Retardation in the Elastic Scattering of Photons by Atomic Hydrogen

M. Gavrila*
Joint Institute for Laboratory Astrophysics,† *Boulder, Colorado* 80302
and
A. Costescu
Department of Physics, Bucharest University, Bucharest, Romania
(Received 18 March 1970)

The matrix element for the elastic scattering of photons by atomic hydrogen is evaluated analytically in closed form in the nonrelativistic case including retardation. The method of integration is similar to the one used previously by one of the authors for the dipole approximation. It is based on the possibility of expressing the matrix element in terms of the Green's function for the Coulomb field, for which the momentum-space integral representation derived by Schwinger and others is then used. Some integrations yield the matrix element in terms of essentially two hypergeometric functions of Appell's type F_1, with variables and parameters depending on the photon energy. A discussion of the result is given.

I. INTRODUCTION

Some time ago an exact analytic formula was derived for the matrix element of the elastic scattering of photons by atomic hydrogen in the nonrelativistic dipole approximation. The result was expressed in terms of the Gauss hypergeometric function ${}_2F_1$, with variable and parameters depending on the photon energy.[1] We shall now extend this result to include the effect of retardation.[2,3] The method used is similar to the one of Ref. 1. It is based on expressing the matrix element of the process in terms of the Coulomb Green's function and then performing the integrations in momentum space.

The nonrelativistic elastic scattering of a photon by a bound atomic electron, retardation included, is described by the Kramers-Heisenberg-Waller matrix element[4,5]:

$$\mathfrak{M} = (\vec{s}_1\cdot\vec{s}_2)[e^{i(\vec{\kappa}_1-\vec{\kappa}_2)\cdot\vec{r}}]_{00}$$

$$-\frac{1}{m}\mathop{S}_{n}\frac{[e^{-i\vec{\kappa}_2\cdot\vec{r}}(\vec{s}_2\cdot\vec{P})]_{0n}[e^{i\vec{\kappa}_1\cdot\vec{r}}(\vec{s}_1\cdot\vec{P})]_{n0}}{E_n-(E_0+\kappa+i\epsilon)}$$

$$-\frac{1}{m}\mathop{S}_{n}\frac{[e^{i\vec{\kappa}_1\cdot\vec{r}}(\vec{s}_1\cdot\vec{P})]_{0n}[e^{-i\vec{\kappa}_2\cdot\vec{r}}(\vec{s}_2\cdot\vec{P})]_{n0}}{E_n-(E_0-\kappa)}. \quad (1)$$

Here $\vec{\kappa}_1$ and $\vec{\kappa}_2$ denote the initial and final momenta of the scattered photon of energy $\kappa=\kappa_1=\kappa_2$, $\vec{s}_1$ and $\vec{s}_2$ are its initial and final polarizations, $\vec{P}$ is the electron momentum operator, E_n are the eigenvalues of the energy spectrum, and E_0 is the ground-state energy.

The matrix element $\mathfrak{M}$ may be expressed in terms of the Green's function $G(\vec{r}_2, \vec{r}_1; \Omega)$ of the atomic field, which is considered to be of Coulomb type in our case. Indeed, taking into account the expansion in energy eigenfunctions of the Green's function [see Ref. 1, Eq. (3)], Eq. (1) becomes

$$\mathfrak{M} = (\vec{s}_1\cdot\vec{s}_2)\,\mathcal{O} - \sum_{i,j} s_{1i}s_{2j}[\Pi_{ij}(\Omega)+\tilde{\Pi}_{ij}(\Omega_2)], \quad (2)$$

where

$$\mathcal{O} = \int |u_0(r)|^2 e^{i(\vec{\kappa}_1-\vec{\kappa}_2)\cdot\vec{r}}\,d\vec{r}, \quad (3)$$

$$\Pi_{ij}(\Omega) = (1/m)\int\int u_0(r_2)\,e^{-i\vec{\kappa}_2\cdot\vec{r}_2}P_{2j}\,G(\vec{r}_2, \vec{r}_1; \Omega)$$

$$\times P_{1i}\,e^{i\vec{\kappa}_1\cdot\vec{r}_1}u_0(r_1)\,d\vec{r}_1 d\vec{r}_2, \quad (4)$$

and $\tilde{\Pi}_{ij}(\Omega)$ is obtained from $\Pi_{ij}(\Omega)$ by interchanging i,j and $\vec{\kappa}_1$, $-\vec{\kappa}_2$. The quantities Ω_1 and Ω_2 are given by

$$\Omega_1 = E_0+\kappa+i\epsilon = -|E_0|+\kappa+i\epsilon,$$
$$\Omega_2 = E_0-\kappa = -|E_0|-\kappa. \quad (5)$$

Because $G(\vec{r}_2, \vec{r}_1; \Omega) = G(\vec{r}_1, \vec{r}_2; \Omega)$, $\vec{P}$ is Hermitian, and $u_0(r)$ is real, we have

$$\tilde{\Pi}_{ij}(\Omega) = \Pi_{ij}(\Omega).$$

Therefore, (2) becomes

$$\mathfrak{M} = (\vec{s}_1\cdot\vec{s}_2)\,\mathcal{O} - \sum_{i,j} s_{1i}s_{2j}[\Pi_{ij}(\Omega_1)+\Pi_{ij}(\Omega_2)]. \quad (6)$$

