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Self-Induced Transparency in Two-Pass Attenuators and Related Phenomena

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A theory describing two sequential electromagnetic pulses interacting with a resonant, inhomogeneously broadened attenuator is presented. The resulting equations are solved on a computer. Limiting cases are considered which indicate the conditions under which the second pulse is expected to increase in energy. The self-induced transparency effect in a two-pass attenuator is treated using this theory. It is shown that there are qualitative as well as quantitative differences between the single-pass and double-pass cases. For example, curves of output versus input energy in the two-pass case may have observable structure that is sensitive to the presence or absence of atomic coherence. It is found that under a wide range of circumstances, the output for the second pulse is independent of whether the pulse is going in the same or in the opposite direction from the first. This result is seen to be related to more general considerations involving the interaction of a pulse with spatially inhomogeneous systems such as a chain of amplifiers and attenuators. Also discussed are the measurement of T_1 using two-pass techniques, and the possibility of using these phenomena in a practical application to optical logic circuitry.

I. INTRODUCTION

Recently, there has been considerable interest in the self-induced transparency effect (SIT) first predicted by McCall and Hahn,[1,2] and the major aspects of the problem have been treated in detail from a theoretical standpoint.[3,4] However, there have been difficulties in the experimental verification of the effect. These difficulties are, in part, due to the limited number of optical systems available in which the conditions are suitable for SIT. In practice, only two media have been used to date: first, ruby,[1,3] in which the experimental conditions present great difficulty; and second, SF_6,[5] in which the resonances used are probably highly degenerate.[6] In this paper, the single-pass single-pulse theory of an electromagnetic field interacting with an inhomogeneously broadened ensemble of two-level atoms which was developed in a prior publication[7] is generalized to cover two-pass (or two-pulse) problems. The original motivation for doing this was to deal with SIT using ultrasonic techniques. An ultrasonic pulse interacts with paramagnetic impurities in a crystal[8] in a manner analogous to that of optical pulses interacting with an inhomogeneously broadened spin-$\frac{1}{2}$ system so that the same equations of motion apply to both cases. Thus, one can expect to observe SIT in an ultrasonic system. There are experimental advantages that result from using ultrasonics: One has wide ranges of available carrier frequencies; the atomic system can be easily tuned; and large amplitude pulses can be produced. However, in the ultrasonic case, it is easiest to use the same transducer both for producing the input and for monitoring the output. Thus, the pulse passes down the crystal, reflects off the open end, and passes back through the same region of the crystal with which it had previously interacted. Some computer calculations were made by the author at a previous time in connection with such an experiment,[9] using the model presented in this paper. This model differs from the actual experimental situation in that the pulse passes completely out of the attenuator before being reflected back on the second pass. Nonetheless, the calculations agreed qualitatively with the experiment and indicate that the major modifications in the SIT effect due to the second pass can be understood without worrying about the fact that the pulse overlaps itself in some part of the crystal. Those calculations with an extensive discussion of the agreements and discrepancies between experiment and theory plus a discussion of the relevance of the overlap (standing wave) problem are given in Ref. 9.

In order to maintain connection with previous work, the calculations presented in this paper will be described in terms of optical systems. The theory is presented in Sec. II, and involves equations that are too complicated to be readily solved in closed form. Hence, the equations are solved by numerical integration. The theory is much more general than just two-pass problems, and treats the whole range of questions that arise when a second pulse interacts with the spatially inhomogeneous line shape[10] produced by the interaction of a previous pulse with an attenuator. This paper goes on to deal with several of these questions. The conditions for gain and loss of energy in the case of an inhomogeneously broadened attenuator "pumped" by a π pulse is discussed in Sec. III. Section IV presents the expected modifications of SIT in a two-pass system and indicates how these are, in turn, changed by introducing finite decay times. In Sec. V A, the question of the direction of propagation of the second pulse is considered, and this leads to general conclusions about the interaction of pulses with complicated systems of amplifiers and attenuators. In Sec. V B, the utility of these phenomena for measuring T_1 are considered, and in Sec. V C, a practical application of this problem to logic circuits is presented.[11]

II. DERIVATION OF EQUATIONS

In this section, the theory of two sequential pulses interacting with a resonant inhomogeneously broadened medium is presented. This problem can involve photon echoes,[12] interference phenomena,[13] and other considerations which make the general analysis very lengthy. However, so long as the inverse inhomogeneous linewidth (T_2^*) is much less than the separation in time between the pulses (Δt), and so long as the pulses are well resolved (i.e., if $\hat{t}$ is the pulse width, then $\Delta t \gg \hat{t}$), the particular effects related to having a phase memory time (T_2) that is long compared to the pulse separation time appear, as in the case of the photon echo, at a time after the second pulse has passed. The first two pulses evolve in a manner independent of the relative magnitudes of T_2 and Δt.[14] In this paper, we are considering only the behavior of these two pulses, and the derivation of the working equations is greatly simplified by assuming that $T_2 \ll \Delta t$. In this limit, the problem becomes a simple generalization of the single-pass single-pulse case, and the reader is referred to a prior paper on that problem[7] for all of the details of the derivation.

The atoms in the medium are described by 2×2 density matrices ρ, where

$$\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix} \quad . \tag{1}$$

Each atom is assigned a dipole matrix element $\wp$, decay times T_2 (phase memory time) and T_1 (atomic lifetime), and natural frequency ω_0. The frequencies of the atoms are distributed according to a Gaussian $\sigma(\omega)$ which is resonant with the field (i.e., has a center at $\omega = \nu$, where ν is the carrier frequency of the radiation), and which has a standard deviation $2/T_2^*$. T_2^* is then the dephasing time of the atomic ensemble. The two pulses, labeled $E_\lambda(t, z)$, $\lambda = 1, 2$, are taken to be plane polarized waves of uniform cross section, passing either to the right or to the left along a single axis. The fields are described by slowly varying amplitudes[7] $\mathcal{E}_\lambda$ such that if $t_1(z)$ is a time before $\mathcal{E}_1$ arrives at the point z, then $t_2(z)$ is a time after it leaves, $t_3(z)$ is a time before $\mathcal{E}_2$ arrives, and $t_4(z)$ a time after $\mathcal{E}_2$ has passed:

$$E_\lambda(t, z) = \mathcal{E}_\lambda(t, z) \cos(k_\lambda z - \nu t) \quad , \tag{2}$$

$$\begin{aligned} &\mathcal{E}_1(t, z) = 0, \quad t > t_2(z) \;\; t < t_1(z) , \\ &\mathcal{E}_2(t, z) = 0, \quad t < t_3(z) \;\; t > t_4(z) , \end{aligned} \tag{3}$$

where $k_\lambda = k$ if the field is going to the right and $k_\lambda = -k$ if it is going to the left. The convention used here will have the first pulse always going to the right (i.e., $k_1 = k$). We have ignored the phase shift that occurs between the passes, since it has no consequences for this problem.

