not depend on the preceding detections. It must be emphasized that this result is quite unexpected. In fact, it depends on the weakness of the beam density ($\langle\langle N\rangle\rangle\, \delta \ll 1$), but this condition is not sufficient. It is only for the particular shape of the correlation function given by Eq. (5.8) that the independence between successive waiting times occurs.

The renewal process is characterized by the probability $p(t)$ that the waiting time will be equal to t. This probability can be calculated from the expression of $\mathrm{P}_p[\{a_i\}; \{t_i\}]$. We have

$$p(t) = 2\langle I\rangle (1 - 2\langle I\rangle\tau)^{-1/2} e^{-t/\tau} \sinh[t(1 - 2\langle I\rangle\tau)^{+1/2}/\tau]. \tag{5.10}$$

In this expression, we have introduced the mean number of particles passing through the section of the beam per time unit $\langle I\rangle$ which is given by the relation $\langle I\rangle = \langle\rho\rangle v$.

We notice that for $p(t)$ to be a probability, it is necessary that $2\langle I\rangle\tau \leq 1$. This is merely the expression of the Pauli exclusion principle. When $\tau \to 0$, the process becomes a Poisson process, and we have $p(t) \to p_0(t) = \langle I\rangle \exp[-\langle I\rangle t]$.

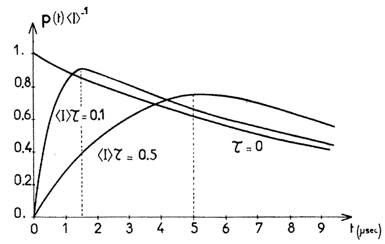

FIG. 2. Variation with the time t of the probability $p(t)$ defined by Eq. (5.10), for a given value $\langle I\rangle = 10^5$ particles/sec, and several values of τ. The two limiting cases, $\langle I\rangle\tau = 0.5$ and $\tau = 0$, are drawn. The curve representing $p(t)$ for the possible values of τ varies between these two limits, as shown for $\langle I\rangle\tau = 0.1$.

As can easily be calculated, the probability $p(t)$ is maximum for a value t_0 of the time t. This value t_0 depends on the coherence time τ and the mean intensity $\langle I\rangle$. We find that $\partial t_0/\partial\tau > 0$ and $\partial t_0/\partial\langle I\rangle < 0$, results which are easily understood. Moreover, for any value of $\langle I\rangle$, satisfying the inequality $2\langle I\rangle\tau \leqslant 1$, we have the relation $t_0 \geqslant \tau$. The larger the quantity $\langle I\rangle\tau$, the smaller the difference $(t_0 - \tau)$. For the limiting case where $2\langle I\rangle\tau = 1$, we have $t_0 = \tau$. The function $p(t)$ is drawn on Fig. 2 for a given value of $\langle I\rangle$ and several values of τ.

VI. CONCULSION

The study of the statistical properties of a set of particles in the wave-packet formalism, in the most general conditions, yields results which are too complicated to be easily used. Some conditions (stationarity, incoherence, weakness of density) must be fulfilled in order for the difficulties in this formalism arising from the nonorthogonality of the wave packets to be solved.

Under these conditions, the wave-packet formalism gives, in a simple way, several interesting new results. In particular, it shows how bunching and antibunching effects come from the detection of indistinguishable particles. It enables us to determine the detection process of a weak beam of incoherent fermions.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor B. Picinbono for his support throughout this work. We are also indebted to Professor C. Cohen-Tannoudji for very stimulating discussions, and to Doctor O. Macchi for her cooperation so necessary for the study of certain aspects of the fermion detection process.

APPENDIX A

Let us consider the purely spatial correlation function $\gamma(x_1 - x_2)$ such that

$$\gamma(x_1 - x_2) = \langle \eta_i(x_1, t)\eta_i^*(x_2, t)\rangle = \gamma^*(x_2 - x_1). \quad \text{(A1)}$$

The functions $\eta_i(x, t)$ are identical and stationary stochastic functions. They are defined by the relation $\eta_i(x, t) = L^{1/2}\Phi_i(x, t)$. The quantities L and $\Phi_i(x, t)$ are defined in Sec. IV.