Rotational invariance arguments indicate that $\Pi_{ij}(\Omega)$ should have the following form:

$$\Pi_{ij}(\Omega) = \delta_{ij}P(\Omega)+\nu_{2i}\nu_{1j}Q(\Omega)+\nu_{1i}\nu_{2j}R(\Omega)$$
$$+\nu_{1i}\nu_{1j}S(\Omega)+\nu_{2i}\nu_{2j}T(\Omega),$$

where $\vec{\nu}_1$ and $\vec{\nu}_2$ are the unit vectors of $\vec{\kappa}_1$ and $\vec{\kappa}_2$. It then follows that

$$\sum_{i,j} s_{1i}s_{2j}\Pi_{ij}(\Omega) = (\vec{s}_1\cdot\vec{s}_2)P(\Omega) + (\vec{s}_1\cdot\vec{\nu}_2)(\vec{s}_2\cdot\vec{\nu}_1)Q(\Omega). \quad (7)$$

With Eq. (7), the matrix element $\mathfrak{M}$ of Eq. (6) can be written

$$\mathfrak{M} = (\vec{s}_1\cdot\vec{s}_2)M + (\vec{s}_1\cdot\vec{\nu}_2)(\vec{s}_2\cdot\vec{\nu}_1)N, \quad (8)$$

where

$$M = \mathcal{O} - P(\Omega_1) - P(\Omega_2), \tag{9}$$

$$N = -[Q(\Omega_1) + Q(\Omega_2)]. \tag{10}$$

The calculation of $\mathfrak{M}$ is thus reduced to the evaluation of $\mathcal{O}, P(\Omega), Q(\Omega)$, which will be carried out in Sec. II.

II. CALCULATION OF $\mathcal{O}, P(\Omega), Q(\Omega)$.

The evaluation of $\mathcal{O}$ is immediate. Introducing the expression of the ground-state energy eigenfunction of a hydrogenlike atom $u_0(r)$ in Eq. (3), one finds

$$\mathcal{O} = \{1 + [(\vec{\kappa}_1 - \vec{\kappa}_2)^2/4\lambda^2]\}^{-2} = \{1 + (\kappa^2/\lambda^2)\sin^2 \tfrac{1}{2}\theta\}^{-2}, \tag{11}$$

where $\lambda = \alpha Z m$, and we have taken into account that

$$(\vec{\kappa}_1 - \vec{\kappa}_2)^2 = 4\kappa^2 \sin^2 \tfrac{1}{2}\theta. \tag{12}$$

In order to determine $P(\Omega)$ and $Q(\Omega)$ we begin by expressing $\Pi_{ij}(\Omega)$ as a momentum-space integral. We get from Eq. (4)

$$\Pi_{ij}(\Omega) = (1/m) \int\int p_{1i} p_{2j} u_0(\vec{p}_2 - \vec{\kappa}_2) G(\vec{p}_2, \vec{p}_1; \Omega) \times u_0(\vec{p}_1 - \vec{\kappa}_1) d\vec{p}_1 d\vec{p}_2, \tag{13}$$

where $u_0(p)$ and $G(\vec{p}_2, \vec{p}_1; \Omega)$ are the Fourier transforms of $u_0(r)$ and $G(\vec{r}_2, \vec{r}_1; \Omega)$.

We shall use for $G(\vec{p}_2, \vec{p}_1; \Omega)$ the Schwinger integral representation. This may be written as[6]

$$G(\vec{p}_2, \vec{p}_1; \Omega) = \frac{m}{2\pi^2} X^3 \left(\frac{i e^{i\pi\tau}}{2\sin\pi\tau}\right) \int_1^{(0+)} \rho^{-\tau} \frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho} \frac{1}{[X^2(\vec{p}_1 - \vec{p}_2)^2 + (p_1^2 + X^2)][(p_2^2 + X^2)(1-\rho)^2/4\rho]^2}\right) d\rho, \tag{14}$$

where

$$\tau = (\lambda/X), \quad X^2 = -2m\Omega, \tag{15}$$

and X is chosen so that

$$\mathrm{Re} X > 0. \tag{16}$$

Introducing Eq. (14) and the expression for $u_0(p)$ [Ref. 1, Eq. (14)] into (13) we find, after interchanging the order of integrations,

$$\Pi_{ij}(\Omega) = \frac{4\lambda^5}{\pi^4} X^3 \left(\frac{i e^{i\pi\tau}}{2\sin\pi\tau}\right) \int_1^{(0+)} \rho^{-\tau} \frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho} T_{ij}\right) d\rho, \tag{17}$$

with

$$T_{ij} = \iint \frac{p_{1i} p_{2j}}{[(\vec{p}_2 - \vec{\kappa}_2)^2 + \lambda^2]^2 [X^2(\vec{p}_1 - \vec{p}_2)^2 + \alpha(p_1^2 + X^2)][p_2^2 + X^2)]^2 [\vec{p}_1 - \kappa_1)^2 + \lambda^2]^2} d\vec{p}_1 d\vec{p}_2. \tag{18}$$