The response of the medium is contained in a susceptibility χ which is the cosine Fourier transform over ω of the instantaneous line shape,[10] i.e.,

$$\chi(T, t, z) = \frac{1}{2\pi\sigma(\nu)} \int_{-\infty}^{\infty} d\omega\, \sigma(\omega) \times [(\rho_{bb}(\omega, t, z) - \rho_{aa}(\omega, t, z)] \cos((\omega - \nu)T). \tag{4}$$

Then the equations which describe the system on the first pass are derived in Ref. 7 and are shown to be[15]

$$\frac{k_\lambda}{|k|} \frac{\partial \mathcal{E}_\lambda}{\partial z} + \frac{1}{c} \frac{\partial \mathcal{E}_\lambda}{\partial t} = -a \int_{-\infty}^{t} dt'\, \mathcal{E}_\lambda(t', z) e^{-(t-t')/T_2} \chi(t - t', t', z) \quad , \tag{5}$$

$$\frac{\partial \chi(T, t, z)}{\partial t} = \frac{[D(T) - \chi(T, t, z)]}{T_1} - \frac{\wp^2}{2\hbar^2} \int_{-\infty}^{t} \times dt'\, \mathcal{E}_\lambda(t, z) \mathcal{E}_\lambda(t', z) e^{-(t-t')/T_2} \times \{\chi(T + t - t', t', z) + \chi(T - t + t', t', z)\}, \tag{6}$$

where $\lambda = 1$, $k_1 = k$, and c is the velocity of light in the inert background. The initial condition for χ is

$$\chi(T, t_1, z) = D(T) \quad , \tag{7}$$

where $D(T)$ is the normalized Fourier transform of $\sigma(\omega)$. If ϵ is the dielectric constant, and N is the total number of atoms,[15] then

$$\alpha = \wp^2 \nu N \pi \sigma(\nu)/c\epsilon\hbar \quad . \tag{8}$$

As was noted earlier, we wish to place two constraints on the interval $\Delta t = t_3 - t_2$ between the two pulses. The first is that the pulses be well resolved; that is, if $\hat{t}_\lambda(z)$ is the pulse width, taken to be the full width at half-height of the envelope $\mathcal{E}_\lambda(t, z)$, then[16]

$$\Delta t \gg \max[\hat{t}_1(z), \hat{t}_2(z)], \quad \text{for all } z. \tag{9}$$

Also, in order to simplify the derivation,

$$\Delta t \gg T_2 \quad . \tag{10}$$

In the absence of radiation, off-diagonal elements of the density matrix ρ_{ab} and ρ_{ba} decay to zero in a time T_2. If Eqs. (9) and (10) hold, then one sees that $\rho_{ab}(\omega, t_3, z) = 0$. At the same time, the line shape is $\sigma(\omega) \times [\rho_{bb}(\omega, t_3, z) - \rho_{aa}(\omega, t_3, z)]$. It is readily seen then, that the active medium at $t = t_3$ has the same general properties as it had at $t = t_1$. In that case, the off-diagonal elements were also zero, and one had a line shape

$$\sigma(\omega)[\rho_{aa}(\omega, t_1, z) - \rho_{bb}(\omega, t_1, z)] = \sigma(\omega).$$

Thus, the only consequence of the first pulse is to produce a new line shape which is the initial condition for the active medium on the second pass. Hence, the equations of motion for the second pulse are the same as for the first pulse, i.e., one sets $\lambda = 2$ in Eqs. (5) and (6), and one has an initial condition for χ given by

$$\chi(T, t_3, z) = D(T)\,(1 - e^{-\Delta t/T_1}) + \chi(T, t_2, z) e^{\Delta t/T_1} \, . \tag{11}$$

In summary, the problem of two pulses in an active medium is equivalent to two consecutive solutions of the single-pulse problem, where one uses Eq. (11) as the initial condition for the medium on the second pass. These equations [(5)–(7) and (11)] properly describe all the radiation if the inequality in Eq. (10) holds. They also describe the behavior of $\mathcal{E}_1$ and $\mathcal{E}_2$ even if the inequality does not hold, but one must remember that there may be radiation appearing at a later time which is not included in this formulation.

III. ENERGY AND AREA DEVELOPMENT OF A SECOND PULSE

The equations given in Sec. II are not readily solved except by numerical methods. However, there are analytic formulas that can be useful in understanding how pulses evolve. In particular, one can generalize the area theorem of McCall and Hahn, and in the limit that the medium is optically thin, one can use rectangular pulses to determine the conditions under which the second pulse gains or loses energy. From these, one can draw inferences as to the behavior of delay times and pulse widths. First, one defines the area under the pulse envelope as

$$\theta_\lambda = \frac{\wp}{\hbar} \int_{-\infty}^{\infty} dt' \; \mathcal{E}_\lambda(t', z), \tag{12}$$

and if[16] $T_2 \gg \hat{t}_\lambda$ and $T_1 \gg \hat{t}_\lambda$, $\lambda = 1, 2$, then it is readily seen that

$$\frac{d\theta_1}{dz} = -\tfrac{1}{2}\alpha \sin\theta_1, \tag{13a}$$

$$\frac{k_2}{k}\frac{d\theta_2}{dz} = -\tfrac{1}{2}\alpha \sin\theta_2 \, [(1 - e^{-\Delta t/T_1}) + e^{-\Delta t/T_1}\cos\theta_1] . \tag{13b}$$

Equation (13b) describes the results of "inverting" the attenuator. If $\cos[\theta_1(z)] < 0$ and $T_1 \gg \Delta t$, then $\theta_2(z)$ will converge on $(2n+1)\pi$ as it would in an amplifier. However, as has been shown in previous problems,[7] area theorems are, by themselves, not sufficient to predict how pulses will evolve. In this problem, for example, if $\cos[\theta_1(z)] < 0$ and $T_1 \gg \Delta t$, it is not necessarily true that the second pulse will gain in energy. To study how the energy behaves, it is instructive to use thin-medium perturbation theory.[17] One takes a medium that is optically so thin that pulses propagating through it do not change significantly in their passage. One uses rectangular pulses of area θ_λ and width $\hat{t}_\lambda$, and describes the energy with a parameter $\tau_\lambda(z)$ where

$$\tau_\lambda(z) = \frac{\wp^2}{\hbar^2} \int_{-\infty}^{\infty} dt' \; \mathcal{E}_\lambda^2(t', z). \tag{14}$$

Analytic solutions of this problem, as well as a more detailed description of the theory, can be found elsewhere.[17] In Ref. 7, this problem is solved with an emphasis on the polarization induced by the pulse, and the medium response was given in terms of $\chi(T, t, z)$. In the present case, it is more useful to look directly at the line shape.[10] One writes the population difference left behind by $\mathcal{E}_1$ as $f(\hat{t}_1, \theta_1, \delta) = \rho_{bb}(\omega, t_2, z) - \rho_{aa}(\omega, t_2, z)$ and we have set $\delta = \omega - \nu$. Then one can readily see, e.g., by using the equations of Ref. 7 and taking the inverse of Eq. (4), that

$$f(\hat{t}_1, \theta_1, \delta) = [\, \delta^2 t_1^2 + \theta_1^2 \cos(\delta^2 \hat{t}_1^2 + \theta_1^2)^{1/2}] / (\delta^2 \hat{t}_1^2 + \theta_1^2) . \tag{15}$$

The function f goes asymptotically to one for large δ, and dips down to the value $\cos\theta$ at the center. The width of this dip is inversely proportional to the pulse width. When θ is near π, the center of the line is negative[15] (i.e., is inverted) and the range in frequency that is inverted determines whether the second pulse will gain in energy. This value of f is also proportional to the energy absorbed by the medium at the frequency δ, so that

$1-f(\hat{t}_2, \theta_2, \delta)$ is proportional to the energy gained or lost by the second pulse at that frequency. It is convenient to use a function

$$F(\hat{t}_1, \theta_1; \hat{t}_2, \theta_2; T_2^*) = \frac{2\hat{t}_2}{\sigma(\nu)\pi\theta_2^2} \times \int_{-\infty}^{\infty} d\omega\, \sigma(\omega) f(\hat{t}_1, \theta_1, \delta)[1-f(\hat{t}_2, \theta_2, \delta)], \quad (16)$$

which is proportional to the change in energy of the second pulse, and then define a gain parameter

$$\Gamma(\hat{t}_1, \theta_1; \hat{t}_2, \theta_2; T_2^*) = \frac{-F(\hat{t}_1, \theta_1; \hat{t}_2, \theta_2; T_2^*)}{F_0(\hat{t}_1, \hat{t}_2, T_2^*)}, \quad (17)$$

where $F_0(\hat{t}_1, \hat{t}_2, T_2^*) = F(\hat{t}_1, 0; \hat{t}_2, 0; T_2^*)$ refers to the condition in which θ_1 and θ_2 are so small (i.e., much less than 1) that the nonlinearities of the problem are unimportant. This procedure normalizes Γ so as to remove from it the decrease in absorption or gain which occurs when the spectral width of the pulse is greater than the width of $\sigma(\omega)$. The change in energy for the second pulse[18] is given by

$$\frac{d\tau_2}{dz} = -\alpha F_0(\hat{t}_1, \hat{t}_2, T_2^*)\Gamma(\hat{t}_1, \theta_1; \hat{t}_2, \theta_2; T_2^*)\tau_2. \quad (18)$$

