range is described in Sec. IV B of the present paper. Although the electron collision frequency for excitation of the bending and symmetric stretch vibrations is much less than the momentum-transfer collision frequency in the 3–5-eV range, the total inelastic cross section (Ref. 13) in this energy range is comparable in magnitude to the momentum-transfer cross section, thereby violating the condition established by Holstein (Ref. 6) for the validity of the two-term expansion of the distribution function. The effect of this result on the present analysis is not expected to be significant but should be investigated further.

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Electromagnetic Instability of Counterstreaming Plasmas in a Magnetic Field

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Using the Vlasov equation we calculate σ_{zz} of the conductivity tensor for counterstreaming plasmas in a magnetic field. Using the dispersion relations for ordinary waves, we consider the electromagnetic instability. The spectrum of the instability and its growth rate have been calculated.

I. INTRODUCTION

The study of instabilities in plasma beams having equal but opposite velocities has received a great deal of attention in the literature.[1–3] These instabilities play an important role in some of the basic physical processes occurring in plasmas, as well as in their possible applications to microwave amplification and plasma heating devices. The possibility of plasma beam instabilities against electromagnetic excitations has been pointed out by Gold[2] and been the subject of work by Momota.[3] Here the "electromagnetic instability" indicates excited fields obeying the full Maxwell equations, as opposed to the "electrostatic instability" where the excited electric fields can be described as gradients of time-dependent potentials.

Thus the electromagnetic instability is characterized by the existence of the ac magnetic field and therefore is typically smaller by a factor of v_{drift}/c than an electrostatic instability. The observation of any electromagnetic instability requires a system which at the same time is stable against growing electrostatic fluctuations.

Recently Lee[4] calculated the electromagnetic instability in counterstreaming plasmas in a magnetic field. His calculations consist of the cold-plasma model and the first-order nonlocal corrections, i.e., the terms proportional to $kv_{\mathrm{th}}/\omega_c$. This correction (to the first order in kr_c, where $r_c = v_{\mathrm{th}}/\omega_c$ is the radius of gyration) cannot describe the instability in its entirety. It is easy to establish that when v_{th} decreases the wave number of maximum instability increases, thus the limit of $kr_c < 1$ is unrealistic.

In this paper we calculate the electromagnetic instability for counterstreaming plasmas in a magnetic field. Our starting point for the electron dynamics is the Vlasov equation and we obtain a solution for the conductivity tensor which is applicable for all values of kr_c. For a plasma beam having average velocity v_0 in the z direction, the conductivity components σ_{xz} and σ_{yz} are proportional to v_0. Therefore for two counterstreaming electron-plasma beams we obtain $\sigma_{xz} = \sigma_{yz} = 0$. Adding a static homogeneous magnetic field in the z direction does not mix electronic motion in the $x(y)$ with the z directions and we again obtain $\sigma_{xz} = \sigma_{yz} = 0$. We therefore obtain that two counterstreaming electron-plasma beams can separately sustain ordinary and extraordinary electromagnetic waves.[4] While the extraordinary wave is not affected by the relative streaming, the ordinary wave is modified. Moreover, under certain conditions, the plasma becomes unstable against excitation of ordinary electromagnetic fields, i.e., growing electric field in the z direction, with magnetic field and wave number k in the x, y plane.

In Sec. II we solve the diagonal part of the conductivity in the z direction and establish the propagation condition of ordinary waves in our system. In Sec. III, we consider the condition for electromagnetic instability for arbitrary kr_c and discuss in detail the analytic solution for the extreme cases of $kr_c > 1$. Section IV is reserved for discussion and conclusions.