We set

$$\gamma(x_1 - x_2) = \int A(k)\, e^{2i\pi k(x_1 - x_2)}\, dk. \quad \text{(A2)}$$

The function $A(k)$, being the Fourier transform of a correlation function, is real and non-negative.[34] It is assumed to have a finite maximum $A(k_0) = A_M$ and to vary monotonically before and after this maximum, as is the case if $A(k)$ is the spectral density of a quasimonochromatic field. Let us introduce the quantity Δk which is related to the width l of the correlation function $\gamma(x_1 - x_2)$ by

$$\Delta k = l^{-1}, \quad \text{(A3)}$$

and let us call $A_M m^{-1}$ the value of the function $A(k)$ for $k = k_0 \pm \Delta k/2$. (The value of m is of the order of 2.)

From the normalization condition $(\Phi_i, \Phi_i) = 1$, it follows that

$$\gamma(0) = \int A(k)\, dk = 1. \quad \text{(A4)}$$

From Eq. (A3) it follows that

$$\int A(k)\, dk \geqslant A_M m^{-1} l^{-1}, \quad \text{(A5)}$$

and from Eqs. (A4) and (A5), we deduce that

$$A_M \leqslant ml, \quad \text{(A6)}$$

and that

$$\int A^2(k)\, dk \leqslant ml. \quad \text{(A7)}$$

Thus we can set

$$\int |\gamma(x_1 - x_2)|^2\, dx_2 = \int A^2(k)\, dk = \beta l, \quad \text{(A8)}$$

where $\beta \leqslant m$.

We now show that, if the preceding conditions are fulfilled, we have

$$(\beta l)^{p-1} \leqslant \mathcal{C}_p \leqslant (ml)^{p-1}, \quad \text{(A9)}$$

where we have set

$$\mathcal{C}_p = \int_{-\infty}^{+\infty} \gamma(x_1 - x_2)\gamma(x_2 - x_3)\cdots\gamma(x_i - x_j)\gamma(x_j - x_k)\cdots \times\gamma(x_p - x_1)\, dx_2\, dx_3 \cdots dx_p.$$

We have

$$\mathcal{C}_p = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(k_1)\, e^{2i\pi k_1(x_1 - x_2)} A(k_2)\, e^{2i\pi k_2(x_2 - x_3)}\cdots \times A(k_p)\, e^{2i\pi k_p(x_p - x_1)}\, dx_2 - dx_p\, dk_1\, dk_2 \cdots dk_p \quad \text{(A10)}$$

or

$$\mathcal{C}_p = \int_{-\infty}^{+\infty} A(k_1)A(k_2)\cdots A(k_p)\,\delta(k_2 - k_1)\,\delta(k_3 - k_2)\cdots$$

$$\times\ \delta(k_p - k_1)\,dk_1\,dk_2\cdots dk_p, \qquad \text{(A11)}$$

that is to say,

$$\mathcal{C}_p = \int_{-\infty}^{+\infty} A^p(k)\,dk\ . \qquad \text{(A12)}$$

This relation still holds if, instead of integrating from $-\infty$ to $+\infty$, we integrate over an interval L very large compared to l, as we do in Sec. IV.

Therefore, we have to establish the relation

$$(\beta l)^{p-1} \leq \int_{-\infty}^{+\infty} A^p(k)\,dk \leq (ml)^{p-1}\ . \qquad \text{(A13)}$$

We have

$$\int A^p(k)\,dk \leq \int A^{p-1}(k) A_M\,dk \leq ml \int A(k)^{p-1}\,dk. \qquad \text{(A14)}$$

Since $\int A^2(k)\,dk \leq ml$, it follows that

$$\int A^p(k)\,dk \leq (ml)^{p-1}\ . \qquad \text{(A15)}$$

To show that $(\beta l)^{p-1} \leq \int A^p(k)\,dk$, we use the Tschebycheff inequality[35]: If $B(k)$ and $C(k)$ are two functions of k such that, for every k_1 and k_2, $[B(k_1) - B(k_2)][C(k_1) - C(k_2)] > 0$, and if $P(k)$ is a probability density, we can write

$$\int_{-\infty}^{+\infty} P(k)B(k)\,dk \int_{-\infty}^{+\infty} P(k)C(k)\,dk \leq \int_{-\infty}^{+\infty} P(k)B(k)C(k)\,dk. \qquad \text{(A16)}$$

