Radiation with Solids, edited by Adli Bishay (Plenum, New York, 1967), pp. 693–701. I am obliged to N. A. Kurnit for the objections to the sorts of measurements indicated in Sec. V B. He also pointed out that the value of T_1 probed by this measurement is not necessarily the same as the effective T_1 measured by the stimulated echo. For a discussion of this point, see the paper by these authors in the same source as Ref. 17. In L. O. Hocker, M. A. Kovacs, C. K. Rhodes, G. W. Flynn, and A. Javan [Phys. Rev. Letters 17, 233 (1966)], various phenomena involving the relaxation processes in CO_2 are investigated using pulse techniques and fluorescence measurements. These phenomena are involved in producing the lifetime T_1 used in the text.

[30]The fact that the second output pulse is independent of direction strongly indicates that the individual (perhaps infinitesimally thin) segments of a complex system commute with one another provided they conform to the restrictions given in the text (see Sec. V A). This commutability can be proved using rate equations, and also for more general T_2 provided that the line shapes are the same. This suggests, in turn, the following theorem which can be proved using rate equations and which has been confirmed to a limited extent using numerical techniques. One starts with two different systems A and B which conform to the restrictions indicated in Sec. V A, but which may be pumped in very different ways. A pulse $\mathcal{E}_1$ interacts with these systems producing output pulses $(\mathcal{E}_1)^A_{\text{out}}$ from A and $(\mathcal{E}_1)^B_{\text{out}}$ from B and new systems A' and B' that results from this interaction. Then, it is sufficient that $(\mathcal{E}_1)^A_{\text{out}} = (\mathcal{E}_1)^B_{\text{out}}$ in order that a second pulse $(\mathcal{E}_2)_{\text{in}}$ interacts with A' and B' to produce outputs $(\mathcal{E}_2)^{A'}_{\text{out}} = (\mathcal{E}_2)^{B'}_{\text{out}}$. This result holds independently of the direction of the pulses, and implies that a series of N pulses would pass through an amplifier or an attenuator (it is now essential that $T_2 \ll \Delta t$) producing outputs that are independent of the relative directions of the pulses. Provided one ignores noise, dispersion, scattering losses and stability requirements, one can conclude from the above that any steady-state solution of the unidirectional ring laser that is describable as a single, well-resolved ultrashort pulse will also be a solution of some suitably chosen linear laser.

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Model for the Quantum Scattering of a He^3 Impurity from a Rectilinear Vortex in Liquid He II

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A model is given that describes the quantum scattering of a He^3 impurity from a rectilinear vortex in liquid He II; the principal assumption is the existence of an impurity wave function that satisfies a Schrödinger equation. Various possibilities for the impurity-vortex interaction are discussed. For certain interactions, the theoretical results are consistent with the experimental work of Rayfield and Reif provided spatial variations in the superfluid density are considered. A T-matrix formalism is also developed and applied to the vortex scattering of impurities, as well as phonons and rotons. The results are compared to existing theoretical and experimental work; the main discrepancies occur for roton scattering.

I. INTRODUCTION

The scattering of quasiparticles (phonons, rotons, and He^3 impurities) by vortices in liquid He II has been observed in various experiments.[1–4] The main purpose of this paper is to derive and investigate a formalism that describes the quantum scattering of a He^3 quasiparticle from a quantized rectilinear vortex in superfluid helium. Models already exist which have been applied to the corresponding classical scattering problem.[5–8] The basic model is presented in Sec. II; it assumes the existence of a quasiparticle wave function that satisfies a Schrödinger equation. In Sec. III, the model is applied to impurity scattering when the superfluid density ρ_s is constant. Various possibilities for the impurity-vortex interaction are discussed. Some peculiarities related to the Bohm-Aharonov problem occur[9]; the scattering amplitude appears to diverge as an infinite series in partial waves, and the incident state is found to consist of a plane wave modified by a phase factor. The approximate effects of spatial variations in ρ_s are considered in Sec. IV. In Sec. V, the frictional force is derived for the scattering results of Secs. I–IV; a comparison is made to the experimental work of Rayfield and Reif.[4] Finally, in Sec. VI, a T-matrix formalism is developed and applied to the vortex scattering of He^3 impurities, phonons, and rotons. Results for constant ρ_s are derived. In the case of impurities, these results agree with those of Sec. III; for phonons and rotons, the results are compared to existing theoretical[10–14] and experimental work.[1–4]

II. BASIC MODEL

The model used to describe the quantum scattering of a He^3 quasiparticle from a quantized rectilinear vortex in liquid He II is based on the assumption that all information can be obtained from a quasiparticle wave function $\Psi(\vec{R}, t)$ that satisfies the Schrödinger equation ($\hbar = 1$)

$$[i\partial/\partial t - H(-i\nabla, \vec{R})]\Psi(\vec{R}, t) = 0 \ . \quad (1)$$

The wave function and its first derivatives must be periodic in the angular variables, regular at the origin of the force, and continuous everywhere. In addition, the boundary conditions for scattering require that $\Psi(\vec{R}, t)$ asymptotically consists of an incident wave [corresponding to an initial state with momentum $\vec{P}$ and energy $E(P)$, normalized to one particle per unit volume] and an outgoing scattered wave. The quantity $H(\vec{P}, \vec{R})$ is the analogous classical Hamiltonian symmetrized in momentum-dependent operators. In the absence of any interactions,[15]

$$H = E_0 + (2m^*)^{-1} P^2 \ . \quad (2)$$

The effective mass m^* is found from experiment to range from 2.2 to 2.8 He^3 masses.[16,17]

It will also be assumed that when no impurities are present, the superfluid state consists of a normal fluid with zero velocity $\vec{v}_n$ and constant density ρ_n, and a superfluid with velocity $\vec{v}_s$ and density ρ_s describing a quantized rectilinear vortex. In terms of a *preferred* cylindrical coordinate system (r, ϕ_r, z) (origin on the vortex axis and z axis oriented along the direction of vortex vorticity),[18]

$$\vec{v}_s = (\bar{\kappa}/r)\hat{\phi}_r, \quad \rho_s \approx \rho_s^0 r^2/(r^2 + a^2) \ . \quad (3)$$

The quantity a is a core radius of order one angstrom, the constant ρ_s^0 is the superfluid density far from the vortex core, and $\bar{\kappa}$ is $n\hbar/m_4$ (n is a positive integer, m_4 is the mass of a He^4 atom, and $\hbar$ is Planck's constant divided by 2π). The existence of such a superfluid state is an oversimplification; the normal fluid, viewed as a gas of phonons and rotons, is certainly influenced by the vortex.

III. SCATTERING FOR CONSTANT DENSITY

It is necessary to determine the Hamiltonian for the classical scattering problem in order to obtain $H(\vec{P}, \vec{R})$. In general, this determination is difficult. However, if the superfluid density were constant and the impurity corresponded to a spherical point particle on which no viscous forces act, the classical Hamiltonian would be[8]

$$H = E_0 + (2m^*)^{-1} (\vec{P} + \alpha_1 m^* \vec{v}_s)^2 - \tfrac{1}{2} \alpha_2 m^* \vec{v}_s^{\,2}, \quad (4)$$

$$\text{where} \quad \alpha_1 = \alpha_2 = 3\frac{\rho_s^0}{\rho^0}\frac{\delta m}{m^*}, \quad \text{(classical)}. \quad (5)$$

Here ρ_s^0 and ρ^0 are the superfluid and total fluid densities, δm is $m^* - m_3$, and m_3 is the mass of a He^3 atom. Thus, at least when ρ_s is nearly constant, Eq. (4) gives an approximate expression for $H(\vec{P}, \vec{R})$. However, other possibilities exist. On the basis of Galilean relativity, Bardeen, Baym, and Pines[17] (BBP) have also predicted a Hamiltonian of form (4) with constants

$$\alpha_1 = \alpha_2 = \delta m/m^*, \quad (\text{BB}\vec{\text{P}}) \ . \quad (6)$$

Further, Eq. (4) can be derived with constants

$$\alpha_1 = \alpha_2 = 1, \quad (\vec{P} \cdot \vec{v}_s) \quad (7)$$

by applying to Eq. (1) a Galilean transformation with velocity $\vec{v}_s$ ($\vec{v}_s$ slowly varying) from the particle rest frame to the coordinate system fixed on the vortex. In both these latter cases, the density ρ_s is essentially constant.

The scattering results for Hamiltonian (4) will be investigated in this section for the three choices of α_1 and α_2 given in the preceding paragraph; the assumption is one of constant ρ_s. If variations in ρ_s are not important, a comparison of theory to experiment should indicate which set of constants is best. It is interesting to note that regardless of the values of α_1 and α_2, Hamiltonian (4) is identical to that of an electron of mass m^* and charge q interacting with a vector potential $-q^{-1}c\,\alpha_1 m^* \vec{v}_s$ and scalar potential $q^{-1}(E_0 - \tfrac{1}{2}\alpha_2 m^* \vec{v}_s^{\,2})$. Here, c is the speed of light.

Consider Hamiltonian (4). The corresponding wave function

$$\Psi(\vec{R}, t) = e^{-it(E + E_0)}\Psi(\vec{R}) \ , \quad (8)$$

where $\Psi(\vec{R})$ is a solution to

$$[-(2m^*)^{-1}\nabla^2 - i\alpha_1 \vec{v}_s \cdot \nabla + \tfrac{1}{2}(\alpha_1^2 - \alpha_2)m^* \vec{v}_s^{\,2} - E] \times \Psi(\vec{R}) = 0 \ , \quad (9)$$

must represent a scattering state with initial momentum $\vec{P}$ and energy $E(P) = (2m^*)^{-1}P^2$. In the *preferred* coordinate system of Sec. II, the interaction in (9) is independent of z and momentum is conserved in the z direction. The time-independent wave function must be of the form $e^{ip_z z}\,\psi(r, \phi_r)$ and the scattering is two dimensional in nature. [The cylindrical components of $\vec{P}$ will be denoted by (p, ϕ_p, p_z).]