II. ANALYSIS OF DISPERSION RELATIONS

We consider an electron-plasma system with smeared positive background in a static homogeneous magnetic field $\vec{B} = \hat{z} B$ having a drift velocity in the $\hat{z}$ direction, i.e., $\langle v_z \rangle = v_0$. The steady-state electron distribution function $f_0(v_x^2 + v_y^2,\ (v_z + v_0)^2)$ is taken for simplicity to be Maxwellian. The response of the plasma to "small" electromagnetic fields is described by the linearized Vlasov equation

$$\left(\omega - \vec{k}\cdot\vec{v} - i\omega_c(\vec{v}\times\hat{z})\cdot\frac{\partial}{\partial\vec{v}}\right) f(\vec{v}\vec{k}\omega) = \frac{ie}{m}\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{H}\right)\frac{\partial f_0}{\partial\vec{v}} \ . \qquad (1)$$

Here $\vec{E}$ and $\vec{H}$ are, respectively, the electric and magnetic components of the electromagnetic field obeying the Maxwell equations. The integration of Eq. (1) is well known[5] and we now give the results: The conductivity components σ_{xz} and σ_{yz} are proportional to v_0 and therefore for counterstreaming electron-plasma systems $\sigma_{xz} = \sigma_{yz} = 0$. For the diagonal component of the dielectric function in the $\hat{z}$ direction, ϵ_{zz}, which is of interest to us, we obtain

$$\epsilon_{zz} = 1 - \frac{\omega_p^2}{\omega^2} e^{-\lambda}\left[\left(1 + \frac{v_0^2}{v_{th}^2}\right) I_0 - \frac{v_2^0}{v_{th}^2} e^{\lambda} + 2\left(1 + \frac{v_0^2}{v_{th}^2}\right)\omega^2 \sum_{n=1}^{\infty} \frac{I_n(\lambda)}{\omega^2 - n^2\omega_c^2}\right] , \qquad (2)$$

where the I_n's are the modified Bessel function. Here we have used the relation $\epsilon_{zz} = 1 + (4\pi i/\omega)\sigma_{zz}$. In Eq. (2) ω_p is the electron-plasma frequency of our counterstreaming plasma system, ω_c is the electron cyclotron frequency, v_{th} is the thermal velocity (which for simplicity we assume to be isotropic) and $\lambda = (kv_{th}/\omega_c)^2$. Here $\sqrt{\lambda}$ is a measure of the ratio of the radius of gyration to the wavelength of the electromagnetic field.

We consider next the dispersion relations for "ordinary" electromagnetic waves, i.e., $\vec{E} = \hat{z} E$, $\vec{H} = \hat{y} H$, and $\vec{k} = \hat{x} k$, where the plasma drifts in the $\pm\hat{z}$ directions. Their dispersion relations are given by

$$k^2c^2/\omega^2 = \epsilon_{zz} \ . \qquad (3)$$

We first discuss the limit of small k (long wavelength) or vanishingly small thermal velocity. Using the asymptotic limit for the I_n's in Eq. (2) and substituting ϵ_{zz} in Eq. (3) gives the dispersion relations:

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{k^2(v_0^2 + v_{th}^2)}{\omega^2 - \omega_c^2}\right) . \qquad (4)$$

Here ω^2 obeys the quadratic equation

$$\omega^4 - \omega^2(\omega_c^2 + \omega_p^2 + k^2c^2) + \omega_c^2(\omega_p^2 + k^2c^2) - k^2\omega_p^2(v_0^2 + v_{th}^2) = 0 \ . \qquad (5)$$

We consider k to be real and the solution for ω indicates a possible instability when

$$k^2 > \frac{\omega_c^2}{v_0^2(1 - c^2\omega_c^2/\omega_p^2 v_0^2) + v_{th}^2} \ , \qquad (6)$$

with the condition $\omega_p^2 v_0^2 > c^2\omega_c^2$. In the limit of $v_{th} = 0$ we are in agreement with Ref. 4. However, our results are different for the finite but small thermal velocity case. We find that the smallest possible k, k_{min}, for the starting of the instability decreases with the increase of v_{th}. However, that does not mean that the spectrum of unstable k increases. On the contrary, one must also impose the restriction which was used in arriving at Eq. (6), i.e., $\omega_c^2/v_{th}^2 > k^2$, which indicates that the range of unstable k decreases as $v_{th} \to \infty$ or $v_0 \to 0$. It is essential to point out here that for finite drift velocity v_0, the thermal velocity v_{th} cannot be assumed to be small for realistic cases. As is well known, the system will be unstable against electrostatic excitation[6] if $v_0 > 1.37 v_{th}$. Moreover, the solution of Eq. (5), substituting $\omega = \pm ix$, gives for the growth rate x the result