In Eq. (18) we have abbreviated

$$\alpha = (1-\rho)^2/4\rho. \tag{19}$$

Now, because

$$\vec{\kappa}_1 \cdot \vec{s}_1 = \vec{\kappa}_2 \cdot \vec{s}_2 = 0, \tag{20}$$

it is easy to derive the equality

$$\sum_{i,j} s_{1i} s_{2j} T_{ij} = \tfrac{1}{4} \sum_{i,j} s_{1i} s_{2j} \frac{\partial^2 J(X^2; \lambda, \lambda)}{\partial \kappa_{1i} \partial \kappa_{2j}}, \tag{21}$$

which contains the parameter-dependent integral $J(X^2; \lambda, \mu)$ defined in Eq. (A1) of Appendix A. This enables us to write Eq. (17) in the form

$$\sum_{i,j} s_{1i} s_{2j} \Pi_{ij}(\Omega) = \frac{\lambda^5}{\pi^4} X^3 \left(\frac{i e^{i\pi\tau}}{2\sin\pi\tau}\right) \int_1^{(0+)} \rho^{-\tau} \sum_{i,j} s_{1i} s_{2j} \frac{\partial^2}{\partial\kappa_{1i}\partial\kappa_{2j}} \frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho} J(X^2; \lambda, \lambda)\right) d\rho. \tag{22}$$

In Eqs. (21) and (22) it is understood that after taking the derivatives of $J(X^2; \lambda, \lambda)$ with respect to κ_{1i}, κ_{2j}, one should set $\kappa_1 = \kappa_2 = \kappa$.

The calculation of the integral $J(X^2; \lambda, \mu)$ is performed in Appendix A, the result being given by Eqs. (A6)–(A8). Noting that b of Eq. (A8) is independent of ρ and taking into account Eq. (19), it is easy to show that

$$\frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho} J(X^2; \lambda, \mu)\right) = -\frac{8\pi^4}{X^2}\frac{1}{a^2-b}\frac{da}{d\rho} . \tag{23}$$

Furthermore, using Eq. (20) one finds that

$$\sum_{i,j} s_{1i} s_{2j} \frac{\partial^2}{\partial\kappa_{1i}\partial\kappa_{2j}} \frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho} J(X^2; \lambda, \lambda)\right) = 8\pi^4\left((\vec{s}_1\cdot\vec{s}_2)\frac{1}{\rho^3}\frac{[(X+\lambda)^2+\kappa^2]^4}{(a^2-b)^2} + (\vec{s}_1\cdot\vec{\kappa}_2)(\vec{s}_2\cdot\vec{\kappa}_1)\frac{4X^2}{\rho^4}\frac{[(X+\lambda)^2+\kappa^2]^6}{(a^2-b)^3}\right) . \tag{24}$$

Here we have set $\kappa_1 = \kappa_2 = \kappa$ after performing the derivatives with respect to κ_{1i}, κ_{2j}. Using Eqs. (A7) and (A8), with $\kappa_1 = \kappa_2 = \kappa$, $\lambda = \mu$, the quantity $a^2 - b$ appearing in (24) may be expressed in the form

$$a^2 - b = (1/4\rho^2)\,[(X+\lambda)^2+\kappa^2]^4\,(1 - s\rho + p\rho^2), \tag{25}$$

where

$$s = 2\frac{(X-\lambda)^2+\kappa^2}{(X+\lambda)^2+\kappa^2} - \frac{4X^2(\vec{\kappa}_1-\vec{\kappa}_2)^2}{[(X+\lambda)^2+\kappa^2]^2} , \tag{26}$$

$$p = \left[\frac{(X-\lambda)^2+\kappa^2}{(X+\lambda)^2+\kappa^2}\right]^2 . \tag{27}$$

We now return to Eq. (22) and insert Eqs. (24) and (25) in it. We get

$$\sum_{i,j} s_{1i} s_{2j} \Pi_{ij}(\Omega) = 128\lambda^5 X^3\left(\frac{ie^{i\pi\tau}}{2\sin\pi\tau}\right)\left(\frac{(\vec{s}_1\cdot\vec{s}_2)}{[(X+\lambda)^2+\kappa^2]^4}\int_1^{(0+)}\rho^{1-\tau}(1-s\rho+p\rho^2)^{-2}d\rho + \frac{16X^2(\vec{s}_1\cdot\vec{\kappa}_2)(\vec{s}_2\cdot\kappa_1)}{[(X+\lambda)^2+\kappa^2]^6}\int_1^{(0+)}\rho^{2-\tau}(1-s\rho+p\rho^2)^{-3}d\rho\right). \tag{28}$$

The result may be expressed in terms of Appell's hypergeometric function F_1. This has the following integral representation[7]:

$$F_1(a; b_1, b_2; c; x_1 x_2) = -\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\left(\frac{ie^{-i\pi a}}{2\sin\pi a}\right)\int_1^{(0+)}\rho^{a-1}(1-\rho)^{c-a-1} \times (1-x_1\rho)^{-b_1}(1-x_2\rho)^{-b_2}\,d\rho , \tag{29}$$

provided that $\mathrm{Re}c > \mathrm{Re}a$.