In Fig. 1, one sees a family of curves for Γ when $\theta_1 = \pi$, $\theta_2 \ll 1$, $T_2^* = 1$ nsec., and $\hat{t}_1$, $\hat{t}_2$ are variable. The corresponding value of $F_0(\hat{t}_1, \hat{t}_2, T_2^*)$ is the broken curve in the figure. One can then see that the second pulse will increase in energy if $\hat{t}_1 \lesssim T_2^*$ or if $\hat{t}_2 \gtrsim \hat{t}_1$. Otherwise, the pulse will actually lose energy even though $\cos\theta_1 = -1$ and Eq. (13b) implies that the system is an amplifier. In the limit that both $\hat{t}_1$ and $\hat{t}_2$ are much greater than T_2^*, then the curves of Γ versus pulse width become functions of the ratio $\hat{t}_1/\hat{t}_2$. Under these circumstances, the threshold for gain ($\Gamma = 0$) is $\hat{t}_2 = 0.63\hat{t}_1$.

A family of curves for $\theta_2 = \pi$ was also produced using Eq. (16). In that case, the value of Γ was found to lie between ± 0.4. From previous work, it has been found that the expected value of Γ should lie between ± 0.5 when one uses more realistic pulse shapes.[19] Thus, one expects an error of 20% in applying Eq. (18) to real systems even without considering the complications that are left out of the theory entirely (e.g., mode shape and self-focusing). In view of this error, one can use an estimate of Γ for $0 < \theta < \pi$, which differs from Eq. (16) by much less than 20%, by taking the value from Fig. 1 and multiplying by $(4/\theta_2^2)\sin^2(\frac{1}{2}\theta_2)$. The reason this estimate works is that the full width at half-height of the curve $1-f(\hat{t}_2, \theta_2, \delta)$ is not very sensitive to the value of θ_2, and the integral in Eq. (15) for fixed θ_1, $\hat{t}_1$, and $\hat{t}_2$ will be roughly proportional to $1-f(\hat{t}_2, \theta_2, 0)$. The percent error in this estimate is large only when $\Gamma \approx 0$; however, the threshold for gain for the case

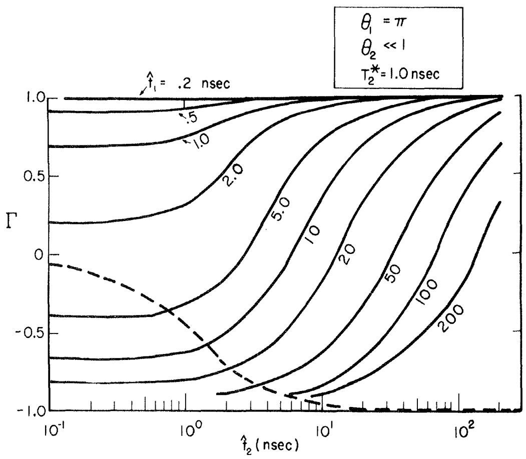

FIG. 1. In this figure, there is a family of curves for the gain parameter Γ of a weak pulse interacting with an attenuator "pumped" by a previous π pulse. The parameter is calculated using rectangular pulses interacting with an optically thin medium. The curves are given for fixed values of the width of the first pulse ($\hat{t}_1$) as a function of the width of the second ($\hat{t}_2$). The broken curve is the function F_0 (see text for definition) which was used to normalize Γ to eliminate the effects on the gain due solely to the relative spectral widths of the medium and the pulse. The actual gain goes as $-\Gamma \times F_0$. Methods for using these curves to estimate gain constants when θ_2 is large (i.e., comparable to π) are given in the text.

$\theta_2 = \pi$ differs from the small-signal threshold (i.e., $\hat{t}_2 = 0.63\hat{t}_1$) by less than 10%. In the case of SIT in two passes, it is typically true that when the input pulse is π, then $\hat{t}_2(z) \geq \hat{t}_1(z)$. Thus, one will expect to find that the pulse gains in energy on the second pass. This, in turn, will have a strong effect on the "knee" of the curve of output versus input energy.

A second interesting case is the one in which $\theta_1 = \frac{1}{2}\pi$. In this case, $\Gamma \leq 0$ since $f(\hat{t}_1, \pi/2, \delta) \leq 0$. If $\hat{t}_1 > T_2^*$, then the second pulse will always lose energy. However, in this case, Eq. (13b) implies that θ_2 is fixed ($\cos\theta_1 = 0$). In previous papers involving attenuators,[4,6] it has been noted that conditions of fixed θ and decreasing τ mean an increase in $\hat{t}$ and substantial delay times [the delay time $t_D(z)$ is defined as the change in the position of the maximum amplitude, or peak, of the pulse relative to the velocity of light in the inert background].[20] This follows from the formula $\hat{t}\tau \approx \theta^2$. It is clear that one expects pulse delays to occur if θ increases and τ is fixed or decreases. In the case of two-pass SIT, it is typical that when θ_{in}[21] lies between $\frac{1}{2}\pi$ and π, the pulse will encounter such a situation somewhere in the rod. That is, there is a segment in the rod for which $\cos\theta_1 < 0$

(θ_2 increases) but θ_1 is not large enough to produce gain (τ_2 decreases). Thus, one expects to find delay times occurring on the second pass when $\theta_{in} < \pi$. In the single-pass case, one has delays only if $\theta_{in} > \pi$.[20]

Having seen that one expects significant differences in the SIT effect between the single- and two-pass cases when pulses are less than or near π, one notes that no changes are expected when the pulses are near 2π. In that case, previous work[3,4] has shown that little or no energy is lost by the pulse, even when it has not yet evolved into an hyperbolic secant, and one must infer that the line shape has not been sufficiently disturbed to have any effect on the pulse on the second pass. Moreover, if $\cos\theta_1(z) > 0$ for all z and the pulse has already evolved into the 2π hyperbolic secant form by the time it gets to the end of the rod on the first pass, then on the second pass it will propagate without further change. Hence, there will be a considerable range of input areas around $\theta_{in} = 2\pi$ for which no difference is expected between the one- and two-pass problems.

It is usually the case when dealing with SIT that one must take into account the effect of a finite phase-memory time T_2 in order to deal with experimental circumstances. Unfortunately, in this case, closed-form formulas are not readily obtained, even with the thin-medium approximation. One can get some information by solving the thin-medium problem at resonance ($\omega = \nu$) which will measure the degree to which the medium is inverted. At that frequency, the value of f is

$$f(\hat{t}_1, \theta_1, 0) = e^{(-\hat{t}/2T_2)}/[\theta^2 - (\hat{t}/2T_2)^2]^{1/2} \times\{(\hat{t}/2T_2)\sin[\theta^2 - (\hat{t}/2T_2)^2]^{1/2} + [\theta^2 - (\hat{t}/2T_2)^2]^{1/2}\cos[\theta^2 - (\hat{t}/2T_2)^2]^{1/2}\}. \quad (19)$$

When $\theta < \hat{t}/2T_2$, the terms under the radical are reversed and the trigonometric functions become hyperbolic. It was seen earlier that with $T_2 \gg \hat{t}$, the medium was inverted to its maximum extent by a π pulse. However, when T_2 is finite, the area that gives maximum inversion is $\theta_1 = \frac{1}{2}[\pi^2 + (\hat{t}/2T_2)^2]$, which is greater than π. At that value of the area, one has $f = -e^{-\hat{t}/T_2}$, which means that the medium cannot be completely inverted at the center frequency (or at any other frequency). Thus, the presence of a finite T_2 decreases the ability of the medium to amplify a second pulse. In the limit that $T_2 \to \infty$, one sees that the medium cannot be inverted no matter how large the first pulse is, and hence, the second pulse can never be amplified.