$$x^2 = \frac{k^2[\omega_p^2(v_0^2 + v_{th}^2) - \omega_c^2c^2] - \omega_c^2\omega_p^2}{\omega_c^2 + \omega_p^2 + k^2c^2} \ , \qquad (7)$$

which is a monotonic increasing function of k, having the maximum growth rate at $k = \infty$, $x_\infty^2 = \omega_p^2(v_0^2 + v_{th}^2)/c^2 - \omega_c^2$. This result, giving large growth rates at $k = \infty$, is inconsistent with the assumption originally used in the cold-plasma or small-v_{th} cases, i.e., $kv_{th}/\omega_c < 1$. We must therefore solve for the instability using ϵ_{zz}, which is valid for large λ, in order to investigate the most important spectral part of the instability, i.e., the case of large k.

We next calculate ϵ_{zz} in the limit of $\lambda \gg 1$. Here, using the asymptotic forms for the I_n's, we obtain to dominant order in λ^{-1}

$$\epsilon_{zz} = 1 - \frac{\omega_p^2}{\omega^2}\left[-\frac{v_0^2}{v_{th}^2} + \frac{1}{\sqrt{2\pi\lambda}}\left(1 + \frac{v_0^2}{v_{th}^2}\right)\frac{\pi\omega}{\omega_c}\cot\frac{\pi\omega}{\omega_c}\right] . \qquad (8)$$

The dispersion relations for the ordinary waves are determined now by substituting Eq. (8) into Eq. (3):

$$\omega^2 = k^2c^2 + \omega_p^2\left(\frac{1+v_0^2/v_{\rm th}^2}{\sqrt{2\pi\lambda}}\ \frac{\pi\omega}{\omega_c}\cot\frac{\pi\omega}{\omega_c} - \frac{v_0^2}{v_{\rm th}^2}\right). \qquad (9)$$

The modified dispersion relations for the ordinary waves, in the presence of the drift, are given by Eq. (9) in the limit of $\lambda = k^2v_{\rm th}^2/\omega_c^2 \gg 1$. The solution of $\omega(k, v_0)$ can be obtained only numerically and compared with experimental values. However, in this paper we limit ourselves to the instability conditions, which we discuss in Sec. III.

III. CALCULATIONS OF INSTABILITY

We consider now the conditions under which the plasma will be unstable. As we have pointed out earlier, the dominant region of the spectrum is in the large-k regime. The solution for the growth rate can be calculated numerically using our Eqs. (2) and (3). We would like, however, in order to obtain some analytical results, to consider the dominant contribution in the large-k regime, i.e., $\lambda > 1$. Our starting point is Eq. (9) using the assumption[7] that $\omega = \pm ix$ while k is a real quantity. The solution for the growth rate x is governed by the equation

$$\frac{1}{\sqrt{2\pi}}\ \frac{\alpha^2(\beta_0^2+\beta_{\rm th}^2)}{\lambda^{3/2}}\ y\coth y = \left(\frac{\alpha^2\beta_0^2}{\lambda} - 1\right) - \frac{1}{\pi^2}\ \frac{\beta_{\rm th}^2}{\lambda}\ y^2. \qquad (10)$$

Here, $\alpha = \omega_p/\omega_c$, $\beta_0 = v_0/c$, $\beta_{\rm th} = v_{\rm th}/c$, and $y = \pi x/\omega_c$. In seeking solutions for x, as functions of λ, we are interested in real solutions only. Thus, for a real y, $y\coth y$ is always positive and Eq. (10) imposes an upper limit on the wave number of the instability given roughly by $\alpha^2\beta^2/\lambda - 1 > 0$. The region of unstable k is determined by setting $y = 0$ in Eq. (10) and solving the cubic equation for $\sqrt{\lambda}$, given by

$$(\sqrt{\lambda})^3 - \alpha^2\beta_0^2\sqrt{\lambda} + (1/\sqrt{2\pi})(\beta_{\rm th}^2+\beta_0^2)\,\alpha^2 = 0\ . \qquad (11)$$