The periodicity requirement for the wave function can be satisfied by expanding $\psi(r,\phi_r)$ in partial waves

$$\psi(r,\phi_r)=\sum_m e^{im\chi}\, a_m\psi_m(r)\ , \tag{10}$$

where the sum extends over all integers m and $\chi=\phi_r-\phi_p$. The Fourier coefficients $\psi_m(r)$ are solutions to Bessel's equation[19]

$$[D(r)-\nu_m^2 r^{-2}+p^2]\psi_m(r)=0\ , \tag{11}$$

where $D(r)$ is the differential operator

$$D(r)=\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\ , \tag{12}$$

and
$$\nu_m^2=(m+v)^2-w^2\ ,\quad v=\alpha_1 m^*\overline{\kappa}\ ,\qquad w^2=\alpha_2(m^*\overline{\kappa})^2\ . \tag{13}$$

One notes from Eq. (11) that $a_m\psi_m(r)$ is a linear combination of Bessel functions $J_{\nu_m}(pr)$ and $Y_{\nu_m}(pr)$. The linear combination must be chosen so that the asymptotic form of $\psi(r,\phi_r)$ consists of an incident wave (corresponding to a particle with momentum $\vec{p}$, normalized to one particle per unit area) and an outgoing scattered wave. Since the scattering is two dimensional, one would normally expect

$$\psi(r,\phi_r)\sim e^{i\vec{p}\cdot\vec{r}}+r^{-1/2}f(p,\chi)\,e^{i(pr-\pi/4)}, \tag{14}$$

where χ becomes the scattering angle in the asymptotic region and $f(p,\chi)$ is the scattering amplitude. If $f(p,\chi)$ and $e^{i\vec{p}\cdot\vec{r}}$ are expanded in partial waves, and Eq. (14) is used for an asymptotic evaluation, it is not difficult to show that

$$\psi(r,\phi_r)=\sum_m e^{im\chi}e^{i\pi|m|/2}e^{i\delta_m}\psi_m(r)\ , \tag{15}$$

$$f(p,\chi)=(2\pi p)^{-1/2}\sum_m e^{im\chi}(e^{2i\delta_m}-1)\ . \tag{16}$$

The phase shifts δ_m are defined by the asymptotic condition

$$\psi_m(r)\sim(2/\pi pr)^{1/2}\cos(pr-\tfrac{1}{4}\pi-\tfrac{1}{2}\pi|m|+\delta_m). \tag{17}$$

Consider the case $\alpha_2=0$. For $\psi_m(r)$ to simultaneously satisfy (11), (17), and the regularity condition at the origin, one must have

$$\psi_m(r)=J_{|m+v|}(pr)\ . \tag{18}$$

The asymptotic behavior of Bessel functions for large arguments[19] then yields

$$\delta_m=\tfrac{1}{2}\pi(|m|-|m+v|)\ . \tag{19}$$

With phase shift (19), the scattering amplitude in (16) diverges as an infinite series in m. (The geometric series $\sum_1^\infty z^m$ diverges for all $|z|\geq 1$.[20]) The difficulty lies in an incorrect asymptotic evaluation. Indeed in Appendix A, the asymptotic behavior of $\psi(r,\phi_r)$ is evaluated without resorting to a partial-wave expansion for $f(p,\chi)$. The results for $\chi_0<\chi<2\pi-\chi_0$ are

$$\psi(r,\phi_r)=e^{iv(\pi-\chi)}e^{i\vec{p}\cdot\vec{r}}+r^{-1/2}f(p,\chi)e^{i(pr-\pi/4)}\ , \tag{20}$$

$$f(p,\chi)=(2\pi p)^{-1/2}e^{i(m_0+1/2)\chi}\sin(\pi v)\csc(\tfrac{1}{2}\chi). \tag{21}$$

Here $\chi_0=(8/pr)^{1/3}$ and m_0 is the integer that satisfies the inequality $-1\leq m_0+v<0$. It should however be noted that scattering amplitude (21) is identical to that obtained from (16) and (19) if the infinite series in (16) is summed in the Abel sense[21]

$$\sum_m=\lim_{\epsilon\to 0^+}\sum_m e^{-\epsilon|m|}\ . \tag{22}$$

The additional phase in the incident asymptotic state is a result of the long-range nature of the $\vec{P}\cdot\vec{v}_s$ component of the potential. Indeed if one assumes a better representation of the three-dimensional incident state is a plane wave multiplied by the factor $\exp(i\int\vec{F}\cdot d\vec{R})$, where the integral is path independent, a direct substitution into Eq. (9) yields

$$\Psi_{\rm inc}(r,\phi_r,z)=e^{iv(\pi-\chi)}e^{i\vec{P}\cdot\vec{R}},\ 0<\chi<2\pi\ . \tag{23}$$

Equation (23) is valid even when $\alpha_2\neq 0$ and physically represents a particle whose probability current is $(m^*)^{-1}\vec{P}$.

It thus appears that plane waves are not valid incident states for the impurity-vortex problem; the impurity always feels the long-range nature of the force. However, it is important to note that identical scattering results are obtained with incident plane waves provided infinite sums are interpreted in the Abel sense.

With Hamiltonian (4), an impurity interacts with a vortex field in the same way an electron interacts with certain electromagnetic fields. The additional phase in the incident state should thus appear whenever an electron is scattered by a vector potential proportional to $\hat{\phi}_r/r$. Bohm and Aharonov have studied this latter problem when the scalar potential vanishes.[9] They also were able to find an explicit expression for the asymptotic wave function, similar to (20), without expanding the scattering amplitude in partial waves. The work in Appendix A is an alternative derivation of their results.

Consider the case $\alpha_2\neq 0$. The wave function can be decomposed into two parts, one part corresponding to the solution when $\alpha_2=0$. With this decomposition, $\psi(r,\phi_r)$ is given by Eq. (15) with phase-shift definition (17). The asymptotic behavior of $\psi(r,\phi_r)$ is given by (20) where the scattering amplitude $f(p,\chi)$ is the sum of (21) and

$$f'(p, \chi) = (2\pi p)^{-1/2} \sum_m e^{im\chi} \times [e^{2i\delta_m} - e^{i\pi(|m| - |m+v|)}] . \quad (24)$$

Equation (16) may also be used as a representation of $f(p, \chi)$ provided the sum is evaluated in the Abel sense.

In order to satisfy (11) and (17),

$$\psi_m(r) = \cos\delta'_m J_{\nu_m}(pr) - \sin\delta'_m Y_{\nu_m}(pr) , \quad (25)$$

where $\delta'_m = \frac{1}{2}\pi(\nu_m - |m|) + \delta_m$. The constant δ'_m is determined by the regularity condition at the origin. As long as ν_m is real, regularity demands that δ'_m vanish; a unique solution for $\psi_m(r)$ is obtained with phase shift

$$\delta_m = \tfrac{1}{2}\pi(|m| - \nu_m) . \quad (26)$$

However, when ν_m is imaginary, a unique solution for $\psi_m(r)$ is not obtained from the regularity condition; the constant δ'_m is arbitrary and an infinite number of solutions is possible, some even corresponding to the absorption and emission of particles. The difficulty with imaginary ν_m also occurs in the classical scattering problem[7,8] where for the corresponding values of angular momenta the particle spirals into the center of the vortex and is removed from the incident beam. It shall thus be assumed in the quantum problem that capture occurs for those values of m where ν_m is imaginary; elsewhere the phase shifts are given by (26). The classical differential cross section ($\hbar \to 0$) is derived in Appendix B from quantum expressions (21), (24), and (26).

IV. SCATTERING WITH VARIABLE DENSITY

When Eq. (4) was used for $H(\vec{P}, \vec{R})$, no unique solutions existed for certain partial-wave coefficients $\psi_m(r)$ and a capture assumption had to be invoked. The difficulty probably lies in the fact that (4) was derived for a constant superfluid density; in actuality ρ_s has the approximate spatial dependence given in Eq. (3). The author has been unable to derive the classical Hamiltonian for the variable-density case. However, since ρ_s is slowly varying, it may be reasonable that $H(\vec{P}, \vec{R})$ is still given by (4) where constants α_1 and α_2 are altered by replacing ρ_s^0/ρ^0 by

$$\rho_s/\rho = (\rho_s^0/\rho^0)\, r^2/(r^2 + b^2) . \quad (27)$$

The quantity b is a core radius $a(\rho_n^0/\rho^0)^{1/2}$. However it is also conceivable in analogy to existing work on phonon-vortex scattering[11–14] that $\alpha_1 \vec{P} \cdot \vec{v}_s$ should remain proportional to r^{-2} for all r. In this latter case, only α_2 could have spatial dependence (27). The first of these approximations will be referred to as model (i), the second as model (ii).

With either model, an examination of differential equation (9) shows that a unique solution for $\psi_m(r)$ is obtained from the regularity condition for all m. Since the solutions do not exist in closed form, the following cruder approximations, based on the replacement

$$(r^2 + b^2)^{-1} = r^{-2}, \quad r \geq b$$
$$= b^{-2}, \quad r \leq b \quad (28)$$

are made for the partial-wave equations

for $r \geq b$:

$$[D(r) - \nu_m^2 r^{-2} + p^2]\, \psi_m(r) = 0 , \quad \text{(i, ii)} \quad (29)$$

for $r \leq b$:

$$[D(r) - m^2 r^{-2} + p^2 + (w^2 - 2mv)\, b^{-2}] \times \psi_m(r) = 0, \quad \text{(i)} \quad (30)$$

$$[D(r) - (m+v)^2 r^{-2} + p^2 + w^2 b^{-2}] \times \psi_m(r) = 0, \quad \text{(ii)} . \quad (31)$$

Here $D(r)$ is differential operator (12). The solution for $r \geq b$ consistent with (17) is given by Eq. (25). For $r \leq b$, regularity demands

$$\psi_m(r) = A_m J_{|m|}\{r[p^2 + (w^2 - 2mv)b^{-2}]^{1/2}\}, \quad \text{(i)} \quad (32)$$

$$\psi_m(r) = B_m J_{|m+v|}\,[r(p^2 + w^2 b^{-2})^{1/2}] , \quad \text{(ii)}. \quad (33)$$

The phase shifts are determined by the continuity of the logarithmic derivative of $\psi_m(r)$ at $r = b$. When ν_m is real,

$$\delta'_m = \tfrac{1}{2}\pi + \arg[pb\, H'^{(2)}_{\nu m}(pb) - \Lambda_m b H^{(2)}_{\nu_m}(pb)], \quad (34)$$

where $H^{(2)}_{\nu m}(pb)$ is the Hankel function of the second kind[19] and Λ_m is the logarithmic derivative of $\psi_m(r)$ evaluated as $r \to b$ from below. When ν_m is imaginary,

$$\tan(\tfrac{1}{2}\pi|m| - \delta_m) = \tanh(i\tfrac{1}{2}\pi\nu_m)\cot K_m , \quad (35)$$
$$K_m = \arg[pbJ'_{\nu_m}(pb) - \Lambda_m b J_{\nu_m}(pb)] .$$

The expressions for Λ_m are obtained from (32) and (33).