IV. SIT IN A TWO-PASS ATTENUATOR

In this section, the general theory of Sec. II is solved by numerical (computer) techniques with a configuration that describes two-pass SIT. The pulse passes out of the rod, reflects off a 100% mirror, and then returns through the same region of the rod with which it had just interacted. Thus, in the computer solution, the input pulse for the second pass is the same as the output from the first pass [i.e., $\mathcal{E}_2(L, t - 2L_m/c) = \mathcal{E}_1(L, t)$, where c is the velocity of light in a vacuum] and the second pulse propagates in the opposite direction from the first (i.e., $k_2 = -k$). The input pulse is of the form $\mathcal{E}(t, 0) = C_1 t e^{-t^2/C_2}$ with C_1 varied to give a range of input θ's and C_2 adjusted to give an initial pulse width of 10 nsec. The criteria for an "ideal" two-pass attenuator, i.e., one in which a 2π hyperbolic secant is expected to propagate without distortion or loss in energy, are $T_2 \gg \hat{t}$, $T_1 \gg \Delta t$, $T_1 \gg \hat{t}$ (also, no detuning and no scattering loss)[7]; we set $T_1 = 10^7$ nsec., $T_2 = 10^6$ nsec., and the length L_m between the rod and mirror is 50 m so that $\Delta t > 333$ nsec. We set $T_2^* = 1$ nsec and we let the single-pass small-signal ($\theta \ll 1$) attenuation of the rod, i.e., τ_{out}/τ_{in}, be 0.0341 (the two-pass small-signal attenuation is ~0.0012). The rod is taken to be 9 cm in length with an index of refraction of 1.5. Calculations involving an 18-cm rod in single-pass with the same values of α, T_2, T_1, T_2^*, etc., as the 9-cm rod are presented as a means of demonstrating the differences in the nonlinear behavior between the one- and two-pass case. The 18-cm rod will, in one pass, have the same effect on a small pulse as the 9-cm rod does in two passes, and is referred to as the equivalent single-pass attenuator.

In Fig. 2, one sees[21] τ_{out}/τ_{in} as a function of θ_{in} for the 9-cm rod for one pass (short-dashed curve), and two passes (solid line), and for the equivalent single-pass attenuator (long-dashed curve). One sees that input pulses with $\frac{1}{2}\pi < \theta_{in} < \frac{3}{2}\pi$ receive much less attenuation in the two-pass system than in the equivalent single-pass case, resulting in a curve that rises more gradually, has a lower transition point,[22] and has some structure near $\theta_{in} = \pi$. A comparison of the solid curve and the dotted curve shows that for $0.8\pi < \theta_{in} < 1.2\pi$, there is an actual increase in energy on the second pass. This is in accord with the discussion in Sec. III, since it is typically true that $\hat{t}_2(z) > \hat{t}_1(z)$ when θ_{in} is near π, and that was seen as a criterion for gain.

The corresponding delay times are shown in Fig. 3. Because pulses for $\theta_{in} \gtrsim \pi$ gain in energy on the return pass, there is no large maximum delay as in the equivalent single-pass attenuator. For $1.3\pi > \theta_{in} > \pi$, the delay occurs mostly during the first pass. For pulses with $\theta_{in} < \pi$, one sees large delays that do not occur in the equivalent

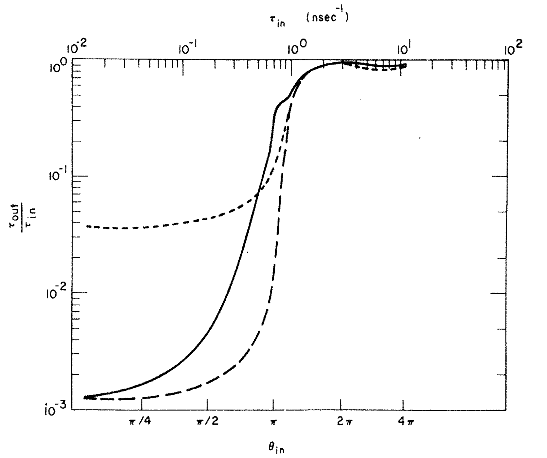

FIG. 2. This figure gives the attenuation versus input energy curves for three cases. The solid curve is for a two-pass attenuator (τ_{out} is the output of the second pass, τ_{in} is the input to the first pass). The short-dashed is the attenuation on the first pass. The long-dashed is the attenuation for a rod of twice the length of the two-pass attenuator, operated as a single-pass system. In these curves, one has $T_2^* \ll$ pulse width $\ll T_2$, and $T_1 \gg$ pulse separation.

single-pass system.[20] The reason for these delays is that one has a case, as was shown in Sec. III, where θ increases and τ decreases which is a condition that typically produces pulse delays. The detailed structure of the solid curve is not shown in Fig. 3, but is shown instead in Fig. 6. The delay curve for two-pass SIT has two maxima, one corresponding to the maximum for $\theta_{in} > \pi$ which occurs on the first pass, and the other to the maximum delay for $\theta_{in} < \pi$ which occurs on the second pass. In the case of larger θ_{in} (i.e., $2.5\pi > \theta_{in} > 1.5\pi$), it is seen that, as expected, there is little or no difference in either Fig. 2 or Fig. 3 between the two-pass and the equivalent single-pass attenuator.

In Fig. 4, one sees one particular example of pulse evolution with the envelope $\mathcal{E}_\lambda(t, z)$ shown at the input and at the output of the first and second passes. The output area is π, which, in the case of a rod of only a few absorption lengths, develops on the first pass in a manner characteristic of pulses with $\theta_{in} < \pi$. It develops in an highly asymmetric fashion on the first pass with a long reradiated trailing edge and very little motion of the peak. On the second pass, instead of the peak advancing and increasing the asymmetry of the pulse as would be expected from an amplifier, one sees that the trailing edge of the pulse is built up and the peak is delayed. The resulting "resymmetrization" on the second pass of pulses with $\theta_{in} < \pi$ is characteristic of two-pass SIT. Since output pulses with θ_{in} larger than π are expected to be symmetrical anyway, one finds that one expects to see output pulses which are roughly symmetrical for all values of θ_{in}. The undershoot at the trailing edge is a normal[7] effect in an

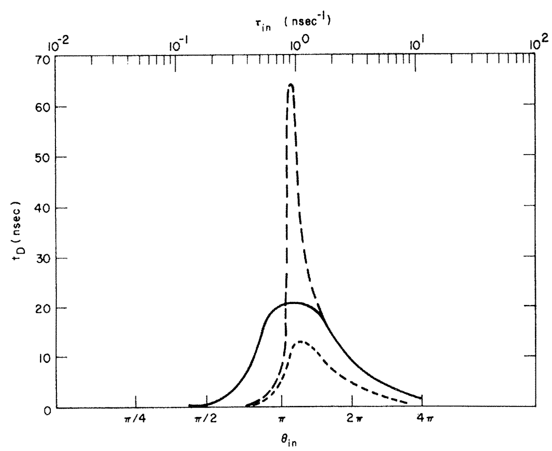

FIG. 3. In this figure, the delay time $t_D(z)$ is given for a two-pass attenuator (solid curve), for the first pass of the two-pass attenuator (short-dashed curve), and for the equivalent single-pass attenuator (long-dashed curve). The detailed structure of the solid curve is given in Fig. 6(a).