Hence, Eq. (28) becomes

$$\sum_{i,j} s_{1i} s_{2j} \Pi_{ij}(\Omega) = 128\lambda^5 X^3 \times\left((\vec{s}_1\cdot\vec{s}_2)\frac{F_1(2-\tau; 2, 2; 3-\tau; x_1, x_2)}{(2-\tau)[(X+\lambda)^2+\kappa^2]^4} + (\vec{s}_1\cdot\vec{\kappa}_2)(\vec{s}_2\cdot\vec{\kappa}_1)\,16X^2\frac{F_1(3-\tau; 3, 3; 4-\tau; x_1, x_2)}{(3-\tau)[(X+\lambda)^2+\kappa^2]^6}\right). \tag{30}$$

The variables x_1, x_2 of the Appell functions F_1 are obtained from

$$1 - s\rho + p\rho^2 = (1 - x_1\rho)(1 - x_2\rho) , \tag{31}$$

and are equal to

$$x_{1,2} = \Big((X^2-\lambda^2-\kappa^2)^2 + 4X^2(\vec{\kappa}_1\cdot\vec{\kappa}_2) \pm\{-4X^2(\vec{\kappa}_1-\vec{\kappa}_2)^2[X^2(\vec{\kappa}_1+\vec{\kappa}_2)^2 + (X^2-\lambda^2-\kappa^2)^2]\}^{1/2}\Big)\Big/[(X+\lambda)^2+\kappa^2]^2 . \tag{32}$$

Comparing Eqs. (7) and (30) we find the desired expressions of $P(\Omega)$ and $Q(\Omega)$ are[8]

$$P(\Omega) = \frac{128\lambda^5 X^3}{[(X+\lambda)^2+\kappa^2]^4}\,\frac{F_1(2-\tau;\ 2, 2;\ 3-\tau;\ x_1,\ x_2)}{2-\tau} , \tag{33}$$

$$Q(\Omega) = \frac{2048\lambda^5 X^5\kappa^2}{[(X+\lambda)^2+\kappa^2]^6}\,\frac{F_1(3-\tau;\ 3, 3;\ 4-\tau;\ x_1,\ x_2)}{(3-\tau)} . \tag{34}$$

III. DISCUSSION

We shall now discuss some of the properties of the matrix element derived in Eqs. (8)–(11), (33), and (34). With retardation included, $\mathfrak{M}$ is considerably more complex than in the dipole approximation, both concerning the angular and the photon energy dependence. Thus $\mathfrak{M}$ now depends separately on κ and Z, whereas in the dipole approximation, the dependence on these quantities was concentrated in the unique variable

$$k = \frac{\kappa}{(\lambda^2/2m)} = \frac{\kappa}{Z^2\mathfrak{R}} , \tag{35}$$

where $\mathfrak{R}$ is the rydberg. Furthermore, M and N are angle dependent.

In order to achieve a better understanding of our result we shall use series expansions for the Appell functions F_1 contained in $P(\Omega)$, $Q(\Omega)$. Note that these have the parameters b equal: $b_1 = b_2$. Various expansion formulas have been established especially for this case.[9] However, as the one convenient for us does not seem to have been considered before, we have derived it in Appendix B [see Eq. (B5)].

The evaluation of the quantities contained in Eq. (B5) yields, on account of Eqs. (31), (26), (27), and (12):

$$x_1+x_2-2(x_1x_2)^{1/2}=s-2p^{1/2}=-\frac{16\kappa^2X^2\sin^2\frac{1}{2}\theta}{[(X+\lambda)^2+\kappa^2]^2}\ ,$$
$$1-(x_1x_2)^{1/2}=1-p^{1/2}=\frac{4\lambda X}{(X+\lambda)^2+\kappa^2}\ , \tag{36}$$

where the convenient sign of $(x_1x_2)^{1/2}$ has been chosen. Furthermore, we shall denote

$$u=(x_1x_2)^{1/2}=[(X-\lambda)^2+\kappa^2]/[(X+\lambda)^2+\kappa^2]\ . \tag{37}$$

Thus, using formula (B5) in Eqs. (33) and (34) we get

$$P(\Omega)=\frac{2\lambda^2}{(X+\lambda)^2+\kappa^2}\sum_{p=0}^{\infty}\frac{(2)_p}{p!}\left[-\frac{\kappa^2}{\lambda^2}\sin^2\tfrac{1}{2}\theta\right]^p \times\frac{{}_2F_1(1,\ -1-p-\tau,\ 3+p-\tau;\ u)}{2+p-\tau}\ , \tag{38}$$

$$Q(\Omega)=\frac{2\kappa^2}{(X+\lambda)^2+\kappa^2}\sum_{p=0}^{\infty}\frac{(3)_p}{p!}\left[-\frac{\kappa^2}{\lambda^2}\sin^2\tfrac{1}{2}\theta\right]^p \times\frac{{}_2F_1(1,\ -2-p-\tau,\ 4+p-\tau;\ u)}{3+p-\tau}\ . \tag{39}$$

To obtain the convergence condition of these expansions we note that on account of Eqs. (5), (15), and (16), the values X of interest are X positive real and X pure imaginary (negative). Now, for X positive real, $|u|<1$ and for X imaginary, $|u|=1$. Consequently, as discussed in Appendix B, the appropriate convergence condition is given by Eq. (B9). With Eqs. (36) this becomes, for both $P(\Omega)$ and $Q(\Omega)$,

$$(\kappa^2/\lambda^2)\ \sin^2\tfrac{1}{2}\theta<1\ . \tag{40}$$

It follows that the series for $P(\Omega)$ and $Q(\Omega)$ are convergent for *all* θ if

$$\kappa^2/\lambda^2<1\ . \tag{41}$$

Now, the ratio κ/λ characterizes the magnitude of retardation. Therefore expansions (38) and (39) exhibit explicitly the succesive orders of the retardation corrections. The dipole approximation corresponds to setting $\kappa/\lambda=0$ in Eqs. (11) and (37)–(39) (however not in X). Then $P(\Omega)$ reduces to the expression given in Ref. 1, Eq. (54), and $Q(\Omega)=0$, so that $\mathfrak{M}$ of Eq. (8) reduces to that of Ref. 1, Eq. (7), as should be.