V. FRICTIONAL FORCE

The theoretical quantity that can be compared to experiment in quasiparticle-vortex scattering is the frictional force

$$\vec{F}(\vec{u}) = (2\pi)^{-3}\int d^3P_i n(E_i - \vec{P}_i \cdot \vec{u}) \left|\frac{\partial E_i}{\partial P_i}\right| (2\pi)^{-3} \times \int d^3P_f (\vec{P}_i - \vec{P}_f)\, d\sigma_{fi} \quad (36)$$

for a small velocity $\vec{u}$. Here $d\sigma_{fi}$ is the differential cross section for the scattering of a particle with group velocity $|\partial E/\partial P|$ from an initial state i (momentum $\vec{P}_i$, energy E_i) to a final state f (momentum $\vec{P}_f$, energy E_f); the total cross section would be $(2\pi)^{-3}\int d^3P_f\, d\sigma_{fi}$. The quantity $n(E_i)$ is the initial quasiparticle distribution function and is such that

$$(2\pi)^{-3}\int d^3P\, n(E) \tag{37}$$

is the number of quasiparticles per unit volume. Physically, $\vec{u}\cdot\vec{F}(\vec{u})$ is proportional to the energy loss per distance for a gas of quasiparticles with drift velocity $\vec{u}$ interacting with a stationary vortex.

For the impurity-vortex scattering considered in Secs. III and IV, the energy and z component of momentum are conserved, and the scattering amplitude $f(p_i, \chi)$ is a function only of p_i and $\chi=\phi_f-\phi_i$. Consequently,

$$d\sigma_{fi}=(2\pi)^3P_i^{-1}\,\frac{\partial E_i}{\partial p_i}\,L\,|f(p_i,\chi)|^2 \times\delta(E_f-E_i)\,\delta(p_{zf}-p_{zi})\,, \tag{38}$$

where L is the infinite length of the rectilinear vortex. Using (36) and (38), one finds to lowest order in u

$$F_c=-uL\pi(2\pi)^{-3} \times\int dp_z\int dp_i\left(\frac{\partial n(E_i)}{\partial E_i}\right)\frac{\partial E_i}{\partial p_i}\,p_i^3\,\sigma_c(p_i). \tag{39}$$

The real part of the complex quantity F_c is the force in the $\vec{u}$ direction while the imaginary part is the force in the $\vec{z}\times\vec{u}$ direction. The p_z integration limits are $\pm\infty$ and those of p are 0 and $+\infty$. The function $\sigma_c(p)$ is the complex transport cross section

$$\sigma_c(p)=\int_0^{2\pi}d\chi(1-e^{i\chi})\,|f(p,\chi)|^2\,. \tag{40}$$

The impurity distribution

$$n(E)=2\{\exp[(E-\mu)/k_BT]+1\}^{-1} \tag{41}$$

is the Fermi function for spin-$\frac{1}{2}$ particles at chemical potential μ and temperature T. Here $E=E_0+(2m^*)^{-1}P^2$ and k_B is Boltzmann's constant. The total number of impurities per unit volume n_3 is given by Eq. (37). The impurity contribution to the normal fluid density is[22]

$$\rho_{ni}=-\tfrac{1}{3}(2\pi)^{-3}\int d^3P\,\frac{\partial n(E)}{\partial E}\,P^2=m^*n_3\,. \tag{42}$$

When $\mu'=(E_0-\mu)/k_BT>1$, Eq. (37) may be integrated; in particular,

$$n_3=2(2\pi)^{-3}(2\pi m^*k_BT)^{3/2}\textstyle\sum_s(-1)^{s+1}s^{-3/2}e^{-\mu's}, \tag{43}$$

where the sum extends over zero and all positive integers. Equation (43) relates the chemical potential to the temperature and particle density; it may be inverted to give μ' as an expansion in the Fermi degeneracy temperature $T_d=(3\pi^2n_3)^{2/3}\times(2m^*k_B)^{-1}$,

$$e^{-\mu'}=\tfrac{1}{2}(2\pi)^3(2m^*k_BT)^{-3/2}\,n_3 \times[1+\tfrac{1}{3}(2/\pi)^{1/2}(T_d/T)^{3/2}+\cdots]\,. \tag{44}$$

The frictional force can also be expressed as

$$\frac{F_c}{uLn_3}=\frac{\int dp_z\int dp\,n(E)\partial[p^2g_c(p)]/\partial p}{\int dp_z\int dp\,n(E)2p} \tag{45}$$

$$=\frac{\sum_s(-1)^{s+1}s^{-3/2}e^{-\mu's}\bar{g}_c(s)}{\sum_s(-1)^{s+1}s^{-3/2}e^{-\mu's}}\,, \tag{46}$$

where

$$g_c(p)=p\sigma_c(p)\,, \tag{47}$$

$$\bar{g}_c(s)=2\int_0^\infty q^3\,dq\,e^{-q^2}g_c[(2m^*k_BT/s)^{1/2}q]\,. \tag{48}$$

To express $g_c(p)$ in terms of phase shifts, one must use the decomposed form for the scattering amplitude. Equation (16), even interpreted as an Abel sum in ϵ, is not uniformly convergent in ϵ and cannot be squared and integrated term by term with weight factor $1-e^{i\chi}$. However these operations can be applied to $f'(p,\chi)$ since (24) converges uniformly on $0<\chi<2\pi$. Using Eqs. (21) and (24), one finds after some algebra

$$g_c(p)=i\sin(2\pi v)+\textstyle\sum_m[1-e^{2i(\delta_m-\delta_{m+1})}]\,. \tag{49}$$

It is worthwhile to digress and consider the case where phase shifts are determined by introducing a cutoff in the potential (for r greater than some large r_0, the velocity $\vec{v}_s$ is equal to zero). One then finds

$$\psi(r,\phi_r)\sim e^{i\vec{p}\cdot\vec{r}}+r^{-1/2}f(p,\chi,r_0)\,e^{i(pr-\pi/4)}\,, \tag{50}$$

where $f(p,\chi,r_0)$ is given by (16). The resulting phase shifts are such that

$$\delta_{m,r_0}=\delta_m[1-C(m)/pr_0],\quad C(m)\ll pr_0$$
$$\propto(epr_0/2|m|)^{2|m|}\,,\qquad |m|\gg pr_0\,. \tag{51}$$

The function $C(m)$, at least when $|m|$ is large, is linear in m; the phase shift δ_m is that calculated without cutoff. When (47) is determined with the cutoff phase shifts,

$$g_c(p,r_0)=\textstyle\sum_m\{1-\exp[2i(\delta_{m,r_0}-\delta_{m+1,r_0})]\}\,. \tag{52}$$

Equation (52) does not converge uniformly in r_0, and in general $g_c(p,\infty)$ does not equal $g_c(p)$ [Eq. (49)] calculated without cutoff. Indeed if consecutive phase-shift differences are vanishingly small, it is easy to see that

$$g_c(p)=i[-2\pi v+\sin(2\pi v)],\quad g_c(p,\infty)=0\,. \tag{53}$$

It appears in general that with phase shifts (51),

$$g_c(p)=g_c(p,\infty)+i[-2\pi v+\sin(2\pi v)] . \quad (54)$$

Consequently, when a cutoff is introduced, an additional transverse transport cross section

$$\sigma_c'(p)=ip^{-1}[-2\pi v+\sin(2\pi v)] \quad (55)$$

must be added to that calculated from cutoff if the results are to agree with scattering in an infinite medium. This problem also appears in classical impurity-vortex scattering[8] and is not strictly a quantum effect. It is evident from (45) that Eq. (55) is equivalent to a transverse frictional force

$$F_c'=iuLn_3[-\alpha_1 m^*\kappa+\sin(\alpha_1 m^*\kappa)] . \quad (56)$$

When $\alpha_1=1$, the first term in (56) would be identical to the Iordanskii force[14] $-iuL\rho_{ni}\kappa$ since $\rho_{ni}=m^*n_3$; Iordanskii calculated the frictional force by introducing a cutoff.

The only experimental results to compare with are those obtained from Rayfield and Reif's work with vortex rings.[4] (A large vortex ring may be approximated by two parallel rectilinear vortices with opposing circulations.) The experiments were carried out at tempertures T and impurity (molar) concentrations X such that $T_d \ll T$. (As long as $X<10^{-2}$ and $T>0.1\,^\circ$K, it can be shown that $T_d<T$.) It is convenient to define F_c by

$$F_c=uLn_3\bar{g}_c . \quad (57)$$

Table I gives the values of $\mathrm{Re}(\bar{g}_c)$ obtained from Rayfield and Reif's work. The imaginary part of $\bar{g}_c$ vanishes by symmetry for vortex rings. The errors quoted originate from uncertainties in the measurements of energy losses and molar concentrations. For the 0.61 °K value, it was necessary to subtract out the experimental roton contribution to F_c; consequently, the corresponding error in $\mathrm{Re}(\bar{g}_c)$ may be larger than that indicated. For the other two values, the experimental phonon and roton contributions were negligible.

With the scattering results of Sec. III, the phase shifts are independent of momentum. Thus, Eqs. (45) and (49) predict that $\bar{g}_c=g_c$ for all T below the λ transition. Numerical results for $\bar{g}_c$ are given in Table II and correspond to $T<1\,^\circ$K, values (5)–(7) for α_1 and α_2, and $M=m^*/m_3$ values 2.2, 2.5, and 2.8. Figure 1 illustrates the temperature dependence of $\bar{g}_c$ when constants α_1 and α_2 are given by (5), $X\ll 1$, and $M=2.2$; the relative maxima and minima for large T arise from the capture assumption.

TABLE I. Rayfield and Reif experimental values for $\mathrm{Re}(\bar{g}_c)$ as a function of temperature T and impurity (molar) concentration X.

T (°K)	X	$\mathrm{Re}(\bar{g}_c)$
0.28	2.84×10^{-5}	7.16 ± 0.22
0.28	7.55×10^{-6}	7.75 ± 0.44
0.61	7.55×10^{-6}	10.15 ± 0.36

TABLE II. Theoretical values of $\bar{g}_c$ for temperatures less than 1 °K, several effective-mass ratios, and various choices [Eqs. (5)–(7)] for the interaction constants α_1 and α_2.

	$\vec{P}\cdot\vec{v}_s$		BBP		Classical	
m^*/m_3	$\mathrm{Re}(\bar{g}_c)$	$\mathrm{Im}(\bar{g}_c)$	$\mathrm{Re}(\bar{g}_c)$	$\mathrm{Im}(\bar{g}_c)$	$\mathrm{Re}(\bar{g}_c)$	$\mathrm{Im}(\bar{g}_c)$
2.2	6.60	−0.31	4.32	−0.71	7.60	−0.67
2.5	7.46	+0.69	4.81	+1.03	9.32	+0.63
2.8	7.63	+1.64	5.69	+0.51	10.48	−0.70

In comparing Tables I and II, one notes that the experimental results do not agree with those obtained from the BBP constants. However, provided $M\approx 2.5$, the $\vec{P}\cdot\vec{v}_s$ constants give results consistent with the 0.28 °K values. The results for the classical constants are consistent with the 0.28 °K values when $M\approx 2.2$ and with the 0.61 °K value when $M\approx 2.8$. However, in all cases, the theoretical results are temperature independent for a given m^*; the temperature increase in $\mathrm{Re}(\bar{g}_c)$ indicated by experiment is not predicted. This discrepancy could be due to an underestimation of the experimental errors; more likely it results from an inadequacy of Hamiltonian (4) to consider the spatial dependence of ρ_s.