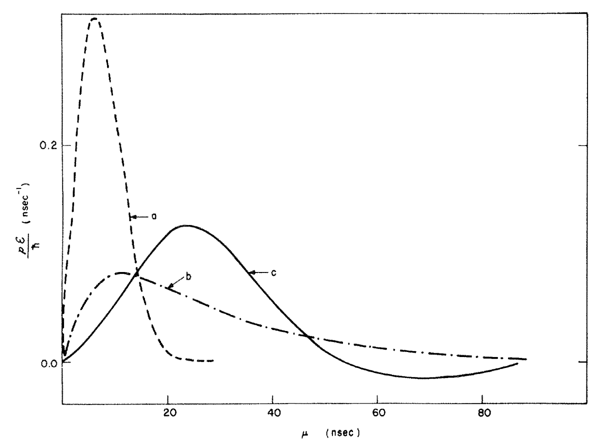

FIG. 4. In these curves, the pulse envelope $\wp\mathcal{E}_\lambda(\mu, z)/\hbar$ is shown. The broken curve is the input pulse $\wp\mathcal{E}_1(\mu, 0)/\hbar$, $\mu = t$. The dot-dashed curve is the output pulse from the first pass $\wp\mathcal{E}_1(\mu, L)/\hbar$; $\mu = t - L/c$, and the solid curve is the output pulse from the second pass $\wp\mathcal{E}_2(\mu, 0)$; $\mu = t - 2L/c - 2L_m/c_{vac}$. Here, L_m is the distance from the mirror, c is the velocity of light in the inert background, and c_{vac} is the velocity of light in a vacuum.

amplifier, and occurs to varying degrees in the two-pass attenuator with this instance showing the undershoot at its maximum. Even though it is small in energy, the undershoot can have important consequences since a pulse with an undershoot can exhibit SIT[2] in a single-pass attenuator[23] even though it has an area less than π.

It is usually necessary in experimental situations to take into account the effect of a finite phase memory time. One may have cases where the pulse width is expected to increase by more than an order of magnitude[4] so that even if $\hat{t}_{in} \ll T_2$, it may not be true that $\hat{t}_{out} \ll T_2$. The effects of a finite T_2 (i.e., $T_2 \sim \hat{t}$) on this problem have been investigated by numerical solution. The medium and the input field are the same as described before except that $T_2 = 50$ and 20 nsec, respectively, and α is adjusted[7] to keep $\tau_{out}/\tau_{in} = 0.0341$ for one pass when τ_{in} is small. The curves of τ_{out}/τ_{in} versus τ_{in} are shown in Fig. 5, with a rate-equation calculation $(T_2 \ll \hat{t})^{24}$ given for comparison. These curves are in accord with the thin-medium calculations done in Sec. III. As T_2 becomes shorter, the structure associated with the gain in energy on the second pass occurs for higher values of the input area. The particular value of the area

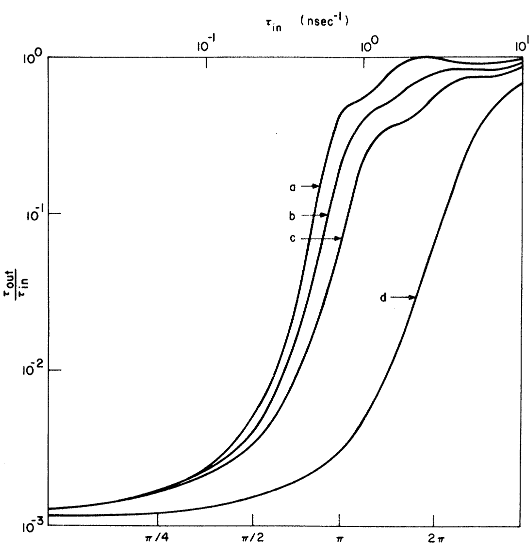

FIG. 5. In this figure, the absorption τ_{out}/τ_{in} for the two-pass attenuator is given as a function of θ_{in} for different values of T_2, with (a) $T_2 = \infty$, (b) $T_2 = 50$ nsec, (c) $T_2 = 20$ nsec, and (d) $T_2 \ll \hat{t}$ where the pulse width $(\hat{t})$ is initially 10 nsec. In each of these cases, one has $T_1 \gg$ pulse separation. Curves (a), (b), and (c) come from numerical integration of the working equations. Curve (d) is an analytic calculation using rate equations.

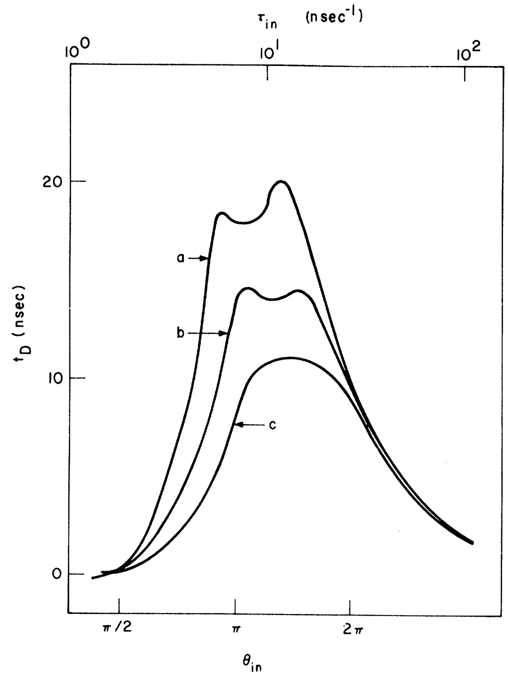

FIG. 6. This figure gives the delay times t_D versus input area (or energy) for a two-pass attenuator with different values of T_2. One has : (a) $T_2 = \infty$, (b) $T_2 = 50$ nsec, and (c) $T_1 = 20$ nsec with an initial pulse width of 10 nsec.

at which this structure appears corresponds very closely with the transition point[22] for the single-pass (9-cm-rod) case. Thus the differences in the value of τ_{out}/τ_{in} near that value of θ_{in} arise almost entirely from the fact that there is less gain on the second pass as T_2 becomes smaller. The structure observed is clearly associated with the coherent inversion of the line, since in the case $T_2 \ll \hat{t}$, the structure disappears. The delay times for the different values of T_2 indicated above are shown in Fig. 6. One sees that the effect of T_2 is to eliminate the structure in the delay curves. For this particular example, there are not enough absorption lengths in the single-pass attenuator to produce output pulse widths that are very much larger than the input widths. Hence, there is no drastic reduction in delay times in contrast to the single-pass case discussed in Ref. 4.

The effect of a finite atomic lifetime (T_1) is straightforward, and may be of importance from an experimental standpoint. When T_1 is comparable to the time interval between passes (Δt), one sees from Eq. (11) that the line relaxes back towards its original distribution between the passes, and if $\Delta t \gg T_1$, then the two-pass and the equivalent single-pass cases are the same. Thus, changing T_1 will result in a family of curves that take the solid curves over into the broken ones in Figs. 2 and 3. Rather than present these curves, we will consider a somewhat similar problem in Sec. V in which we consider using these effects to measure T_1. Because of Eq. (9), the case $T_1 \sim \hat{t}$ will occur only if $\Delta t \gg T_1$, so that if one is dealing with that situation, one is dealing with a single-pass problem (see Ref. 4).

V. RELATED PROBLEMS

In this section, we will consider a series of problems that involve pulses interacting with line shapes[10] produced by a previous pulse. In V A, the question of the direction of propagation of the second pulse is considered. It turns out that this leads, in turn, to considering a much more general problem of pulses passing through nonuniform rods. In V B, we return to the question of the effects of T_1 and, more specifically, to how one might measure it using the phenomena discussed in Secs. II and III. And in Sec. VC, we contemplate the possibility of applying this problem to optical logic circuits.[11]