Setting $B(k) = A(k)$, $P(k) = A(k)$, and $C(k) = A^{p-2}(k)$, we obtain

$$\int A^2(k)\,dk \int A^{p-1}(k)\,dk \leq \int A^p(k)\,dk, \qquad \text{(A17)}$$

which, according to Eq. (A8), can be rewritten as

$$\beta l \int A^{p-1}(k)\,dk \leq \int A^p(k)\,dk. \qquad \text{(A18)}$$

Since we have $\int A^2(k)\,dk = \beta l$, it follows from Eq. (A18) that

$$(\beta l)^{p-1} \leq \int A^p(k)\,dk \quad \text{(QED)}.$$

Let us now consider the quantity

$$F_p = \int \chi(x_1 - x_2)\cdots\chi(x_i - x_j) \times \chi(x_j - x_k)\cdots\chi(x_p - x_1)\ dx_1\,dx_2\cdots dx_p,$$

where

$$\chi(x_1 - x_2) = L^{-1}\gamma(x_1 - x_2) = \langle \Phi_i(x_1, t)\ \Phi_i^*(x_2, t)\rangle.$$

From Eq. (A9), we deduce that

$$(\beta l L^{-1})^{p-1} \leq F_p \leq (ml L^{-1})^{p-1}, \qquad \text{(A19)}$$

a relation which is identical to Eq. (4.24).

APPENDIX B

This appendix shows that σ_D^2 verifies Eq. (4 9) if $\langle D\rangle$ verifies Eq. (4.8).

According to Eq. (2.7), $D(\omega, \omega')$ is given by the relation

$$D(\omega, \omega') = 1 + \sum_{k=1}^{N(\omega)-1} P^k_{N(\omega)}\,(\pm)^k f(\{\alpha_j\})\ , \qquad \text{(B1)}$$

where

$$f(\{\alpha_j\}) = \prod_{j=1}^{N(\omega)} \int \Phi_j(x_j)\,\Phi^*_{\alpha_j}(x_j)\,dx_j, \qquad \text{(B2)}$$

and $\{\alpha_j\}$ is any k-order permutation of $\{1, 2, \cdots N(\omega)\}$. Expressing $\{\alpha_j\}$ as a product of l circular permutations of q_i given elements $(1 \leq i \leq l)$, we can set

$$f(\{\alpha_i\}) = \prod_{i=1}^{l} F_{q_i}\ , \qquad \text{(B3)}$$

where

$$\prod_{i=1}^{l} q_i - l = q - l = k,$$

and F_{q_i}, integral of a circular permutation of q_i functions Φ,[36] is given by the expression

$$F_{q_i} = \int \Phi_1^i(x_1^i, t)\,\Phi_1^{i*}(x_2^i, t)\,\Phi_2^i(x_2^i, t)\cdots \times \Phi^i_{q_i}(x^i_{q_i}, t)\,\Phi_1^{i*}(x_1^i, t)\,dx_1^i\cdots dx^i_{q_i}\ . \qquad \text{(B3}'\text{)}$$

Therefore, $D(\omega, \omega')$ can be written as

$$D(\omega, \omega') = 1 + \sum_{k=1}^{N(\omega)-1} (\pm)^k \sum_{l=1}^{k} \sum_{\{q_i\}} \mathcal{S}(\{q_i\}) \prod_{i=1}^{l} F_{q_i}. \qquad \text{(B4)}$$

The symbol $\mathcal{S}(\{q_i\})$ indicates that we take the sum over all the equivalent permutations $\prod_{i=1}^{l} F_{q_i}$, where l and q_i are given. Let us note that we have

$$\langle F_{q_i}\rangle = F_{q_i}\ . \qquad \text{(B5)}$$

where F_{q_i} is the quantity defined by Eq. (4.12).