The work in Sec. IV attempted to consider spatial variations of ρ_s in an approximate fashion; Hamiltonian (4) was used with constants (5) altered by replacing ρ_s by its approximate analytic form given in Eq. (3). With further approximations, the phase shifts δ_m, and consequently $g_c(p)$, were found to depend on the momentum only through the product pb, where $b=a(\rho_n^0/\rho^0)^{1/2}$. In terms of the dimensionless integration parameter q of Eq. (48),

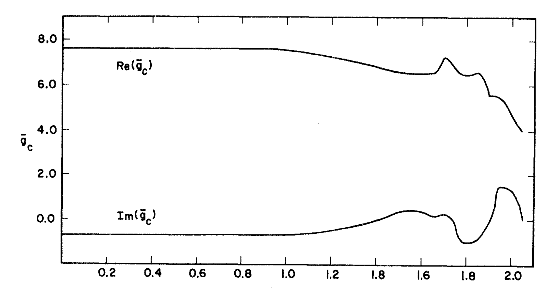

FIG. 1. Temperature dependence of $\bar{g}_c$ for 0 °K< T < 2 °K, classical coefficients, constant ρ_s, and molar concentrations less than 10^{-2}. The relative maxima and minima are due to the capture assumption.

the product pb equals ζq, where $\zeta = (2m^* k_B T b^2)^{1/2}$. [When b is expressed in angstroms and T in °K, the constant ζ is equal to $0.351(TMb^2)^{1/2}$.] Assuming ζ is small enough so that δ_m and $g_c(p)$ may be expanded in powers of ζq, one finds the following approximate behavior for the phase shifts:

$$\delta_m \approx \tfrac{1}{2}\pi(|m| - \nu_m) + b_m(\tfrac{1}{2}\zeta q)^{\nu_m}, \quad \nu_m \text{ real} \tag{58}$$

$$\delta_m \approx c_m \ln(\tfrac{1}{2}\zeta q) + d_m, \quad \nu_m \text{ imaginary} . \tag{59}$$

One would thus expect to lowest order in ζ that

$$\bar{g}_c = \bar{g}_c \text{ (Sec. III)} + [\text{oscillating function of } \ln(\tfrac{1}{2}\zeta)] . \tag{60}$$

Figure 2 illustrates a typical behavior for $\bar{g}_c$ as a function of $\ln(\frac{1}{2}\zeta)$ to order $(\frac{1}{2}\zeta)^2$ when $M = 2.38$ and $T_d \ll T < 1$ °K. [For these temperatures, $\bar{g}_c$ equals $\bar{g}_c(1)$ of Eq. (48).] Results for both model (i) and (ii) are included; the horizontal lines represent $\bar{g}_c$ (Sec. III). The expansion breaks down for $\frac{1}{2}\zeta > 1$. The exact analytic expressions used for a numerical evaluation of $\bar{g}_c$ are omitted; they are rather involved and offer no new insight into the physics of the scattering problem. A comparison with experiment (Table I) indicates that the best agreement for model (i) occurs when the core radius a is about 0.4 Å and for model (ii) when a is about 9 Å. These results are summarized in Table III and are consistent with experiment considering the possible experimental errors quoted in Table I.

It thus appears from the preceding work that with the proper effective mass, core radius, and model describing spatial variations in ρ_s, Hamiltonian (4) with constants (5) yield results consistent with the experimental work of Rayfield and

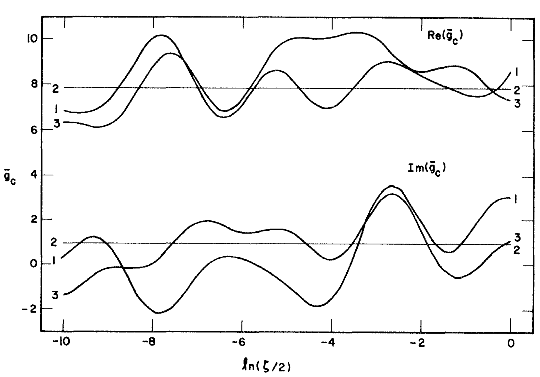

FIG. 2. Behavior of $\bar{g}_c$ for the classical constants as a function of $\ln(\frac{1}{2}\zeta)$, where $\zeta = (2m^* k_B T b^2)^{1/2}$ and $T_d \ll T < 1$ °K. Curves 1 and 3 are the results for models (i) and (ii) (Sec. IV); curve 2 is the result for constant ρ_s (Sec. III).

TABLE III. Best values of $\bar{g}_c$ (for comparison to Table I) using models (i) and (ii) of Sec. IV when the effective-mass ratio is 2.38. The quantity a is the radius of the vortex core.

T (°K)	Model (i), $a = 0.4$ Å		Model (ii), $a = 9$ Å	
	Re($\bar{g}_c$)	Im($\bar{g}_c$)	Re($\bar{g}_c$)	Im($\bar{g}_c$)
0.28	6.78	+0.50	7.50	+1.96
0.61	9.74	−1.77	10.03	+0.64

Reif. However, as is shown elsewhere,[23] consistent results can also be obtained with the $\vec{P}\cdot\vec{v}_s$ interaction provided a vortex core of about 0.45 Å rotates with a velocity proportional to r while ρ_s remains constant. It would thus be interesting to investigate the many-body formulation for impurity-vortex scattering in order to properly consider spatial density variations and to determine the effective interaction in a quantum-mechanical fashion.

For better comparison to experiment, it would be advantageous to repeat Rayfield and Reif's work over a broader range of temperatures and concentrations. Second-sound experiments[1-3] could also be done in the presence of impurities. A rough estimate for the second-sound coefficients yields $B \approx 1$ and $B' \approx |0.2|$ when $X \lesssim 10^{-4}$ and $T \lesssim 0.5$ °K; the exact values depend on the model used to describe the spatial variations of ρ_s.

VI. T-MATRIX FORMALISM

General Results

The purpose of this section is to develop a T-matrix formalism to describe the quantum scattering of a quasiparticle from a rectilinear vortex in a superfluid with constant ρ_s. The results will be applied to He^3 impurities and to the quasiparticles of pure He^4 (phonons and rotons).

The quasiparticle is assumed to be stable against decay and have a state vector $|\Psi(t)\rangle$ satisfying the Schrödinger equation

$$\left(i\frac{\partial}{\partial t} - H\right)|\Psi(t)\rangle = 0 . \tag{61}$$

A δ-function normalization is chosen corresponding to one particle per unit volume. The Hamiltonian H consists of an unperturbed part H_0 and an interaction V. The unperturbed energy spectrum for impurities is given by Eq. (2). The energy spectrum for a quasiparticle of pure He^4 has been measured experimentally.[24] The regions of the spectrum most important for thermal averages occur for $P < 0.75$ Å^{-1} and $P \approx 1.9$ Å^{-1}. In the former case,

$$E = cP, \quad c = 237 \text{ m/sec}, \tag{62}$$

and the quasiparticle is called a phonon; in the latter case, the quasiparticle is called a roton and has an energy spectrum

$$E = \Delta + (2\mu)^{-1}(P - P_0)^2, \tag{63}$$

with constants

$$\Delta/k_B = 8.6\,^\circ\mathrm{K}, \quad \mu/m_4 = 0.16, \quad P_0 = 1.91\ \text{Å}^{-1}. \tag{64}$$

The potential V for impurities is assumed to be given by the appropriate part of Eq. (4). For the quasiparticles of pure He^4, the interaction is taken to be a symmetrized $\vec{P}\cdot\vec{v}_s$; this choice would result from a Galilean transformation on Eq. (61) and is consistent with existing work on phonon-vortex scattering.[11-14] Any interaction among the various quasiparticles themselves is neglected.

The details of the scattering process are as follows. Initially the quasiparticle has momentum $\vec{P}_i$ and energy $E_i(P_i)$; its time-independent state vector $|\Psi_0(i)\rangle$ satisfies

$$(H_0 - E_i)|\Psi_0(i)\rangle = 0. \tag{65}$$

As time progresses, the quasiparticle interacts with the potential and is scattered into a final state with time-independent state vector $|\Psi_0(f)\rangle$. The scattering may be studied by using the Lippmann-Schwinger equations[25]

$$\begin{aligned}|\Psi^{\pm}(i)\rangle &= |\Psi_0(i)\rangle + (E_i - H_0 \pm i\epsilon)^{-1}V|\Psi^{\pm}(i)\rangle \\ &= |\Psi_0(i)\rangle + (E_i - H \pm i\epsilon)^{-1}V|\Psi_0(i)\rangle,\end{aligned} \tag{66}$$

which follow from (61) and (65). The states $|\Psi^+(i)\rangle$ and $|\Psi^-(i)\rangle$ are the time-independent portions of the state vectors that develop in time according to the full Hamiltonian H, but equal $|\Psi_0(i)\rangle$ in the remote past and distant future, respectively.

It is not difficult to show from Eq. (66) that the differential cross section $d\sigma_{fi}$ [probability per unit time per incident flux for a transition from $|\Psi_0(i)\rangle$ to $|\Psi_0(f)\rangle$] is

$$d\sigma_{fi} = 2\pi\delta(E_f - E_i)\left|\frac{\partial P_i}{\partial E_i}\right||T_{fi}|^2, \tag{67}$$

where $$T_{fi} = \langle\Psi_0(f)|T(E_i)|\Psi_0(i)\rangle \tag{68}$$

and $$T(E) = V + V(E - H_0 + i\epsilon)^{-1}T(E). \tag{69}$$

Consequently, $d\sigma_{fi}$ is known when the energy-shell values of the T matrix (values for $E_i = E_f$) are determined. The matrix elements T_{fi} satisfy the integral equation

$$T_{fi} = V_{fi} + (2\pi)^{-3}\int_k V_{fk}(E_i - E_k + i\epsilon)^{-1}T_{ki}, \tag{70}$$

which results from (68) and (69). The symbol $\int_k$ represents a sum and integration over the discrete and continuous variables of $|\Psi_0(k)\rangle$. It will be assumed in this section that the spatial representation of $|\Psi_0(k)\rangle$ is the plane wave

$$\langle\vec{R}|\Psi_0(k)\rangle = \exp(i\vec{R}\cdot\vec{P}_k). \tag{71}$$

As was seen in Sec. III, this assumption is not strictly valid; a quasiparticle always feels the presence of a vortex and its incident state is modified correspondingly. However, identical scattering results can be obtained by using plane-wave states and summing resulting partial-wave series in an Abel sense.