A. Direction of Second Pulse

The most immediate question that one can ask about this problem is what effect the direction of propagation has on the output pulse form. In particular, Eq. (11) indicates that the input condition for the medium may be a complicated function of distance which is not symmetrical about the center of the rod. It is clear that a pulse may behave very differently while inside the rod depending on its direction of propagation. However, under a wide range of circumstances, the output pulses are identical[25] and independent of direction. In particular, this result holds for all pulses in any media which are describable by the equations in Sec. II except for the case when cT_1 is comparable to the rod length.[26,27] Further investigation has indicated strongly that this situation is not dependent on how the initial condition in Eq. (11) is produced, i.e., this is not a specific result of the two-pulse problem, but apparently holds for all cases in which a pulse encounters a nonuniform initial line shape and/or a gain constant which is a function of z. A more restricted case of this problem occurs when the rod is made up of an amplifier and an attenuator which differ only in the sign of α, and in that case, the output pulse is always the same as the input. Because of the lack of analytic techniques for solving the equations in Sec. II, the general case is not readily proved. However, one can show that it is true in the case of rate equations ($T_2 \ll \hat{t}$) for which closed form transient solutions exist,[24] and in the case $T_2 \gg \hat{t}$, the result is consistent with Eqs. (13a) and (13b) if the equations are generalized so that α becomes a function of distance. The restricted case is quite easily proved under general circumstances.[28] One notes that if one has a solution to the working equations of the form $\mathcal{E}_a(t-z/c, z)$, $\chi_a(T, t-z/c, z)$, then a field of the form $\mathcal{E}_b[(t-z/c+2(z-L)/c), z] = \mathcal{E}_a(t-z/c, z)$ will induce the same polarization as $\mathcal{E}_a$ in the medium, but when substituted into the left-hand side of Eq. (5), produces a term $-\partial\mathcal{E}_b/c\partial t + \partial\mathcal{E}_b/\partial z$. This is the negative of the derivative of a field going in the opposite direction, and multiplying both sides by -1 changes $+\alpha$ into $-\alpha$. $\mathcal{E}_b$ is a pulse moving in the opposite direction as $\mathcal{E}_a$, the roles of input and output reversed. The substitution indicated above proves that $\mathcal{E}_b$ is a solution of the Eqs. (5) and (6) if $\alpha \to -\alpha$. If one wishes to find circumstances in which these results break down, one must introduce a factor on the right-hand side of Eq. (5) which does not change sign in going from an amplifier to an attenuator. One example is an unsaturable scattering loss[7] which appears in a term $-K\mathcal{E}_\lambda(t, z)$ in Eq. (5) and $-\kappa\theta_\lambda$ in Eqs. (13a) and (13b). When this is introduced, then both assertions are false and the pulse tends to behave in a manner determined by whatever system it encounters last[26] (e.g., if it passes through both an amplifier and an attenuator, it will have more energy if it passes through the amplifier second rather than first). Presumably, other factors such as noise or dispersion will have a similar effect.

From the preceding discussion, it follows that any phenomena that involve a pulse interacting with a line shape produced by a previous pulse will appear in the two-directional problem whenever they occur in the unidirectional case. One example of this is the stimulated echo[29] in which two pulses ($\mathcal{E}_a$ and $\mathcal{E}_b$ separated by Δt_{ab}) pass through a medium in a time short compared to T_2. They produce a term in the population difference that goes as $\cos[(\omega-\nu)\Delta t_{ab}]$. When a third pulse $\mathcal{E}_c$ (at a time Δt_{bc} after the second) comes along, it produces an echo Δt_{ab} later that has an amplitude proportional to $e^{-\Delta t_{bc}/T_1}$. If one lets $\mathcal{E}_1$ be the combination of $\mathcal{E}_a$ and $\mathcal{E}_b$, and $\mathcal{E}_2$ be $\mathcal{E}_c$ and the echo, then in the light of the previous discussion, one notes that the echo will occur independently of the direction of $\mathcal{E}_c$.

B. Measurement of T_1

As was mentioned in Sec. IV, the role of T_1 in this problem is straightforward (so long as $cT_1 \gg$ rod length).[27] In particular, T_1 measures the time in which an attenuator which has been pumped by a "π" pulse will relax back to its original state. This provides a conceptually simple way of measuring T_1, i.e., one passes a π pulse through the rod, bounces it off a low-reflecting mirror positioned a distance L_m away, and returns it through the rod. One measures the enhancement of the pulse on the second pass (τ_{out}/τ_{in})[21] as a function of L_m. This is illustrated in Fig. 7, with the medium and field the same as described in Sec. IV. One sees that, in principle, one can get very large changes in the enhancement of the pulse as a function of L_m. The other method of using pulses to measure T_1 in attenuators involves the stimulated echo effect mentioned in Sec. V A.[29] That effect, which is conceptually very complicated, requires that $\Delta t_{ab} \ll T_2$, or, if the inequality does not hold, that the relative magnitude of these quantities stays fixed. The single-pulse technique requires that $T_2 \gg \hat{t}$ which should be easier to attain. In addition, this experiment would require the producing and monitoring of only one large pulse rather than three (the amplitude of the pulse on the second pass is irrelevant so long as it is small). Thus, the analysis of the results would be much simpler. On the other hand, this technique suffers from the necessity of having a very large L_m (in ruby $L_m \sim 500$ miles if $2L_m \sim cT_1$), since the quantity $\ln(\tau_{out}/\tau_{in})$ goes as $e^{-\Delta t/T_1}$ for small L_m

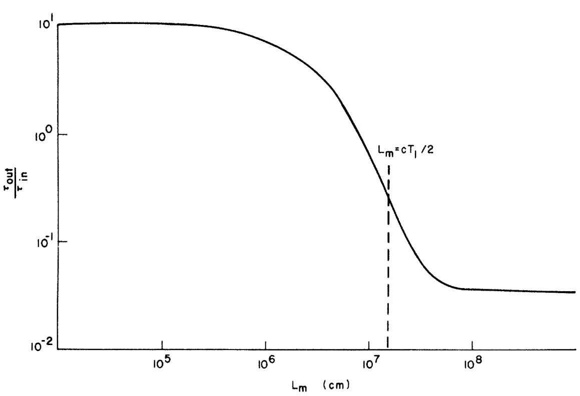

FIG. 7. In this figure, the value of $\tau_{out}/\tau_{in} = \tau_2(0)/\tau_2(L)$ is given as a function of the distance L_m between the rod and the mirror. The input pulse on the first pass has an area $\theta_1(0)=\pi$, and "pumps" the rod so the second-pulse experiences gain. The value of the reflection coefficient was taken to be 0.001%, so that the input pulse on the second pass is weak [i.e., $\theta_2(L) = 0.001 \times \theta_1(L) \ll 1$]. However, the rod relaxes back to its original attenuating state in a time T_1. Thus, changing L_m causes the value of τ_{out}/τ_{in} to go from a value greater than one to the absorption of the attenuator $\tau_{out}/\tau_{in} = 0.0341$. This effect provides a conceptually simpler method of measuring T_1 using pulses compared to using the stimulated-echo effect. The velocity c in the figure refers to that of light in a vacuum.

(in the stimulated echo, the log of the echo energy varies as $\Delta t_{ab}/T_1$). However, there is no practical necessity to use the same pulse on both passes, and one can conceive of many ways of producing a similar situation with two different pulses.

C. Logic Circuits

It has been proposed to use the coherent pumping of an attenuator as a means of building optical logic circuits.[11] This involves comparing the output energy of a second pulse $(\theta_2 \approx \pi)$ as a function of whether the first pulse $(\theta_1 = \pi)$ was present. One enhances the energy difference by using the difference in output for θ_2 predicted by Eq. (13b) and then by passing the pulse into a second attenuator with a different $\wp$ (in practice, one would change θ by focusing or defocusing the beam) so as to have the second pulse greater or less than π in the second attenuator. Because of SIT, one expects the pulse which is less than π to be absorbed and the pulse which is greater than π to be passed with little loss.