For photon energies below the photoelectric threshold, that is $0\leqslant\kappa<1$, from Eqs. (15), (16), and (5) we get

$$\tau_1=\lambda/X_1=1/(1-k)^{1/2},$$
$$\tau_2=\lambda/X_2=1/(1+k)^{1/2}\ . \tag{42}$$

In this case, the quantities M and N are real. Because the Appell function $F_1(a;\ b_1,\ b_2;\ c;\ x_1,\ x_2)$ is a meromorphic function of its parameter c, having poles for $c=0,\ -1,\ -2,\ \dots$, it follows from Eqs. (33) and (34) that $P(\Omega)$ is singular for $\tau=2$ and 3, 4, ..., and $Q(\Omega)$ is singular for $\tau=3$ and 4, 5, ... Taking into account Eqs. (42), one sees that the singularities occur only in $P(\Omega_1)$ and $Q(\Omega_1)$, for photon energies corresponding to the Lyman spectrum, in agreement with the resonant structure of Eq. (1). Below the threshold, the retardation corrections to the dipole approximation result of Ref. 1 are very small for atomic hydrogen, of order $(\alpha Z)^2$ or less.

Above the photoelectric threshold $(k>1)$, we have

$$\tau_1=i/(k-1)^{1/2}\ ,\quad \tau_2=1/(1+k)^{1/2}\ . \tag{43}$$

The variable u_1, obtained by setting $X=X_1$ in Eq. (37), is complex and $|u_1|=1$. When k varies, u_1 moves on the unit circle centered at the origin of the the complex (u) plane. As regards u_2, corresponding to X_2, it is positive real and $0<u_2<1$.

The imaginary values of τ_1, X_1 make $P(\Omega_1)Q(\Omega_1)$, and therefore also $\mathfrak{M}$, complex. Because $\mathcal{O}$, $P(\Omega_2)$, and $Q(\Omega_2)$ are real, we have from Eqs. (8)–(10)

$$\mathrm{Im}\mathfrak{M}=-[(\vec{s}_1\vec{s}_2)\,\mathrm{Im}P(\Omega_1)+(\vec{s}_1\vec{\nu}_2)(\vec{s}_2\vec{\nu}_1)\,\mathrm{Im}Q(\Omega_1)]. \tag{44}$$

We now show that $\mathrm{Im}\mathfrak{M}$ can be expressed in closed form for any scattering angle. Indeed, noting that

$$\tau_1^*=-\tau_1\ ,\quad u_1^*=1/u_1\ ,$$

we find

$$\mathrm{Im}P=\frac{1}{2i}(p-p^*)=\frac{\lambda^2}{i}\sum_{p=0}^{\infty}\frac{(2)_p}{p!}\left(-\frac{\kappa^2}{\lambda^2}\sin^2\tfrac{1}{2}\theta\right)^p \times\left(\frac{{}_2F_1(1,\ -1-p-\tau_1,\ 3+p-\tau_1;\ u_1)}{[(\lambda+X_1)^2+\kappa^2](2+p-\tau_1)} -\frac{{}_2F_1(1,\ -1-p+\tau_1,\ 3+p+\tau_1;\ 1/u_1)}{[(\lambda-X_1)^2+\kappa^2](2+p+\tau_1)}\right). \tag{45}$$

The long expression in large parentheses in Eq. (45) may be transformed using the formula for analytic continuation of a hypergeometric function ${}_2F_1$ of variable u to functions of variable $1/u$.[10] Thus, we have

$$\mathrm{Im}P(\Omega_1)=\frac{\lambda^2}{i}\ \frac{(-u_1)^{-2+\tau_1}(1-u_1)^3}{(\lambda+X_1)^2+\kappa^2}\sum_{p=0}^{\infty}\frac{(2)_p}{p!} \times\frac{\Gamma(2+p-\tau_1)\Gamma(2+p+\tau_1)}{(2p+3)!}\left(\frac{\kappa^2}{\lambda^2}\frac{(1-u_1)^2}{u_1}\sin^2\tfrac{1}{2}\theta\right)^p, \tag{46}$$

with

$$|\arg(-u_1)| < \pi \ .$$

In order to satisfy the preceding condition we must take

$$\arg(-u_1) = 2\chi - \pi \ ,$$

where χ is given by

$$\chi = \arctan\frac{2\lambda |X_1|}{\lambda^2 + \kappa^2 + X_1^2} = \arctan\frac{2(k-1)^{1/2}}{2 - k + \frac{1}{4}k^2(\alpha Z)^2} \ ,$$

and $0 < \chi < \pi$. Consequently, we have

$$(-u_1)^{\tau_1} = \exp[(-2\chi + \pi)|\tau_1|] \ .$$

Because we have

$$\Gamma(2+p-\tau_1)\Gamma(2+p+\tau_1) = (2-\tau_1)_p(2+\tau_1)_p \times \frac{2\pi|\tau_1|(1+|\tau_1|^2)\exp(-\pi|\tau_1|)}{1-\exp(-2\pi|\tau_1|)} \ ,$$