The solution of Eq. (11) is obtained from

$$(\sqrt{\lambda})_s^{\pm} = 2\sqrt{\tfrac{1}{3}\alpha^2\beta_0^2}\ \cos(\tfrac{1}{3}\Phi_\pm + \tfrac{2}{3}\pi s),\quad s = 0, 1, 2 \qquad (12a)$$

where

$$\cos\Phi_\pm = \pm\left(\frac{27}{8\pi}\ \frac{(1+\beta_{\rm th}^2/\beta_0^2)^2}{\alpha^2\beta_0^2}\right)^{1/2}. \qquad (12b)$$

Here the "plus" or "minus" solutions indicate possible positive or negative roots for $\cos\Phi$. Out of these six possibilities we obtain only three different solutions for $\sqrt{\lambda} = kv_{\rm th}/\omega_c$, which must be positive. It is easy to verify that $(\sqrt{\lambda})_1^+$, $(\sqrt{\lambda})_2^+$, and $(\sqrt{\lambda})_1^-$ are always negative and can be discarded. The three roots are therefore given by

$$(\sqrt{\lambda})_0^+ = 2\sqrt{\tfrac{1}{3}\alpha^2\beta_0^2}\ \cos\tfrac{1}{3}\Phi_+\ ,$$
$$(\sqrt{\lambda})_0^- = 2\sqrt{\tfrac{1}{3}\alpha^2\beta_0^2}\ \cos\tfrac{1}{3}\Phi_-\ , \qquad (13)$$
$$(\sqrt{\lambda})_2^- = 2\sqrt{\tfrac{1}{3}\alpha^2\beta_0^2}\ \cos(\tfrac{1}{3}\Phi_- + \tfrac{4}{3}\pi)\ .$$

We next point out that $0 < \Phi_+ < \frac{1}{2}\pi$ and we obtain that $(\sqrt{\lambda})_0^+$ is always larger than $\alpha\beta_0$. As can be seen from our Eq. (10), when $\sqrt{\lambda} > \alpha\beta_0$ no unstable solution for y exists and the root $(\sqrt{\lambda})_0^+$ can be discarded. We are left, therefore, with two solutions for λ; the upper and lower limits, respectively, determine the unstable k region. They are given by

$$\sqrt{\lambda_U} = 2\sqrt{\tfrac{1}{3}\alpha^2\beta_0^2}\ \cos\tfrac{1}{3}\Phi_-\ , \qquad (14a)$$

$$\sqrt{\lambda_L} = 2\sqrt{\tfrac{1}{3}\alpha^2\beta_0^2}\ \cos(\tfrac{1}{3}\Phi_- + \tfrac{4}{3}\pi)\ , \qquad (14b)$$

where Φ_- is defined in Eq. (12a). The upper limit λ_U is proportional to $\alpha^2\beta_0^2$. The lower limit λ_L is only weakly dependent on the value of $\alpha\beta_0$, for $\alpha\beta_0 > 1$. In order to show this point let us define $\Phi_- = \frac{1}{2}\pi + \gamma$. It is a matter of simple algebra to show that

$$\sqrt{\lambda_L} = 2\sqrt{\tfrac{1}{3}\alpha^2\beta_0^2}\ \sin\tfrac{1}{3}\gamma\ ,$$

where

$$\gamma = \sin^{-1}\left(\frac{27}{8\pi}\ \frac{(1+\beta_{\rm th}^2/\beta_0^2)^2}{\alpha^2\beta_0^2}\right)^{1/2}.$$

Thus the dominant part of $\sqrt{\lambda_L}$ is given, for $\alpha^2\beta_0^2 > 1$, by

$$\sqrt{\lambda_L} = \frac{1}{\sqrt{2\pi}}\left(1 + \frac{\beta_{\rm th}^2}{\beta_0^2}\right) + O\left(\frac{1}{\alpha^2\beta_0^2}\right). \qquad (14c)$$