At times it is convenient to determine T_{fi} by introducing the R operator and its matrix elements

$$R(E) = V + \mathrm{PV}[V(E - H_0)^{-1}R(E)], \tag{72}$$

$$R_{fi} = \langle\Psi_0(f)|R(E)|\Psi_0(i)\rangle. \tag{73}$$

The symbol PV implies a Cauchy principal value. The matrix element R_{fi} satisfies the integral equation

$$R_{fi} = V_{fi} + (2\pi)^{-3}\,\mathrm{PV}\int_k V_{fk}(E_i - E_k)^{-1}R_{ki}, \tag{74}$$

and is Hermitian provided $E_i = E_f$. The quantity T_{fi} is related to R_{fi} by the Heitler equation[26]

$$T_{fi} = R_{fi} - i\pi(2\pi)^{-3}\int_k R_{fk}\,\delta(E_i - E_k)\,T_{ki}, \tag{75}$$

derived from (70) and (74).

The energy-shell values of T_{fi} may also be found indirectly by examining the asymptotic behavior of the spatial wave function $\langle\vec{R}|\Psi^{\pm}\rangle$ for large $\vec{R}$; this usually involves solving the integral equation for $\langle\vec{R}|\Psi^{\pm}\rangle$ resulting from (66). However when H_0 is a quadratic function of momentum, a differential equation for the wave function may be obtained by operating on both sides of (66) with $E_i - H_0$ and replacing $\vec{P}$ by $-i\nabla$; this latter method was the approach used in Sec. III.

The remainder of this section is concerned with determining T_{fi} by integral equation (74) for R_{fi}. (In general, perturbation theory will not be valid.) The principal disadvantage is the necessity to find all the values of T_{fi} in order to obtain the desired energy-shell values. The analogous difficulty in the wave-function approach is the need to know $\langle\vec{R}|\Psi^{\pm}\rangle$ for all $\vec{R}$ although only the asymptotic form contains the scattering information.

Consider quasiparticle scattering expressed in the *preferred* coordinate system of Sec. II. Using the fact that V is independent of z and expanding in partial waves, one may write

$$S_{fi} = 2\pi\delta(p_{zf} - p_{zi})\,S'_{fi}(p_f, p_i, p_{zi}, \chi), \tag{76}$$

$$S'_{fi} = (2\pi)^{-1}\textstyle\sum_m e^{im\chi}S_m(p_f, p_i, p_{zi}), \tag{77}$$

where S represents T, R, or V. The scattering angle χ is $\phi_f - \phi_i$. The spherical and cylindrical components of $\vec{P}$ are denoted by (P, θ, ϕ) and (p, ϕ, p_z), respectively. With definitions (76) and (77), it follows from Eqs. (74) and (75) that

$$R_m(p_f, p_i, p_{zi}) = V_m(p_f, p_i) + (2\pi)^{-2}\,\mathrm{PV}\int_0^\infty p_k dp_k \times V_m(p_f, p_k)(E_i - E_k)^{-1} R_m(p_k, p_i, p_{zi}), \tag{78}$$

$$T_m(p_f, p_i, p_{zi}) = R_m(p_f, p_i, p_{zi}) - i\pi \textstyle\sum_k \eta_k \times R_m(p_f, p_k, p_{zi})\, T_m(p_k, p_i, p_{zi}). \tag{79}$$

The sum over k in (79) is over those states where $E_i(p_i, p_{zi})$ equals $E_k(p_k, p_{zi})$. The density of states η_k is defined as

$$\eta_k = (2\pi)^{-2} p_k \left|\frac{\partial p_k}{\partial E_k}\right| . \tag{80}$$

Note that (78) is an integral equation for R_m in variable p_f while (79) is a set of algebraic equations determining T_m in terms of R_m.

The potential V is a linear combination of a symmetrized $\vec{P}\cdot\vec{v}_s$ and $\frac{1}{2}m^*\vec{v}_s^{\,2}$. The partial-wave matrix elements for V may be evaluated using plane waves (71) in which case

$$V_m(p_f, p_k) = 2\pi^2 \overline{\kappa}(p_</p_>)_{fk}^{|m|} \times [\alpha_1 + \tfrac{1}{2}(\alpha_1^2 - \alpha_2)\overline{\kappa} m^*/m]\,\mathrm{sgn}(m). \tag{81}$$

Here sgn(m) is $m/|m|$, and $(p_</p_>)_{fk}$ is the ratio of the lesser of the pair (p_f, p_k) to the greater. For impurities, α_1 and α_2 may take values (5)–(7); for phonons and rotons, it is assumed that $\alpha_1 = \alpha_2 = 1$. The partial-wave potential for $m=0$ is interpreted as the limit of (81) as $m \to 0$ from positive or negative values.

As a function of momentum, the quantity $V_m(x, x')[x = p_f/p_i, x' = p_k/p_i]$ has a discontinuous slope

$$\lim_{\epsilon\to 0^+} x\frac{dV_m}{dx}\bigg|_{x'-\epsilon}^{x'+\epsilon} = -2|m|V_m(1,1) \tag{82}$$

at $x = x'$ and satisfies the differential equation

$$[D(x) - m^2 x^{-2}]V_m(x, x') = -2|m|V_m(1,1)x^{-1}\delta(x - x'), \tag{83}$$

with boundary conditions

$$V_m(x, x') \to 0, \quad \text{as } x \to 0, \infty, \qquad V_m(x, x') \text{ continuous on } 0 \le x \le \infty . \tag{84}$$

Here $D(x)$ is differential operator (12). The right side of (83) is equivalent to slope discontinuity (82).

Using (83) and operating on (78) with $[D(x) - m^2x^{-2}]$, one finds that $R_m(x)$ satisfies the differential equation

$$\{D(x) - m^2x^{-2} + 2|m|V_m(1,1)(2\pi)^{-2}p_i^2[E_i - E_f(x)]^{-1}\} \times R_m(x) = -2|m|V_m(1,1)\delta(x-1), \tag{85}$$

with boundary conditions analogous to (84). The determination of R_m has thus been reduced to the solution of a differential equation in which $R_m(x)$ behaves like a Green's function. To find $R_m(x)$, it is necessary to know the homogeneous solutions of (85) in the regions between the singularities. (The differential equation has at least three singular points 0, 1, and ∞.) By applying the boundary conditions at zero and infinity and the appropriate continuity and discontinuity conditions for $R_m(x)$ at each singularity, all arbitrary constants may be evaluated. However the method requires knowledge of the analytic continuations of the homogeneous solutions between consecutive singular points; this information may be difficult to obtain.

Impurities

Consider impurity-vortex scattering. Since E_i is a quadratic function of momentum [Eq. (2)], the substitutions

$$y = x^2, \quad L_m(y) = x^{1-\gamma}R_m(x), \quad \gamma = 1 + |m| , \qquad \beta = \tfrac{1}{2}(|m| - \nu_m), \quad \alpha = \gamma - \beta - 1, \quad \nu_m^2 = (m+v)^2 - w^2 , \tag{86}$$

reduce the homogeneous part of (85) to the hypergeometric equation[27]

$$\left(y(1-y)\frac{d^2}{dy^2} + [\gamma - (\alpha+\beta+1)y]\frac{d}{dy} - \alpha\beta\right)L_m(y) = 0. \tag{87}$$

When $\alpha - \beta = \nu_m$ is real, the boundary conditions at zero and infinity require

$$R_m(x) = A_0 x^{\gamma-1}F(\alpha, \beta, \gamma; x^2), \qquad 0 \le x \le 1$$
$$= A_\infty x^{\beta-\alpha}F(\alpha, -\beta, \alpha-\beta+1, x^{-2}), \quad 1 \le x \le \infty, \tag{88}$$

where for $|z| < 1$ and γ, neither zero nor a negative integer,

$$F(\alpha, \beta, \gamma; z) = \sum_{l=0}^{\infty} \frac{(\alpha)_l(\beta)_l}{(\gamma)_l l!} z^l, \quad (\alpha)_l = \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)} . \tag{89}$$

Here $\Gamma(\alpha)$ is a gamma function with argument α. If ν_m is imaginary, the boundary condition at infinity does not specify a single solution for $R_m(x)$ on the interval $1 \le x \le \infty$. Thus, just as in Sec. III, a capture assumption is invoked for those values of m where ν_m is imaginary.

The continuity and discontinuity of $R_m(x)$ at $x=1$ determine constants A_0 and A_∞. To apply these conditions, it is necessary to know the following analytic continuation which holds for $\gamma - \alpha - \beta = 1$[27]:

$$\Gamma(\alpha)\Gamma(\beta)F(\alpha, \beta, \gamma; z) = \Gamma(\gamma)(1-z)g(1+\beta, 1+\alpha, 2; 1-z) , \tag{90}$$

where for $|\arg(1-z)| < \pi$ and α and β neither zero nor a negative integer,

$$(1-z)g(1+\beta, 1+\alpha, 2; 1-z) = (\alpha\beta)^{-1}$$

$$+\sum_0^\infty (1-z)^{l+1}\frac{(1+\alpha)_l(1+\beta)_l}{(2)_l l!}[\Psi(1+\alpha+l)$$

$$+\Psi(1+\beta+l)-\Psi(2+l)-\Psi(1+l)+\ln(1-z)]. \quad (91)$$

The quantity $\Psi(z)$ in Eq. (91) is the psi function[28]

$$\Psi(z)=\Gamma'(z)/\Gamma(z)=\Psi(1+z)-z^{-1}=\Psi(1-z)-\pi\cot\pi z\ . \quad (92)$$

In particular, one finds for $0\le x\le 1$,

$$R_m(x)=R_m(1)[\Gamma(1+\alpha)\Gamma(1+\beta)/\Gamma(\gamma)]x^{|m|}F(\alpha,\beta,\gamma;x^2)$$
$$=R_m(1)\alpha\beta x^{|m|}(1-x^2)g(1+\beta,1+\alpha,2;1-x^2), \quad (93)$$

and for $1\le x\le\infty$,

$$R_m(x)=R_m(1)[\Gamma(1+\alpha)\Gamma(1-\beta)/\Gamma(1+\alpha-\beta)]x^{\beta-\alpha}$$
$$\times F(\alpha,-\beta,1+\alpha-\beta;x^{-2})$$
$$=R_m(1)\alpha\beta x^{\beta-\alpha}(x^{-2}-1)g(1-\beta,1+\alpha,2;1-x^{-2})\ , \quad (94)$$

where $R_m(1)=-(\pi\eta_{i_R})^{-1}\tan\delta_m$,

$$\delta_m=\pi\beta=\tfrac{1}{2}\pi(|m|-\nu_m)\ . \quad (95)$$

From a scattering point of view, the only value of $R_m(x)$ needed is the energy-conserving matrix element $R_m(1)$. It then follows from (79) and (95) that the corresponding T-matrix element is

$$T_m(p_i,p_i,p_{zi})=(-\pi\eta_i)^{-1}e^{i\delta_m}\sin\delta_m\ . \quad (96)$$

Equation (96) is consistent with scattering amplitude (16) and phase shifts (26) [see Eqs. (38), (67), (76), and (77)].