$$(2p+3)! = 6\times 2^{2p}(2)_p(\tfrac{5}{2})_p \ ,$$

Eq. (46) becomes

$$\operatorname{Im}P(\Omega_1) = \frac{64\pi}{3i}\frac{|\tau_1|(1+|\tau_1|^2)\exp(-2|\tau_1|\chi)}{1-\exp(-2\pi|\tau_1|)} \times \frac{\lambda^5 X_1^3}{[(X_1^2-\lambda^2-\kappa^2)^2+4\kappa^2X_1^2]^2} \times \sum_{p=0}^{\infty}\frac{(2-\tau_1)_p(2+\tau_1)_p}{(\frac{5}{2})_p\, p!}\left(\frac{4\kappa^2X_1^2\sin^2\frac{1}{2}\theta}{(X_1^2-\lambda^2-\kappa^2)^2+4\kappa^2X_1^2}\right)^p .$$

This may be written

$$\operatorname{Im}P(\Omega_1) = +\frac{64\pi}{3}\frac{\eta^6}{(1+\eta^2)^3}\frac{\exp(-2\eta\chi)}{1-\exp(-2\pi\eta)} \times\{[1-k\tfrac{1}{4}(\alpha Z)^2]^2+(\alpha Z)^2\}^{-2} \times {}_2F_1(2-i\eta,\ 2+i\eta,\ \tfrac{5}{2};v) \ , \tag{47}$$

where the notation has been introduced

$$\eta = |\tau_1| = (k-1)^{-1/2} \ , \tag{48}$$

$$v = -(\alpha Z)^2(k-1)\{[1-k\tfrac{1}{4}(\alpha Z)^2]^2+(\alpha Z)^2\}^{-2}\sin^2\tfrac{1}{2}\theta \ . \tag{49}$$

Equation (47) has been established supposing that condition (41) is satisfied. Nevertheless, by analytic continuation of the ${}_2F_1$ function involved, Eq. (47) remains true for every value of κ.

Proceeding similarly in the case of $\operatorname{Im}Q(\Omega_1)$, one finds

$$\operatorname{Im}Q(\Omega_1) = (\alpha Z)^2\frac{256\pi}{15}\frac{\eta^8(1+\frac{1}{4}\eta^2)}{(1+\eta^2)^5}\frac{\exp(-2\eta\chi)}{1-\exp(-2\pi\eta)} \times k^2\{[1-k\tfrac{1}{4}(\alpha Z)^2]^2+(\alpha Z)^2\}^{-3} \times {}_2F_1(3-i\eta,\ 3+i\eta,\ \tfrac{7}{2};v) \ . \tag{50}$$

With Eqs. (44), (47), and (50), the imaginary part of the matrix element $r_0\mathfrak{M}$ for forward scattering ($\vec{\kappa}_1 = \vec{\kappa}_2$) without change of polarization ($\vec{s}_1 = \vec{s}_2$) is

$$\operatorname{Im}(r_0\mathfrak{M}_0) = -\frac{64\pi}{3}r_0\frac{\eta^6}{(1+\eta^2)^3}\frac{\exp(-2\eta\chi)}{1-\exp(-2\pi\eta)} \times\{[1-k\tfrac{1}{4}(\alpha Z)^2]^2+(\alpha Z)^2\}^{-2} \ . \tag{51}$$

One can thus check the "optical theorem"

$$|\operatorname{Im}(r_0\mathfrak{M}_0)| = (\kappa/4\pi)\sigma_f \ , \tag{52}$$

where σ_f is the nonrelativistic total cross section of the photoeffect, retardation included.[11]

In the case of forward scattering, we have $\mathfrak{M} = (\vec{s}_1\cdot\vec{s}_2)M$; for not too high photon energies, M reduces to its dipole approximation [see Eqs. (11) and (38)]. The retardation corrections come in through the powers of $(\kappa^2/\lambda^2)\sin^2\frac{1}{2}\theta$ and are significant only for $\theta \neq 0$ and higher values of κ/λ. It turns out, however, that for $\kappa \simeq \lambda$, relativistic corrections are also become important. Therefore, a numerical analysis at higher energies should be based on a complete relativistic treatment. The present work represents a preliminary step towards an analytic solution of this problem.

APPENDIX A. BASIC MOMENTUM-SPACE INTEGRAL

The momentum-space integrals considered in the text can be expressed in terms of

$$J(X^2;\lambda,\ \mu) = \int\int\{d\vec{p}_1\,d\vec{p}_2/[(\vec{p}_2-\vec{\kappa}_2)^2+\mu^2] \times[X^2(\vec{p}_2-\vec{p}_1)^2+\alpha(p_1^2+X^2)\times(p_2^2+X^2)] \times[(\vec{p}_1-\vec{\kappa}_1)^2+\lambda^2]\} \ , \tag{A1}$$

where α is a positive real constant.[12]