This situation was simulated on a computer with two attenuators 6 and 18 cm. long, which have parameters the same as in Sec. IV, and pulses with the same input characteristics as before. The first pulse had $\theta_{in} = \pi$, and the system was checked when the second pulse had $\theta_{in} = 0.9\pi(\pi^-)$ with focusing between the rods and $\theta_{in} = 1.1\pi(\pi^+)$ with defocusing. In the case of $\theta_{in} = \pi^-$, the energy output differed by a factor of about 500 between the runs where θ_1 was present and absent. Thus, it should be easy to discriminate between the two outputs. The first attenuator produced a factor of 3 difference which was enhanced in the expected manner. In the case of $\theta_{in} = \pi^+$, there was no substantial difference in output energy between the two runs (a factor of 0.8). The pulse that was less than π in the second attenuator had more energy initially than the one greater than π and it continued to have more energy in its pass through the second attenuator. Because the less energetic pulse is known to evolve eventually into a pulse which induces a transparency, it is reasonable to speculate (the limitation of the computer did not allow for determining the asymptotic forms) that the more energetic pulse would evolve eventually into a nontrivial zero-π pulse[23] and exhibit SIT.[2]

VI. CONCLUSION

In conclusion, we see that when an attenuator is operated as a two-pass system, the SIT effect[2] is modified in several essential ways. The most important effect is that the output-versus-input energy curves become highly structured, and the structure is characteristic of atomic coherence.

It was noted in an earlier publication[4] that the shape of such curves in a single-pass case is relatively insensitive to T_2, and one cannot easily tell from them alone whether one is dealing with atomic coherence. The behavior of delay times as a function of input area is also modified significantly. In this case, one does not expect to see sharp maxima associated with $\theta_{in} = \pi$, nor is the maximum delay expected to coincide with the transition point[22] of the energy curve.[4,6] Instead, one has a wide peak in the delay-versus-θ_{in} curve, with two local maxima under some circumstances. Finally, it is noted that the reradiative phenomena in the two-pass system do not produce the highly asymmetric pulses that are expected in the single-pass case when input areas lie between $\frac{1}{2}\pi$ and π. So long as the input pulse has no extreme asymmetry, one expects reasonably symmetrical outputs for all values of θ_{in}.

When one considers the general problem of a pulse interacting with a line shape produced by a previous pulse (of which two-pass SIT is a particular example), several results appear. The first is that a π pulse, while inverting the medium in the sense that the area of a second pulse converges on $(2n+1)\pi$, does not necessarily produce energy gain for a second pulse. The conditions for gain were investigated in some detail, and it was found that one needs $\hat{t}_2 \geq 0.63\hat{t}_1$, if $\hat{t}_1 \gg T_2^*$, or else $\hat{t}_1 \lesssim T_2^*$ to ensure gain. Particular curves are presented so that the gain or loss of energy for the second pulse can be estimated. Furthermore, it was found that under a wide range of circumstances, the output pulses for the second pulse (or second pass) are independent of whether the pulse is going in the same or opposite direction from the first. This turns out to be a particular case of a more general conclusion that pulses emerge from a nonuniformly pumped system of amplifiers and attenuators (all with the same T_1, T_2, $\wp$, and perhaps T_2^*) independently of the direction of propagation of the pulse. Presumably, it is also true that it makes no difference if the component pieces of the system are interchanged. Moreover, it was noted and a proof was outlined, that an amplifier which differs from an attenuator only by a change in sign of the gain constant will exactly reverse the process of pulse evolution. In order to make the directionality important, it is necessary to introduce something into the medium in addition to the active atoms. A scattering loss has been shown to be sufficient, and presumably, noise or host dispersion will have the same effect.[30]

Finally, a method for measuring T_1 using pulse techniques is presented, and its merits and drawbacks relative to previous techniques are discussed. Also, the possibility of practical application of two-pass phenomena are mentioned briefly in connection with logic circuits.[11] There were two apparently equivalent ways in which one might set up such a circuit, one of which behaved well and the other of which behaved badly. The latter case involved a pulse which, because of phase shifts that were produced in an amplifier, developed subsequently into a nontrivial (nonzero energy) zero-π pulse[23] when passed through an attenuator.

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[1]S. L. McCall and E. L. Hahn, Phys. Rev. Letters 18, 908 (1967).

[2]In this paper, the term SIT is used in two senses. In its general sense, it refers to the entire range of phenomena one expects to find when a coherent pulse interacts nonlinearly with an inhomogeneously broadened attenuator. However, when it is used in reference to one particular pulse, it means that the pulse will evolve into an asymptotic form at which point it will no longer lose energy.

[3]S. L. McCall and E. L. Hahn, Phys. Rev. 183, 457 (1969).

[4]F. A. Hopf and M. O. Scully, Phys. Rev. (to be published).

[5]C. K. N. Patel and R. E. Slusher, Phys. Rev. Letters 19, 1019 (1967).

[6]C. K. Rhodes, A. Szöke, and A. Javan, Phys. Rev. Letters 21, 1151 (1968). Also C. K. Rhodes and A. Szöke, Phys. Rev. 184, 251 (1969); F. A. Hopf, C. K. Rhodes, and A. Szöke (to be published). See also J. P. Gordon, C. H. Wang, C. K. N. Patel, R. E. Slusher, and W. J. Tomlinson [Phys. Rev. 179, 294 (1969)] who contend that SF_6 is nondegenerate.

[7]F. A. Hopf and M. O. Scully, Phys. Rev. 179, 399 (1969).

[8]See for example N. S. Shiren, Phys. Rev. 128, 2103

(1962). Other references are noted in Ref. 3.

[9]See N. S. Shiren (to be published).

[10]The term "line shape" refers to the product of the population difference $(\rho_{bb}-\rho_{aa})$ and the inhomogeneous frequency distribution $\sigma(\omega)$.

[11]The basic ideas for this application were communicated in private to the author by F. Mheran, S. Bjorklund, and J. T. Vanderslice.

[12]I. D. Abella, N. A. Kurnit, and S. R. Hartmann, Phys. Rev. 141, 391 (1966).

[13]For an example of how the theory must be generalized when pulses overlap in the medium, one can compare the equations presented in J. A. Fleck, Jr., Appl. Phys. Letters 13, 365 (1968); and J. A. Fleck, Jr., Phys. Rev. Letters 21, 131 (1968).

[14]It is extremely lengthy and tedious to prove this point for both one and two directions, and since it is a negative conclusion, it does not seem worthwhile to include the proof here. Physically, however, it is reasonable if one considers the following argument: (i) The polarization induced in the medium is proportional to the ensemble average $\langle\rho_{ab}\rangle$ and the resolvability inequality [Eq. (9)] implies it is zero when the second pulse arrives; (ii) the physical reason for $\langle\rho_{ab}\rangle=0$ comes from the fact that the atomic dipoles have dephased owing to their differing natural frequencies; (iii) the interaction of the individual atom with the second pulse is describable by a precession of a vector in a three-dimensional pseudospace, in which the projection on the x, y plane represents the polarization and the z component is the population difference (see Ref. 12 for details). The result of a precession about a set of randomly oriented vectors is a set of vectors which is also randomly oriented at every instant in time while the precession occurs; (iv) Hence, the assertion that the ensemble of off-diagonal elements produced by the first pulse produces a zero $\langle\rho_{ab}\rangle$ in the time of interaction of $\mathcal{E}_2$ (which is the same conclusion one arrives at if $T_2\ll\Delta t$). Tentative calculations using the full set of equations with interference terms included (see Ref. 13) have indicated that there is no photon echo in the two-directional case. If this is the case, the equations in Sec. II will apply for that case even if Eq. (10) does not hold. One only needs the less restrictive conditions in Eq. (9).

[15]In common with Ref. 4 and the third paper in Ref. 6, a sign convention is used to make all of the medium parameters positive, and indicate explicitly that this is an attenuator by the minus sign in Eq. (5). This is in accord with the work of McCall and Hahn, but differs from the conventions of Ref. 7 by a minus sign.

[16]The notation t_1, t_2, t_3, t_4, etc. was introduced in Sec. II in order to make a careful statement of the problem. However, in all but the most pathological cases, it is awkward and superfluous, so the notation is dropped once it has served its purpose. Δt is taken in general to be the peak to peak interval between the pulses [which is equivalent to the original definition provided Eq. (9) holds]. In pathological cases, such as multiple-peaked pulses, one must reintroduce the notation; e.g., Eq. (9) would read $\Delta t\gg\max[t_2(z)-t_1(z),\ t_4(z)-t_3(z)]$ and in Sec. III, the conditions for validity for Eqs. (13a), and (13b) are T_2 and $T_1\gg\max[t_2(z)-t_1(z),\ t_4(z)-t_3(z)]$.