Phonons

Consider a quasiparticle of pure He^4. It is impossible to solve (85) in closed form. One obvious reason is that $E_f(x)$ is a function of $(x^2+\cot^2\theta_i)^{1/2}$ known only graphically except near the phonon and roton regions where analytic forms (62) and (63) may be used. Further, when the initial energy exceeds the roton minimum Δ, there are three different momenta solutions for $E_i=E_f(x)$ (there are three scattering channels that conserve energy) and differential equation (85) has five singularities.

One possible method of approximation would be to define a Green's function $r_m(x,x')$ having the same boundary conditions as $R_m(x)$ and satisfying a differential equation

$$H(x)r_m(x,x')=-2|m|V_m(1,1)x^{-1}\delta(x-x')\ , \quad (97)$$

where $H(x)$ is some differential operator. Equation (85) can then be expressed as an integral equation

$$R_m(x)=r_m(x,1)+\int_0^\infty x'\,dx' r_m(x,x')$$
$$\times[2|m|V_m(1,1)]^{-1}\{D(x')-m^2x'^{-2}+2|m|V_m(1,1)$$
$$\times(2\pi)^{-2}p_i^2[E_i-E_f(x')]^{-1}-H(x')\}R_m(x')\ . \quad (98)$$

The trick is to choose $H(x)$ so that (98) may be solved by interation and, to lowest order, $R_m(x)\approx r_m(x,1)$. For impurities, $H(x)$ was chosen so that the parenthesis in (98) vanished; in this case, $R_m(x)$ is identically equal to $r_m(x,1)$. For a quasiparticle of pure He^4, the problem is more difficult.

Consider phonon-vortex scattering ($P_i<0.75$ Å^{-1}). The only scattering channel of importance is the direct channel $P_f=P_i$. (If $E_i<\Delta$, the direct channel is the only channel.) A good approximation for the energy-conserving R_m-matrix element should thus be obtained by expanding the energy denominator in (98) about $x'=1$ and setting

$$H(x)=D(x)-m^2x^{-2}+2|m|V_m(1,1)(2\pi)^{-2}$$
$$\times[h_1(1-x^2)^{-1}+h_2x^{-2}]\ . \quad (99)$$

Here $h_1=2P_i\dfrac{\partial P_i}{\partial E_i}=2c^{-1}P_i$,

$$h_2=\tfrac{1}{2}\sin^2\theta_iP_i^3\left(\frac{\partial P_i}{\partial E_i}\right)^2\frac{\partial(P_i^{-1}\partial E_i/\partial P_i)}{\partial P_i}$$
$$=-\tfrac{1}{2}c^{-1}P_i(\sin^2\theta_i). \quad (100)$$

With this approximation, the resulting equation for $r_m(x,x')$ is identical to the impurity differential equation provided α, β, and γ are redefined. In particular, to lowest order in perturbation series (98), one finds that $R_m(1)$ and $T_m(1)$ are given by (95) and (96) where now

$$\delta_m=\tfrac{1}{2}\pi\,\mathrm{sgn}\left(\frac{\partial E_i}{\partial P_i}\right)[|m\tau|-(m^2\tau^2+\lambda m)^{1/2}],$$
$$\lambda=2h_1(2\pi)^{-2}V_m(1,1)\,\mathrm{sgn}(m)=2P_i\bar{\kappa}c^{-1}, \quad (101)$$
$$\tau^2-1=-2h_2(2\pi)^{-2}V_m(1,1)/|m|=\lambda\sin^2\theta_i/4m\ ,$$
$$\eta_i=(2\pi)^{-2}P_i\left|\frac{\partial P_i}{\partial E_i}\right|=(2\pi)^{-2}P_ic^{-1}.$$

A unique solution is again possible only if τ and δ_m are real (capture must otherwise be assumed); for phonons this requires that λ must be less than 1, or for a singly quantized vortex, P_i be less than 0.75 Å^{-1}. The thermal averages of interest will require scattering results only for very small P_i and Eq. (101) may be assumed to hold for all m and P_i.

The frictional force (36) is the quantity that must be calculated to compare theory to experiment. Using (67) and (76), and introducing the complex notation F_c, one finds to lowest order in n

$$F_c = -uL\pi(2\pi)^{-3}\int_0^\infty P_i^2\,dP_i \int_{-1}^1 d(\cos\theta_i)\frac{\partial n(E_i)}{\partial E_i} \times \left|\frac{\partial E_i}{\partial P_i}\right| p_i^2 \sigma_c(P_i, \sin\theta_i)\,, \tag{102}$$

where $P_i\sigma_c(P_i, \sin\theta_i) = (2\pi)^3$

$$\times \sum_k \int_0^{2\pi} d\chi [1 - (p_k/p_i)e^{i\chi}]\eta_i\eta_k |T'_{ki}|^2\,. \tag{103}$$

The sum over k in (103) is over the energy-conserving states. The quantity η_i is density of states (80).

In the direct-channel approximation, the complex cross section (103) may be expressed in terms of phase shifts (101). In particular, provided the divergent part of the T matrix is summed in the Abel sense before insertion into (103), one finds

$$P_i\sigma_c(P_i, \sin\theta_i) = -i\sin 4\delta_{+\infty} + \sum_m [1 - e^{2i(\delta_m - \delta_{m+1})}]\,. \tag{104}$$

The phonon contribution to the frictional force may now be evaluated using Eqs. (101)–(104) and the phonon distribution function $[\exp(cP_i/k_BT) - 1]^{-1}$. When T is sufficiently small, the main contribution to F_c comes from small momenta. Thus, δ_m and $\sigma_c(P_i, \sin\theta_i)$ may be expanded in powers of λ, and F_c expanded in powers of $r = 2nk_BT/m_4c^2$. The lowest-order expressions are then

$$\delta_m = -\tfrac{1}{4}\pi\lambda\,\mathrm{sgn}(m), \quad m \neq 0$$
$$= 0, \quad m = 0\,, \tag{105}$$

$$P_i\sigma_c(P_i, \sin\theta_i) = (\tfrac{1}{2}\pi\lambda)^2 - i(\tfrac{1}{2}\pi\lambda)^3\,, \tag{106}$$

$$F_c = \tfrac{1}{4}uL\rho_{np}\,\kappa\pi\left(5r\frac{\zeta(5)}{\zeta(4)} - i15\pi r^2\,\frac{\zeta(6)}{\zeta(4)}\right)\,. \tag{107}$$

The quantity

$$\rho_{np} = 4\pi^{-2}c^{-5}(k_BT)^4\zeta(4) \tag{108}$$

is the phonon contribution to the normal fluid density[29] and $\zeta(n)$ is the Riemann Zeta function of order n.[30] A calculation of the corrections to (105)–(107) indicate the lowest-order expressions are good to 10% when $T < 0.6\,^\circ$K.

In comparing the direct-channel results for phonon-vortex scattering to those previously obtained by other individuals, one finds agreement with Refs. 12 and 13. However, in Ref. 14, Iordanskii finds two important corrections to (105) and (107). The first occurs because density variations of the superfluid near the vortex core significantly alter the $m = \pm 1$ phase shift; this effect decreases F_c by about a factor of 2. The other correction is the addition of an Iordanskii force $-iuL\rho_n\kappa$ onto expression (107). The reason this latter contribution is not obtained in this paper is that it is canceled by the first term in Eq. (104).

Finally the theoretical expression for $\mathrm{Re}(F_c)$ is found to be consistent with Rayfield and Reif's work provided the density effect of Iordanskii is considered. No data are available for the phonon contributions in second-sound experiments; it is extremely difficult to propagate second sound for $T < 1\,^\circ$K unless impurities are present.

Rotons

Consider roton-vortex scattering ($P_i \approx 2\,\text{Å}^{-1}$). Even an approximate solution to (98) is difficult. The principal reason is that when P_i lies in the roton region, the equation $E_i = E_f(x)$ has three solutions corresponding to the energy-conserving channels:

direct roton channel $P_f = P_i$;

indirect roton channel $P_f = 2P_0 - P_i$; (109)

indirect phonon channel $P_f = c^{-1}[\Delta + (2\mu)^{-1}(P_i - P_0)^2]$.

These channels give rise to a complicated singularity structure in any resulting integral or differential equation. Some simplifications result: The phonon channel may be neglected since it is far removed in momentum space from the initial roton channel; also the calculation of thermal averages will require the scattering results only for P_i very close to P_0.

The principal scattering is thus confined to two channels. It is convenient to express the energy-conserving R_m and T_m matrix elements (four each) in terms of two phase shifts δ_m and δ'_m and a mixing parameter Φ_m.[31,32] In particular, with

$$\overline{R}_m(j,k) = (\eta_j)^{1/2}R_m(j,k)(\eta_k)^{1/2}, \quad j, k = 1, 2 \tag{110}$$

one finds

$$-\pi\overline{R}_m(1,1) = \cos^2\Phi_m\tan\delta_m + \sin^2\Phi_m\tan\delta'_m,$$
$$-\pi\overline{R}_m(2,2) = \sin^2\Phi_m\tan\delta_m + \cos^2\Phi_m\tan\delta'_m, \tag{111}$$
$$-\pi\overline{R}_m(1,2) = -\pi\overline{R}_m(2,1) = \tfrac{1}{2}\sin 2\Phi_m(\tan\delta_m - \tan\delta'_m).$$

Here the indices 1 and 2 refer to the channels $P > P_0$ and $P < P_0$, respectively. Further, η_k is the roton density of states $(2\pi)^{-2}\mu|\xi_k|^{-1}$, where $\xi_k = (P_k - P_0)/P_k$. For P_k very close to P_0, the quantities η_1 and η_2 are equal. Similar expressions exist for T_m except that $\tan\delta_m$ is replaced by $\exp(i\delta_m)\sin\delta_m$.