In order to evaluate this integral we shall suppose provisionally that X^2, λ, μ are positive real parameters. The integration over $\vec{p}_1$ occurring in Eq. (A1) can be easily performed[13]:

$$\int\frac{d\vec{p}_1}{[X^2(\vec{p}_2-\vec{p}_1)^2+\alpha(p_1^2+X^2)(p_2^2+X^2)]^2[(\vec{p}_1-\vec{\kappa}_1)^2+\lambda^2]} = \frac{1}{[X^2+\alpha(p_2^2+X^2)]^2}\int\frac{d\vec{p}_1}{[(\vec{p}_1-\vec{q})^2+\Lambda^2]^2[(\vec{p}_1-\vec{\kappa}_1)^2+\lambda^2]} = \frac{\pi^2}{\Lambda}\frac{1}{[X^2+\alpha(p_2^2+X^2)]^2[(\vec{\kappa}_1-\vec{q})^2+(\lambda+\Lambda)^2]} \ , \tag{A2}$$

where

$$\vec{q} = X^2[X^2+\alpha(p_2^2+X^2)]^{-1}\vec{p}_2 \ ,$$
$$\Lambda^2 = \alpha(1+\alpha)X^2(p_2^2+X^2)^2[X^2+\alpha(p_2^2+X^2)]^{-2} \ . \tag{A3}$$

The result of Eq. (A2) is valid provided λ and Λ are taken to be positive.

Taking into account Eqs. (A2) and (A3), (A1) becomes

$$J(X^2;\lambda,\mu)=\frac{\pi^2}{[\alpha(1+\alpha)]^{1/2}\gamma X}\times\int\frac{d\vec{p}_2}{[(\vec{p}_2-\vec{\kappa}_2)^2+\mu^2](p_2^2+X^2)[\vec{p}_1-\vec{k}_1)^2+\nu^2]}\ . \tag{A4}$$

We have abbreviated here

$$\gamma=(1+\alpha)X^2+\alpha(\kappa_1^2+\lambda^2)+2\lambda X[\alpha(1+\alpha)]^{1/2}\ ,$$
$$\vec{k}_1=X^2\gamma^{-1}\vec{\kappa}_1\ , \tag{A5}$$
$$\nu^2=X^2\gamma^{-2}\{[\alpha(1+\alpha)]^{1/2}(X^2+\kappa_1^2+\lambda^2)+(1+2\alpha)\lambda X\}^2\ .$$

The three-denominator integral of Eq. (A4) was evaluated in the general case by Lewis.[14] Using his result and replacing $\vec{k}_1$, ν by their values, we finally find (after an elementary but rather lengthy calculation)

$$J(X^2;\lambda,\mu)=\frac{\pi^4}{[\alpha(1+\alpha)]^{1/2}X^2}\,\frac{1}{b^{1/2}}\ln\frac{a+b^{1/2}}{a-b^{1/2}}\ , \tag{A6}$$

where

$$a=X^2[(\vec{\kappa}_1-\vec{\kappa}_2)^2+(\lambda-\mu)^2]+\mu X[(X+\lambda)^2+\kappa_1^2]+\lambda X[(X+\mu)^2+\kappa_2^2]+\{\alpha+[\alpha(1+\alpha)]^{1/2}\}\times[(X+\lambda)^2+\kappa_1^2][(X+\mu)^2+\kappa_2^2]\ , \tag{A7}$$

$$b=-X^2\{[(\kappa_1^2+\lambda^2-X^2)\vec{\kappa}_2-(\kappa_2^2+\mu^2-X^2)\vec{\kappa}_1]^2+4X^2[\kappa_1^2\kappa_2^2-(\vec{\kappa}_1\vec{\kappa}_2)^2]\}\ . \tag{A8}$$

The principal value of the logarithm appearing in Eq. (A6) should be chosen. Note that the result is independent of the sign of $b^{1/2}$.

We derived Eq. (A6), supposing that X^2, λ, μ were positive real. However, it may be shown that it holds under more general conditions.

First, it may be shown that the integral $J(X^2;\lambda,\mu)$ is an analytic function of X^2 in the (X^2) plane cut along the negative real axis, whatever λ and μ complex, provided that $\mathrm{Re}\lambda\neq 0$, $\mathrm{Re}\mu\neq 0$.[15] Furthermore, it is not difficult to see that $J(X^2;\lambda,\mu)$ is an analytic function of λ in the $\mathrm{Re}\lambda>0$ half-plane, for any X^2 in the cut plane and for any complex μ for which $\mathrm{Re}\mu\neq 0$. A similar property exists with respect to μ. On the other hand, the right-hand side of Eq. (A6) is an analytic function of X^2, λ, μ. Therefore, the equality (A6), which was derived for X^2, λ, μ positive real, will hold by analytic continuation for any X^2 in the (X^2) plane cut along the negative axis and for any λ, μ for which $\mathrm{Re}\lambda>0$, $\mathrm{Re}\mu>0$. It follows that if X^2, λ^2, μ^2 appearing in Eq. (A1) are complex, one should understand by X, λ, μ in Eq. (A6) the square roots for which[16]

$$\mathrm{Re}X>0,\quad \mathrm{Re}\lambda>0,\quad \mathrm{Re}\mu>0\ . \tag{A9}$$

APPENDIX B: EXPANSION OF THE APPEL FUNCTION $F_1(a;b,b;c;x_1,x_2)$.