By using expression (4.1) of $\Phi_i(r, t, \omega_i')$, we can show that Eq. (B4) is equivalent to the relation

$$F_{q_i} = L^{q_i}\,\chi(t_1^i - t_2^i)\,\chi(t_2^i - t_3^i)\cdots\chi(t^i_{q_i} - t_1^i)\ , \qquad \text{(B6)}$$

where $\chi(t - t')$ is the purely temporal correlation function of any wave packet Φ_k, and t_j^i the emission time of the wave pack Φ_j^i. We have

$$\chi(t_j^i - t^i_{j+1}) = \langle \Phi_k(x, t_j^i; \omega_k')\,\Phi_k^*(x, t^i_{j+1}; \omega_k')\rangle = L^{-1}\int \Phi_j^i(x, t)\,\Phi_{j+1}^{i*}(x, t)\,dx\ . \qquad \text{(B7)}$$

Let us now consider σ_D^2. We have

$$\sigma_D^2 = \langle D^2\rangle - \langle D\rangle^2 = \sum_{k=1}^{N(\omega)-1} \sum_{k'=1}^{N(\omega)-1} (\pm)^{k+k'}\,P^k_{N(\omega)}\,P^{k'}_{N(\omega)} \prod_{i=1}^{l} F_{q_i} \prod_{i'=1}^{l'} F_{q_{i'}}$$

$$-\left\langle \sum_{k=1}^{N(\omega)-1} (\pm)^k P^k_{N(\omega)} \prod_{i=1}^{l} F_{q_i} \right\rangle$$

$$\times \left\langle \sum_{k'=1}^{N(\omega)-1} (\pm)^{k'} P^{k'}_{N(\omega)} \prod_{i'=1}^{l'} F_{q_{i'}} \right\rangle. \quad \text{(B8)}$$

In this expression, the terms in which all the functions Φ_j^i are different from the functions $\Phi_j^{i'}$ give a zero contribution.

Let us now show that the terms in which $r(1 \leq r \leq \inf(q, q'))$ functions Φ_j^i are identical to r functions $\Phi_j^{i'}$ are very small compared to 1. We shall call $\hat{\Phi}_s$ these r identical functions $(1 \leq s \leq r)$. We consider a term

$$\prod_{i=1}^{l} F_{q_i} \prod_{i'=1}^{l'} F_{q_{i'}},$$

where r_i functions $\hat{\Phi}_{s_i}$ appear in F_{q_i}, and $r_{i'}$ functions $\hat{\Phi}_{s_{i'}}$ appear in $F_{q_{i'}}$. We have

$$\sum_{i=1}^{l} r_i = \sum_{i'=1}^{l'} r_{i'} = r \quad . \quad \text{(B9)}$$

We call $h_i(h_{i'})$ the number of integrals F_{q_i} $(F_{q_{i'}})$, where there is at least one function $\hat{\Phi}_s$.
We have

$$h_i \leq \inf(r, l); \quad h_{i'} \leq \inf(r, l') \quad . \quad \text{(B10)}$$

By using the relation

$$(1/T^{p-1}) \int L\chi(t_1 - t_2) L\chi(t_2 - t_3) \cdots$$

$$\times L\chi(t_p - t_{p+1})\, dt_2 dt_3 \cdots dt_p \leq (m l/L)^{p-1} L\chi(t_1 - t_{p+1}), \quad \text{(B11)}$$

which can be deduced from Eqs. (A2) and (A15) of Appendix A, we obtain

$$\langle F_{q_i} \rangle_{-s_i} \leq (m l/L)^{q_i - r_i} L\chi(t_1 - t_2) L\chi(t_2 - t_3) \cdots$$

$$\times L\chi(t_{s_i} - t_{s_i+1}) \cdots L\chi(t_{r_i} - t_1), \quad \text{(B12)}$$

where t_{s_i} are the emission times of the r_i functions $\hat{\Phi}_{s_i}$. The symbol $\langle \cdots \rangle_{-s_i}$ indicates that we take the ensemble average of F_{q_i} over the functions Φ_j^i different from $\hat{\Phi}_{s_i}$.