We obtain λ_L to be of order unity and independent of $\alpha\beta_0$, and within the same order of magnitude of λ_L calculated using the small-λ approximation. Thus, the region of unstable λ is given by $\lambda_L > \lambda > \lambda_U$, where λ_L and λ_U are defined by $y(\lambda_L) = y(\lambda_H) = 0$, with $\lambda_L \sim 1$ and $\lambda_U \sim \alpha^2\beta_0^2 > 1$. Now that we expect $y_{\rm max}$ to be in the large-λ region, we find $y_{\rm max}$ using Eq. (10) and approximating $\coth y \approx 1$ and obtain

$$[\alpha^2(\beta_0^2+\beta_{\rm th}^2)/\sqrt{2\pi}]\ y = (\alpha^2\beta_0^2\lambda^{1/2} - \lambda^{3/2}) - \pi^{-2}\beta_{\rm th}^2\lambda^{1/2}y^2. \qquad (15)$$

The last term on the right-hand side of Eq. (15) is small and can be neglected. Simple algebra then gives $\lambda_{\rm max}$, at which y is maximum, and $y_{\rm max}$:

$$\lambda_{\rm max} = \tfrac{1}{3}\alpha^2\beta_0^2\ , \qquad (16a)$$

$$k_{\rm max} = \frac{1}{\sqrt{3}}\ \frac{\omega_p}{c}\ \frac{v_0}{v_{\rm th}} < k_U\ , \qquad (16b)$$

$$y_{\rm max} = \frac{\sqrt{2\pi}}{\sqrt{3}}\ \frac{2}{3}\ \frac{\beta_0^2}{\beta_0^2+\beta_{\rm th}^2}\ \alpha\beta_0 > 1\ , \qquad (16c)$$

$$x_{\rm max} = \frac{\sqrt{2\pi}}{\sqrt{3}}\ \frac{2}{3\pi}\ \frac{v_0^2}{v_0^2+v_{\rm th}^2}\ \frac{\omega_p v_0}{c}\ . \qquad (16d)$$

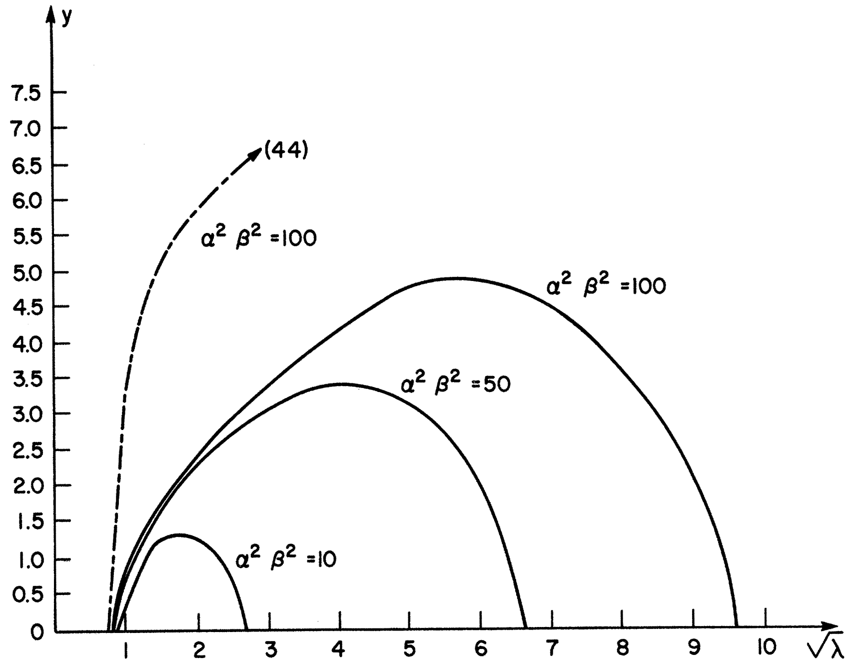

FIG. 1. Plot of $y=\pi x/\omega_c$ as a function of $\sqrt{\lambda}=kv_{\text{th}}/\omega_c$ for $\alpha^2\beta^2=10$, $\alpha^2\beta^2=50$ and $\alpha^2\beta^2=100$. The dash-dotted line represents the solution for y obtained using the small-λ approximation for $\alpha^2\beta^2=100$, which asymptotically increases to $y=44$ at $\sqrt{\lambda}\to\infty$.