For sufficiently large $|m|$, the scattering should be semiclassical and the direct channel should dominate. The approximation used in the phonon case should be valid and analogous phase-shift expressions can be obtained. An order of estimate for the correction terms in (98) indicates that the direct-channel results are valid for rotons provided the parameter

$$\omega_k = |m|\ |\xi_k \csc\theta_k|^2 \lambda_0^{-1}, \quad \lambda_0 = 2\mu\bar{\kappa} \tag{112}$$

is greater than 1. The corresponding phase shifts may be simplified and one finds

$$\delta_m = -\pi\lambda_0 \operatorname{sgn}(m)/4\tau_1|\xi_1|,$$
$$\delta'_m = -\pi\lambda_0 \operatorname{sgn}(m)/4\tau_2|\xi_2|,\ \Phi_m = 0, \tag{113}$$

$$\tau_k^2 = 1 - \operatorname{sgn}(m) P_0(4P_k\omega_k)^{-1}\ . \tag{114}$$

The remaining problem is to determine an approximation for $R_m(x)$ that includes the effects of both scattering channels and is valid at least for $\omega_k < 1$. Perhaps the most direct approximation $[0(\xi) < \theta_i < \pi - 0(\xi)]$ is obtained by expanding the energy denominator in (98) about each singular point and setting

$$H(x) = D(x) - m^2 x^{-2}$$
$$- \lambda_0 m \csc^2\theta_i (x-1)^{-1}(x-x_i)^{-1}. \tag{115}$$

Here x_i is $1-2\xi_i\csc^2\theta_i$. The resulting differential equation for $r_m(x, x')$ has four regular singularities $(0, 1, x_i, \infty)$ and is of the type originally studied by Heun in 1889.[33] Unfortunately, there is insufficient information of the analytic continuations of the solutions to this differential equation to determine a useful expression for $r_m(x, x')$. Some progress can be made by expanding solutions in terms of hypergeometic functions.[34] However, if this expansion technique is used, it becomes necessary to evaluate infinite series that converge very slowly, thereby rendering the procedure inapplicable.

An alternate method for determining $R_m(x)$ arises from the fact that for x and x' close to 1 and x_i, an adequate representation for potential (81) is

$$V_m(x, x') \approx V_m(1,1)\exp[-|m|\ |x-x'|]. \tag{116}$$

Actually, (116) is only valid provided $\frac{1}{2}|m|(x-x')^2$ is much less than 1. However if $\frac{1}{2}|m|(x-x')^2$ is greater than 1, both (81) and (116) are negligible. One would thus expect (116) to give the dominant behavior for $R_m(x)$ close to 1 and x_i. [There is one restriction arising from the Hermiticity of V_m and R_m; when (116) is used, the Hermitian property is only preserved for P_i close to P_0. However since thermal averages will only require results for P_i close to P_0, this difficulty is of no great importance.]

With potential (116), the corresponding R_m matrix elements satisfy the integral equation

$$R_m(x) = r_m(x,1) + \int_{-\infty}^{\infty} dx'\, r_m(x, x')$$
$$\times [2|m|V_m(1,1)]^{-1}\Big(\frac{d^2}{dx'^2} - m^2 + 2|m|V_m(1,1)$$
$$\times (2\pi)^{-2} p_i^2 x'[E_i - E_f(x')]^{-1} - H(x')\Big)R_m(x'), \tag{117}$$

where $r_m(x, x')$ is the solution to the differential equation

$$H(x) r_m(x, x') = -2|m|V_m(1,1)\delta(x-x'), \tag{118}$$

with boundary conditions

$$r_m(x, x') \to 0, \quad \text{as } x \to \pm\infty$$
$$r_m(x, x') \text{ continuous for } -\infty \le x \le \infty\ . \tag{119}$$

Equation (117) is derived in a manner similar to that used to derive (98). The direct-channel limit for large $|m|$ is found by setting

$$H(x) = \frac{d^2}{dx^2} - m^2\tau_i^2 - \frac{\lambda_0 m}{2\xi_i(x-1)}\ . \tag{120}$$

The resulting differential equation can be solved (confluent hypergeometric functions) and gives results identical to those of Eq. (113).

Both channels may be considered by choosing

$$H(x) = \frac{d^2}{dx^2} - m^2 - \lambda_0 m \csc^2\theta_i (x-1)^{-1}(x-x_i)^{-1}. \tag{121}$$

By appropriate substitutions, the resulting equation for $r_m(x, x')$ can be reduced to the spheroidal differential equation.[23,35,36] Sufficient information is available on spheroidal functions to determine the phase shifts and mixing parameter as an expansion in ω_k, valid when ω_k is less than 1. In particular, after much algebra, one finds for P_i close to P_0

$$\delta'_m = \delta_m - \tfrac{1}{2}\pi = \tfrac{1}{4}\pi - \tfrac{1}{2}\pi\lambda_0|\xi_k|^{-1}[-\Omega(-\omega_k)]^{1/2}, \quad m > 0,$$
$$\delta'_m = \delta_m = -\lambda_0[\Omega(\omega_k)]^{1/2}|\xi_k|^{-1} \tag{122}$$
$$\times \ln\{4[\Omega(\omega_k)]^{1/2}/e\omega_k\}, \quad m < 0\ ,$$

$$\Phi_m = \tfrac{1}{4}\pi\ ,$$
$$\Omega(\omega_k) = -\tfrac{1}{4}\xi_k^2\lambda_0^{-2} + \omega_k + \tfrac{1}{2}\omega_k^2 - \tfrac{1}{32}\omega_k^3\ . \tag{123}$$

It is immaterial in (122) and (123) whether k is 1 or 2; only lowest-order results are of interest.

Figures 3 and 4 show how δ'_m varies as a function of ω_1. The phase shifts and mixing parameter are given by (113) when $\omega_k > 0.64$ (m positive) and $\omega_k > 1.0$ (m negative); for the other values of ω_k, Eq. (122) is used. Near $\omega_k = 0.64$ (m positive) and $\omega_k = 1.0$ (m negative), the various approximations break down; the mixing parameter decreases from $\frac{1}{4}\pi$ to 0 in this transition region.

It is convenient in the roton problem to express frictional force (102) in the form

$$F_c = uL\rho_{nr} v_g \langle\sigma_c\rangle\ , \tag{124}$$

where ρ_{nr} is the roton contribution to the normal fluid density[29]

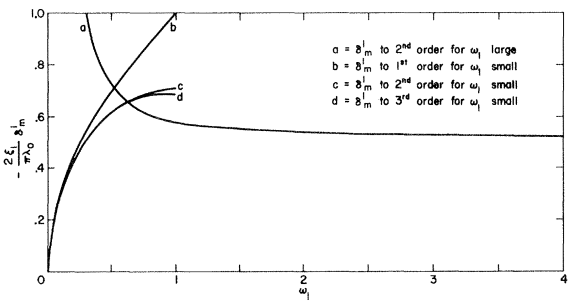

FIG. 3. Behavior of the roton-vortex phase shift δ'_m for positive m. For a roton with initial momentum $\vec{P}_i = (P_i, \theta_i, \phi_i)$ greater than P_0, and energy $E_i = \Delta + (2\mu)^{-1} \times (P_i - P_0)^2$, the parameters are $\lambda_0 = 2\mu\bar{\kappa}$, $\xi_1 = (P_i - P_0)/P_i$, and $\omega_1 = |m\,|\,|\xi_i \csc\theta_i\,|^2/\lambda_0$.

$$\rho_{nr} = \tfrac{2}{3}(2\pi)^{-3/2} P_0^4 (\mu/k_B T)^{1/2} \exp(-\Delta/k_B T), \quad (125)$$

and v_g is the average roton group velocity

$$v_g = (2k_B T/\mu\pi)^{1/2}. \quad (126)$$

The quantity $\langle\sigma_c\rangle$ is given by

$$\langle\sigma_c\rangle = \tfrac{3}{4}\int_{-\infty}^{\infty} dx \exp(-x^2)(1 + x/x_0)^4 |x| \int_{-1}^{1} d(\cos\theta) \times \sin^2\theta\, \sigma_c[(2\mu k_B T)^{1/2}(x + x_0),\ \sin\theta], \quad (127)$$

where $x_0 = P_0(2\mu k_B T)^{-1/2} \approx 11.75 T^{-1/2}$, and for P_i close to P_0,

$$P_0\sigma_c(P_i, \sin\theta_i) = (2\pi)^3 \times \int_0^{2\pi} d\chi (1 - e^{i\chi})[\,|\bar{T}'_{11}(\chi)|^2 + |\bar{T}'_{12}(\chi)|^2]. \quad (128)$$

The quantity x_0 is large for $0.5\,^\circ\mathrm{K} < T < 1.8\,^\circ\mathrm{K}$. Thus within this temperature range, one may expand $\langle\sigma_c\rangle$ in powers of x_0^{-1} and $\sigma_c(P_i, \sin\theta_i)$ in powers of ξ_i. (This is why it was only necessary to find the scattering results as $P_i \to P_0$.) Using the various expressions for the phase shifts and mixing parameter, and converting certain sums to integrals, one finds

$$P_0\sigma_c(P_i, \sin\theta_i) = 2 + \tfrac{1}{4}\pi^2\lambda_0 \csc^2\theta_1 (g_1 + g_2) - i\pi\lambda_0\xi_1^{-1}, \quad (129)$$

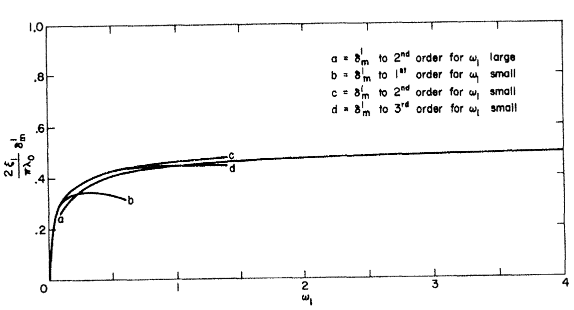

FIG. 4. Behavior of the roton-vortex phase shift δ'_m for negative m.

where

$$g_1 = -0.60 - \ln[\xi_1\lambda_0^{-1}(\lambda_0 \csc^2\theta_1 + \tfrac{1}{4})^{1/2}], \quad g_2 = -\tfrac{4}{3}\pi^{-2}\{\ln[\tfrac{1}{4}e^2\xi_1\lambda_0^{-1}(\lambda_0\csc^2\theta_1 - \tfrac{1}{4})^{1/2}]\}^3. \quad (130)$$

The thermal average $\langle\sigma_c\rangle$ can also be calculated. For a singly quantized vortex in the temperature range between 0.5 and 1.8 °K,

$$\langle\sigma_c\rangle \approx 5.7P_0^{-1} - i\kappa v_g^{-1} \approx (3 - 11i)\,\text{Å}. \quad (131)$$

Also, $F_c = uL\rho_{nr}\kappa(3.2x_0^{-1} - i)$. (132)

If (132) is compared to the theoretical results of Lifshitz and Pitaevskii,[10] it is found that $\mathrm{Re}(F_c)$ in (132) is about four times larger than that in Ref. 10; the $\mathrm{Im}(F_c)$ in (132) agrees in magnitude when the Iordanskii force[14] is excluded but not in sign; when the Iordanskii force is included, Ref. 10 gives $\mathrm{Im}(F_c) = 0$. The work of Ref. 10 is semiclassical in nature. If one would use the semiclassical lowest-order direct-channel results obtained in this paper, one would find that $\mathrm{Im}(F_c)$ is zero. Rayfield and Reif's work indicates that $\mathrm{Re}(\langle\sigma_c\rangle)$ is about 8.4 Å; Lucas's analysis of second-sound data[2] gives $\mathrm{Re}(\langle\sigma_c\rangle) \approx 11$ Å and $\mathrm{Im}(\langle\sigma_c\rangle) \approx -5$ Å.