Therefore, it follows from Eq. (B12) that

$$\langle f(\{\alpha_i\}) \rangle_{-s} \leq (ml/L)^{q-r} L^r \prod_{i=1}^{l} \prod_{s_i=1}^{r_i} \chi(t_{s_i} - t_{s_i+1}), \quad \text{(B13)}$$

and that

$$\langle f(\{\alpha_i\}) f(\{\alpha_{i'}\}) \rangle \leq (ml/L)^{q+q'-2r} (L^{2r}/T^r)$$

$$\int \prod_{i=1}^{l} \prod_{s_i=1}^{r_i} \prod_{i'=1}^{l'} \prod_{s_{i'}=1}^{r_i'} \chi(t_{s_i} - t_{s_i+1}) \chi(t_{s_{i'}} - t_{s_{i'}+1})\, dt_{s_i} \, . \quad \text{(B14)}$$

According to Eq. (A4), we have the relation $\chi(t_1 - t_2) \leq 1/L$. Thus we can write Eq. (B14) in the form

$$\langle f(\{\alpha_i\}) f(\{\alpha_{i'}\}) \rangle$$

$$\leq \left(\frac{ml}{L}\right)^{q+q'-2r} \frac{L^r}{T^r} \int \prod_{i=1}^{l} \prod_{s_i=1}^{r_i} \chi(t_{s_i} - t_{s_i+1})\, dt_{s_i} \, . \quad \text{(B15)}$$

By using the relation

$$\frac{L^{r_i}}{T^{r_i}} \int \chi(t_1 - t_2) \chi(t_2 - t_3) \cdots \chi(t_{r_i} - 1)\, dt_1 \cdots dt_r$$

$$\leq (ml/L)^{r_i - 1}, \quad \text{(B16)}$$

which can be deduced from Eq. (A9), we obtain

$$\langle f(\{\alpha_i\}) f(\{\alpha_{i'}\}) \rangle \leq (m\,l/L)^{q+q'-r-h} \quad , \quad \text{(B17)}$$

where $h = \inf(h_i, h_{i'})$.

Then, by using the method which has enabled us to deduce Eqs. (4.21) and (4.27) from Eq. (4.10), we deduce from Eqs. (B8) and (B17) that

$$\sigma_D^2 \leq \sum_{k,k'=1}^{+\infty} \sum_{l=1,l'=1}^{kk'} \sum_{r=1}^{\inf(q,q')} \langle N(\omega) \rangle^{q+q'-r} \left(\frac{ml}{L}\right)^{q+q'-r-h}. \quad \text{(B18)}$$

The first two terms of σ_D^2, for $k = k' = 1$, are $\langle N(\omega) \rangle^2 \times (ml/L)$ and $\langle N(\omega) \rangle^3 (ml/L)^2$.

From the relations between q, q', l, l', r, and h, we deduce that

$$q + q' - r < 2(q + q' - r - h), \quad . \quad \text{(B19)}$$

Therefore, since the conditions $l/L \ll 1$ and $N(\omega)^2 l/L \ll 1$ are assumed to be fulfilled, Eq. (B18) shows that σ_D^2 is very small compared to 1.

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Correlation-Function Method for the Transport Coefficients of Dense Gases. III. Bulk Viscosity*

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(Received 22 May 1970)

In analogy with the work of Kawasaki and Oppenheim on the shear viscosity, the density expansion of the bulk viscosity is obtained for gas systems which interact with attractive forces and repulsive forces. In the case of repulsive forces, the first term in the density expansion of the bulk viscosity is of order ρ^2. In the case of attractive forces with bound states, the density expansion possesses a term of order ρ.

I. INTRODUCTION

Over the past few years several researchers employing various techniques have investigated the density expansion of the coefficient of bulk viscosity. Choh and Uhlenbeck first attacked the problem using the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy of equations and showed explicitly that for a hard-sphere gas, the bulk viscosity was zero to at least order ρ^2 in the density.[1] Several years later Garcia-Colin,[2] using a kinetic approach, and Ernst,[3] using a correlation-function approach, showed that for systems with repulsive interactions the bulk viscosity is at least of order ρ^2.

We now wish to confirm and extend the results already obtained for the bulk viscosity. We first