For consistency we check the ratio

$$R=\beta_{\text{th}}^2\sqrt{\lambda_{\max}}\,y_{\max}^2/\alpha^2\beta_0^2\sqrt{\lambda_{\max}}$$

and obtain $R\approx\beta_{\text{th}}^2=v_{\text{th}}^2/c^2\ll 1$, which is consistent with our assumption. Our result for the maximum growth rate is proportional, as expected, to $\omega_p v_0/c$, indicating an instability which is characterized by the existence of the ac magnetic field. We next compare our result for $y_{\max}$ with its counterpart y_∞, obtained using the small-λ approximation. We find using Eqs. (7) and (16) that for $\alpha^2\beta_0^2\gg 1$

$$R=\frac{y_{\max}}{y_\infty}=\frac{1}{\sqrt{\pi}}\left(\frac{2}{3(1+\beta_{\text{th}}^2/\beta_0^2)}\right)^{3/2}<1\ .$$

Our solution $y_{\max}$ is smaller by about one order of magnitude than y_∞, the maximum growth rate obtained using small-λ approximation.

We have next calculated numerically y as a function of $\sqrt{\lambda}$ using our Eq. (10) for a realistic case of $\beta_{\text{th}}=\beta_0=\beta$, for various values of $\alpha\beta$. The results, shown in Fig. 1, indicate that the lower limit for the instability, $\sqrt{\lambda_L}$, is roughly independent of the value of $\alpha\beta$ while the upper limit $\sqrt{\lambda_U}$ is roughly proportional to it. The computed values of λ_L and λ_U agree with our calculated results, Eq. (14a). Also, the maximum growth rate $y_{\max}$ and its position $\sqrt{\lambda_{\max}}$ are in a good agreement with our approximately calculated results given by Eq. (16).

IV. DISCUSSION

We have calculated the electromagnetic instability for a counterstreaming electron plasma. The spectrum of the unstable k has a lower as well as upper limit. The lower limits, in the case of small λ or large λ, give us roughly $k_L\sim\omega_c/v_{\text{th}}$ for $\alpha\beta_0=(\omega_p/\omega_c)(v_0/c)>1$, provided $v_0\sim v_{\text{th}}$, which is a necessary condition to avoid the strong electrostatic instability. The upper bound is given by λ_U, where $\sqrt{\lambda_U}\sim\alpha\beta_0$, and therefore $k_U\sim(\omega_p/v_{\text{th}})(v_0/c)$, independent of ω_c for $\alpha\beta_0\gg 1$. The stabilization effect of the magnetic field is seen as follows: After simple algebra our result for k_U, Eq. (14a), may be written as

$$k_U=\frac{\omega_p}{c}\frac{v_0}{v_{\text{th}}}\left(\cos\frac{\gamma}{3}-\frac{1}{\sqrt{3}}\sin\frac{\gamma}{3}\right),$$

where

$$\gamma=\sin^{-1}\left(\frac{\sqrt{27/8\pi}\,(1+v_{\text{th}}^2/v_0^2)}{v_0/c}\,\frac{\omega_c}{\omega_p}\right).$$

The first-order correction in ω_c is given by

$$k_U=\frac{\omega_p}{c}\frac{v_0}{v_{\text{th}}}\left(1-\frac{\omega_c}{\omega_p}\,\frac{\sqrt{27/8\pi}\,(1+v_{\text{th}}^2/v_0^2)}{\sqrt{3}\,3v_0/c}\right).$$

Therefore the increase of the magnetic field will decrease k_U and make the system more stable. Similar conclusions can be obtained for $x_{\max}$, i.e., $x_{\max}$ decreases with increasing ω_c, however it is done numerically and shown in Fig. 2.

For very strong magnetic fields, i.e., for $\alpha^2\beta_0^2<10$ our large-λ approximation fails as does the small-λ approximation. Therefore, the study of the approach to a "stable system" when ω_c increases cannot be treated analytically and one must depend on numerical calculation using our Eqs. (2) and (3).