One thus notes that the theoretical results do not exhibit close agreement with experiment. Improvement could perhaps be made by using better approximations to calculate F_c (sums were converted to integrals). However, the roton-vortex interaction may not be a symmetrized $\vec{P}\cdot\vec{v}_s$ and the structure of the roton may also be important. A theory which is less phenomenological is needed.

VII. CONCLUSION

The model presented in Secs. II–V to describe impurity-vortex scattering gives results consistent with Rayfield and Reif's limited experimental data provided spatial variations in the superfluid density are considered. The model lies between a classical and many-body theory. The T-matrix formalism of Sec. VI gives results consistent with those of Sec. III in the case of impurity-vortex scattering. For phonons, the results agree with existing theoretical and experimental work, except for the question of the Iordanskii force. In the case of rotons, there is apparent agreement between existing theories, but not experiment; the discrepancy could be due to an incorrect roton-vortex interaction and a neglect of roton structure.

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APPENDIX A

Consider the Bohm-Aharonov wave function

$$\psi(r,\chi)=\textstyle\sum_m e^{im(\chi+\pi)}e^{-i\pi|m+v|/2}J_{|m+v|}(pr) \qquad \text{(A1)}$$

obtained from (15), (18), and (19). Since (A1) converges for all χ, Abel's limit theorem[37] can be used to express $\psi(r,\chi)$ as the limit $\epsilon\to 0^+$ of

$$\psi(r,\mathrm{Re}\chi,\epsilon)=\psi_1(z,\chi,v,m_0)+\psi_1(z,-2\pi-\chi^*,-v,-m_0-1), \qquad \text{(A2)}$$

where

$$\psi_1(z,\chi,v,m_0)=\sum_{m_0+1}^{\infty} e^{im\chi}\,e^{i\pi(m-v)/2}J_{m+v}(z)\ . \qquad \text{(A3)}$$

The integer m_0 is defined by the inequality $-1\le m_0+v<0$ while $z=pr$ and $\chi=\mathrm{Re}\chi+i\epsilon$. The $*$ in Eq. (A2) denotes complex conjugation.

Equation (A3) may be explicitly summed by using the integral representation[38]

$$J_\nu(z)=(2\pi)^{-1}\int_{C_1} e^{-iz\sin\phi}\,e^{i\nu\phi}\,d\phi\ , \qquad \text{(A4)}$$

where C_1 is the complex contour illustrated in Fig. 5. In particular,

$$\psi_1(z,\chi,v,m_0)=\int_{C_1} h(\chi,\phi)d\phi\ , \qquad \text{(A5)}$$

$$h(\chi,\phi)=\frac{1}{2\pi}\,e^{-iz\sin\phi}\,e^{iv(\phi-\pi/2)}\,\frac{e^{i(m_0+1)(\phi+\chi+\pi/2)}}{1-e^{i(\phi+\chi+\pi/2)}}\ . \qquad \text{(A6)}$$

If C_1 is deformed into contours C_2 and C_3 (see Fig. 5; contour C_2 is symmetric about $\phi=-\frac{1}{2}\pi$ and C_1 is symmetric about $\phi=\frac{1}{2}\pi$), Eq. (A5) becomes

$$\psi_1(z,\chi,v,m_0)=\int_{C_2,C_3} h(\chi,\phi)d\phi+\textstyle\sum_s e^{iz\cos\chi}\,e^{iv(2\pi s-\chi-\pi)}$$
$$\times\,\Theta(\mathrm{Re}\chi-2\pi s+\pi-\epsilon')\Theta(2\pi s-\mathrm{Re}\chi-\epsilon'), \qquad \text{(A7)}$$

where the sum extends over all integers s. The quantity ϵ' is a small constant that vanishes when $\epsilon\to 0^+$ and $\Theta(x)$ is the step function

$$\Theta(x)=1,\quad x>0$$
$$\Theta(x)=0,\quad x<0\ . \qquad \text{(A8)}$$

Using (A2), (A7), and the symmetry of C_2 and C_3, one finds

$$\psi(r,\chi)=\psi_{\mathrm{inc}}(r,\chi)+\psi_{\mathrm{sc}}(r,\chi)\ , \qquad \text{(A9)}$$

$$\psi_{\mathrm{inc}}(r,\chi)=\textstyle\sum_s e^{iz\cos\chi}\cos(\pi v)\delta_{\chi,2\pi s}$$
$$+\textstyle\sum_s e^{iz\cos\chi}e^{iv(2\pi s-\chi-\pi)}$$
$$\times\,\Theta(\chi-2\pi s+2\pi)\Theta(2\pi s-\chi)\ , \qquad \text{(A10)}$$

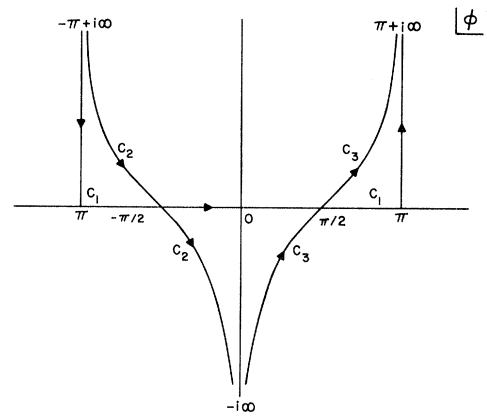

FIG. 5. Complex contours used in an evaluation of the Bohm-Aharonov wave function. The contour C_1 goes from $-\pi+i\infty$ to $-\pi$ to π to $\pi+i\infty$. The contour C_2 ranges from $-\pi+i\infty$ to $-i\infty$ and is symmetric about $\phi=-\frac{1}{2}\pi$. The contour C_3 goes from $-i\infty$ to $\pi+i\infty$ and is symmetric about $\phi=\frac{1}{2}\pi$.

$$\psi_{\mathrm{sc}}(r,\chi)=(2\pi)^{-1}\sin(\pi v)\mathrm{P.V.}\int_{C_2} d\Phi\,e^{iz\cos\Phi}$$
$$\times\,e^{iv\Phi}e^{i(m_0+1/2)(\chi+\Phi)}\csc\tfrac{1}{2}(\Phi+\chi)\ . \qquad \text{(A11)}$$

Here P.V. implies a Cauchy principal value, the symbol $\delta_{\phi,2\pi s}$ is the Kronecker delta, and $\Phi=\phi+\frac{1}{2}\pi$.

The quantity $\psi_{\mathrm{sc}}(r,\chi)$ may be evaluated by the method of steepest descents when z is large,[39] in which case for $0\le\chi\le 2\pi$

$$\psi_{\mathrm{sc}}\sim r^{-1/2}f(p,\chi)\,e^{i(pr-\pi/4)}, \qquad \text{(A12)}$$

$$f(p,\chi)=(2\pi p)^{-1/2}\,e^{i(m_0+1/2)\chi}\sin(\pi v)(\csc\tfrac{1}{2}\chi),$$
$$\chi_0<\chi<2\pi-\chi_0 \qquad \text{(A13)}$$

$$f(p,\chi)=(2\pi p)^{-1/2}i(2m_0+2v+1)\sin(\pi v),\quad \chi=0,\ 2\pi\ . \qquad \text{(A14)}$$

The quantity χ_0 is $(8/pr)^{1/3}$. Thus as long as $\chi_0<\chi<2\pi-\chi_0$

$$\psi(r,\chi)\sim e^{iv(\pi-\chi)}e^{i\vec{p}\cdot\vec{r}}+r^{-1/2}f(p,\chi)e^{i(pr-\pi/4)}, \qquad \text{(A15)}$$

where $f(p,\chi)$ is given by Eq. (A13).

APPENDIX B

Consider the scattering amplitude $(0<\chi<2\pi)$ obtained from Eqs. (21), (24), and (26). In this case,

$$(2\pi p)^{1/2}f(p,\chi)=e^{i(m_0+1/2)\chi}\sin(\pi v)\csc(\tfrac{1}{2}\chi)$$
$$+\textstyle\sum_m' e^{im(\chi-\pi)}[e^{-i\pi\nu_m}-e^{-i\pi|m+v|}]\ , \qquad \text{(B1)}$$

where $\nu_m=[(m+v)^2-w^2]^{1/2}$, the prime indicates the sum is only over those values of m where ν_m is real, and m_0 is defined by the inequality $-1\le m_0+v<0$. Since m equals (angular momentum)/

$\hbar$, the summation in (B1) may be replaced by an integration in the classical limit ($\hbar \to 0$) provided the phase is slowly varying; this latter requirement may be obtained by adding $2\pi m s$ (s an integer) to the phase in (B1).[40] Thus, for each s,

$$\sum_m{}' \approx \int{}' dm\, e^{i2\pi m s} . \tag{B2}$$

The method of stationary phase may now be used to evaluate (B2).[41] For every $s \neq 0$ (there is no stationary phase for $s = 0$), the dominant contribution in (B1) and (B2) comes from the factor $\exp(-i\pi\nu_m)$; in this case, it is not difficult to show that

$$(2\pi p)^{1/2} f(p, \chi, s) = (2w)^{1/2} \times [(2s-1+\chi/\pi)^2 - 1]^{-3/4} e^{ig(s,\chi)} , \tag{B3}$$

where $g(s, \chi) = -v\pi(2s-1+\chi/\pi) + \frac{1}{4}\pi + w\pi[(2s-1+\chi/\pi)^2 - 1]^{1/2}$. (B4)

The classical differential cross section is thus

$$\frac{d\sigma}{d\chi} = \sum_{s\neq 0} |f(p, \chi, s)|^2 = (\pi p)^{-1} w \sum_{s \neq 0} [(2s-1+\chi/\pi)^2 - 1]^{-3/2} . \tag{B5}$$

Equation (B5) agrees with the differential cross section found from classical theory when the scattering angle lies between 0 and 2π.[7]

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