[17]C. L. Tang and B. D. Silvermann, in *Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill, New York, 1966).

[18]We have, of course, set $T_1\gg\Delta t$.

[19]In Refs. 3 and 4, it is noted that the energy of a π pulse should vary like $\tau_{\text{out}}\approx\tau_{\text{in}}e^{-\alpha L/2}$.

[20]This is not the only way of defining $t_D(z)$ and other definitions will result in somewhat modified conclusions. In particular, the absence of delays for $\theta_{\text{in}}<\pi$ in one-pass SIT comes from the fact that the peak does not move. If, instead, one defines $t_D(z)$ relative to the moment when half the pulse energy has passed by, then there will be nonzero (yet still small) delay times for $\theta_{\text{in}}<\pi$.

[21]The subscripts "in" and "out" must always be taken in context. In this case, $\theta_{\text{in}}=\theta_1(0)$ where $z=0$ and $z=L$ are the left and right edges of the rod. In the context of SIT, $\tau_{\text{out}}=\tau_1(L)$ or $\tau_2(0)$ depending on whether one is dealing with one or two passes. In Sec. V B, one has $\tau_{\text{in}}=\tau_2(L)$, and $\tau_{\text{out}}=\tau_2(0)$.

[22]The "knee" of the curve is the region in the curve between the small-signal and "transparent" regimes. The value of τ_{in} (or θ_{in}) such that $\tau_{\text{out}}/\tau_{\text{in}}=e^{-\alpha L}$ for a two-pass attenuator (or $e^{-\alpha L/2}$ for one pass) is called the "transition point."

[23]F. A. Hopf, G. L. Lamb, Jr., C. K. Rhodes, and M. O. Scully (to be published).

[24]L. M. Franz and J. S. Nodvik, J. Appl. Phys. 34, 2346 (1963); N. G. Basov, R. V. Ambartsumzan, V. S. Zeuv, P. G. Kryukov, and V. S. Letokhov, Zh. Eksperim. i Teor. Fiz. 50, 23 (1966) [Soviet Phys. JETP 23, 16 (1966)].

[25]The criteria for identity of two pulses are that within the qualification given below, the following be true: (i) The major parameters of the pulse (width, energy, and area) be the same; (ii) the amplitudes of the envelopes differ by no more than the quantity (peak amplitude) $\times 0.001$. Since the computer truncates the results to the number of significant figures (in this case three) in the output, the qualification is that a difference of a factor of one may occur in the last significant figure, and is not evidence of difference. Hence, for example, areas $\theta_{\text{out}}=3.14$ and $\theta_{\text{out}}=3.15$ would be taken to be the same, whereas $\theta_{\text{out}}=3.14$ and $\theta_{\text{out}}=3.12$ would be taken to be evidence of some difference in the output.

[26]The criterion $T_1\gg L/c$ is needed since otherwise the second pulse may be encountering a basically different $\chi(T, t_3, z)$ depending upon its direction of propagation. This criterion was checked on the computer for a case $T_1=L/c$ and the pulses were not identical according to the criteria of Ref. 23a. However, they did not differ by more than the expected error of the computer program. When pulses are reported as being different in the text, they are different by more than the expected error of the program (5%). Finally, one notes that one expects identical outputs if $T_1\ll\Delta t$, irrespective of the relative magnitudes of cT_1 and L.

[27]The restrictions on the relative magnitudes of the pulse width, the separation between the pulses, and the phase memory time as discussed in Sec. II mean that there is no significance to the relative magnitude of cT_2 to the rod length. Similarly, in Sec. V B, relative magnitude of cT_2 to the rod-mirror separation is unimportant.

[28]S. L. McCall (private communication).

[29]N. A. Kurnit, and S. R. Hartmann, in *Interaction of*

Radiation with Solids, edited by Adli Bishay (Plenum, New York, 1967), pp. 693–701. I am obliged to N. A. Kurnit for the objections to the sorts of measurements indicated in Sec. V B. He also pointed out that the value of T_1 probed by this measurement is not necessarily the same as the effective T_1 measured by the stimulated echo. For a discussion of this point, see the paper by these authors in the same source as Ref. 17. In L. O. Hocker, M. A. Kovacs, C. K. Rhodes, G. W. Flynn, and A. Javan [Phys. Rev. Letters 17, 233 (1966)], various phenomena involving the relaxation processes in CO_2 are investigated using pulse techniques and fluorescence measurements. These phenomena are involved in producing the lifetime T_1 used in the text.

[30]The fact that the second output pulse is independent of direction strongly indicates that the individual (perhaps infinitesimally thin) segments of a complex system commute with one another provided they conform to the restrictions given in the text (see Sec. V A). This commutability can be proved using rate equations, and also for more general T_2 provided that the line shapes are the same. This suggests, in turn, the following theorem which can be proved using rate equations and which has been confirmed to a limited extent using numerical techniques. One starts with two different systems A and B which conform to the restrictions indicated in Sec. V A, but which may be pumped in very different ways. A pulse $\mathcal{E}_1$ interacts with these systems producing output pulses $(\mathcal{E}_1)^A_{\rm out}$ from A and $(\mathcal{E}_1)^B_{\rm out}$ from B and new systems A' and B' that results from this interaction. Then, it is sufficient that $(\mathcal{E}_1)^A_{\rm out} = (\mathcal{E}_1)^B_{\rm out}$ in order that a second pulse $(\mathcal{E}_2)_{\rm in}$ interacts with A' and B' to produce outputs $(\mathcal{E}_2)^{A'}_{\rm out} = (\mathcal{E}_2)^{B'}_{\rm out}$. This result holds independently of the direction of the pulses, and implies that a series of N pulses would pass through an amplifier or an attenuator (it is now essential that $T_2 \ll \Delta t$) producing outputs that are independent of the relative directions of the pulses. Provided one ignores noise, dispersion, scattering losses and stability requirements, one can conclude from the above that any steady-state solution of the unidirectional ring laser that is describable as a single, well-resolved ultrashort pulse will also be a solution of some suitably chosen linear laser.

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Model for the Quantum Scattering of a He^3 Impurity from a Rectilinear Vortex in Liquid He II

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A model is given that describes the quantum scattering of a He^3 impurity from a rectilinear vortex in liquid He II; the principal assumption is the existence of an impurity wave function that satisfies a Schrödinger equation. Various possibilities for the impurity-vortex interaction are discussed. For certain interactions, the theoretical results are consistent with the experimental work of Rayfield and Reif provided spatial variations in the superfluid density are considered. A T-matrix formalism is also developed and applied to the vortex scattering of impurities, as well as phonons and rotons. The results are compared to existing theoretical and experimental work; the main discrepancies occur for roton scattering.

I. INTRODUCTION

The scattering of quasiparticles (phonons, rotons, and He^3 impurities) by vortices in liquid He II has been observed in various experiments.[1–4] The main purpose of this paper is to derive and investigate a formalism that describes the quantum scattering of a He^3 quasiparticle from a quantized rectilinear vortex in superfluid helium. Models already exist which have been applied to the corresponding classical scattering problem.[5–8] The basic model is presented in Sec. II; it assumes the existence of a quasiparticle wave function that satisfies a Schrödinger equation. In Sec. III, the model is applied to impurity scattering when the superfluid density ρ_s is constant. Various possibilities for the impurity-vortex interaction are discussed. Some peculiarities related to the Bohm-Aharonov problem occur[9]; the scattering amplitude appears to diverge as an infinite series in partial waves, and the incident state is found to consist of a plane wave modified by a phase factor. The approximate effects of spatial variations in ρ_s are considered in Sec. IV. In Sec. V, the frictional force is derived for the scattering results of Secs. I–IV; a comparison is made to the experimental work of Rayfield and Reif.[4] Finally, in Sec. VI, a T-matrix formalism is developed and applied to the vortex scattering