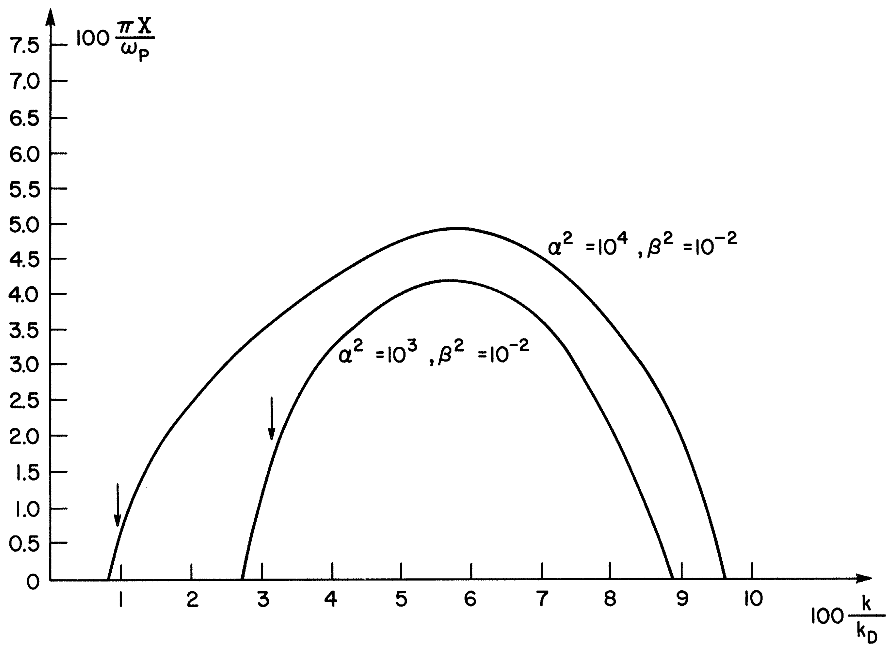

FIG. 2. Plot of x versus k for small magnetic field $\alpha = 10^4$ and larger magnetic field $\alpha = 10^3$, where $a = \omega_p/\omega_c$. The arrows on the curves point to the particular k for which $\lambda = 1$.

We finally would like to point out that in our model the ions were considered to be smeared. For $\alpha\beta_0 > 1$, we obtain the growth rate to be $x_{max} \sim \omega_p v_0/c$, which can be larger than the ion plasma frequency Ω_p. Thus, our smeared ion model is justified. However, when x_{max} decreases when we increase ω_c, as can be seen from Fig. 2, one might have to consider the ion dynamics as well. In order to estimate this process quantitatively we incorporated the effect of the ion motion, neglecting their drift velocity. This results in the addition of the term

$$\omega_p^2\left(\frac{m}{M}\right)^{1/2}\frac{1}{(2\pi\lambda)^{1/2}}\, y \cot\left(\frac{\pi\omega}{\omega_c}\frac{M}{m}\right)$$

to the right-hand side of our Eq. (9), where M is the ionic mass. Carrying this over to the calculations of the growth rates, our Eq. (10) now takes the form

$$\frac{1}{(2\pi)^{1/2}}\frac{\alpha^2(\beta_0^2+\beta_{th}^2)}{\lambda^{3/2}}\, y \coth y + \frac{1}{(2\pi)^{1/2}}\left(\frac{m}{M}\right)^{1/2}\frac{\alpha^2\beta_{th}^2}{\lambda^{3/2}}\, y \coth\left(y\frac{M}{m}\right) = \left(\frac{\alpha^2\beta_0^2}{\lambda}-1\right) - \frac{1}{\pi^2}\frac{\beta_{th}^2 y^2}{\lambda} \quad . \qquad (10')$$

It is easily seen that the boundaries of the instability, obtained by setting $y = 0$, are being shifted by an amount proportional to $(m/M)^{3/2}$. As for the maximum instability for $y > 1$ we obtain a change of y_{max} of the order of $\sqrt{m/M}$. Thus the effects of the ion dynamics are negligible. This result can be expected in our case, since the electromagnetic instability is coupled to current rather than charge fluctuations, to which the massive ions hardly contribute.

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