We shall now derive an expansion of the Appell function for the case in which the parameters b_1 and b_2 are equal $(b_1=b_2=b)$, following a method of Nagel.[9] Supposing provisionally that $\mathrm{Re}c>\mathrm{Re}a>0$, we can use the integral representation[17]

$$F_1(a;b,b;c;x_1x_2)=[\Gamma(c)/\Gamma(a)\Gamma(c-a)]\times\int_0^1\rho^{a-1}(1-\rho)^{c-a-1}[(1-x_1\rho)(1-x_2\rho)]^{-b}d\rho\ . \tag{B1}$$

Taking into account the identity[18]

$$(1-x_1\rho)(1-x_2\rho)=[1-(x_1x_2)^{1/2}\rho]^2\times\left(1-\frac{[x_1+x_2-2(x_1x_2)^{1/2}]\rho}{[1-(x_1x_2)^{1/2}\rho]^2}\right)\ , \tag{B2}$$

we may write

$$[(1-x_1\rho)(1-x_2\rho)]^{-b}=[1-(x_1x_2)^{1/2}\rho]^{-2b}\times\sum_{p=0}^{\infty}\frac{(b)_p}{p!}\left(\frac{[x_1+x_2-2(x_1x_2)^{1/2}]\rho}{[1-(x_1x_2)^{1/2}\rho]^2}\right)^p\ . \tag{B3}$$

The series contained in Eq. (B3) is uniformly convergent in the interval $0\leq\rho\leq 1$, provided that x_1, x_2 are sufficiently small in module. Therefore, when introducing (B3) into (B1), it is possible to integrate term by term. Then, by using the standard integral representation for the Gauss hypergeometric function,[19] we have

$$F_1(a;b,b;c;x_1,x_2)=\sum_{p=0}^{\infty}\frac{(a)_p(b)_p}{(c)_p p!}[x_1+x_2-2(x_1x_2)^{1/2}]^p\times{}_2F_1(a+p,\,2b+2p,\,c+p;(x_1x_2)^{1/2})\ , \tag{B4}$$

where

$$(a)_p=a(a+1)\cdots(a+p-1)\ .$$

Furthermore, using a transformation property of the ${}_2F_1$ functions,[20] Eq. (B4) becomes

$$F_1(a;b,b;c;x_1x_2)=[1-(x_1x_2)^{1/2}]^{c-a-2b}\times\sum_{p=0}^{\infty}\frac{(a)_p(b)_p}{(c)_p p!}\left(\frac{x_1+x_2-2(x_1x_2)^{1/2}}{[1-(x_1x_2)^{1/2}]^2}\right)^p\times{}_2F_1(c-a,\,c-2b-p,\,c+p;(x_1x_1)^{1/2})\ . \tag{B5}$$

It follows from the derivation of (B5) that the series is convergent for x_1, x_2 small enough in module. In order to determine the convergence domain of the expansion we shall consider the asymptotic behavior of its terms for large p. For $p\to\infty$, these become equal to

$$\{[x_1+x_2-2(x_1x_2)^{1/2}]/[1-(x_1x_2)^{1/2}]^2\}^p$$
$$\times {}_2F_1(c-a,\ c-2b-p,\ c+p;(x_1x_2)^{1/2})\ . \qquad \text{(B6)}$$

The asymptotic behavior of the function ${}_2F_1(a,\ b-p,\ c+p;x)$ for large positive values of p has been worked out by Perron.[21] The general result applied to the present case yields the two alternatives: (i) If $|(x_1x_2)^{1/2}|\leq 1$, then the function

$${}_2F_1(c-a,\ c-2b-p,\ c+p;(x_1x_2)^{1/2}) \qquad \text{(B7)}$$

tends to a quantity independent of p. (ii) If $|(x_1x_2)^{1/2}|>1$, the function (B7) behaves as

$$\text{Max}\{1;|[1-(x_1x_2)^{1/2}]^2/4(x_1x_2)^{1/2}|^p\}\ . \qquad \text{(B8)}$$

Factors, independent of p or containing powers of p (without influence on the determination of the radius of convergence), have been ignored in (B8).

Introducing these results into (B6), we conclude that (i) if $|(x_1x_2)^{1/2}|\leq 1$, the series (B5) converges for

$$|[x_1+x_2-2(x_1x_2)^{1/2}]/[1-(x_1x_2)^{1/2}]^2|<1\ ; \qquad \text{(B9)}$$

(ii) if $|(x_1x_2)^{1/2}|>1$, the series (B5) converges for

$$\text{Max}\left(\left|\frac{x_1+x_2-2(x_1x_2)^{1/2}}{[1-(x_1x_2)^{1/2}]^2}\right|;\left|\frac{x_1+x_2-2(x_1x_2)^{1/2}}{4(x_1x_2)^{1/2}}\right|\right)<1\ . \qquad \text{(B10)}$$

Equation (B5) has been established under the condition that $\text{Re}c>\text{Re}a>0$. However, because of the analyticity of the hypergeometric functions with respect to their parameters, it remains true for any values of a and c.

---

*On leave from the Department of Physics, Bucharest University, Bucharest, Romania.

†Of the National Bureau of Standards and the University of Colorado.

[1]M. Gavrila, Phys. Rev. 163, 147 (1967).

[2]Results announced by M. Gavrila and A. Costescu, Phys. Letters 28A, 614 (1969).

[3]Meanwhile, work on the same problem using different methods was done by C. Fronsdal, Phys. Rev. 179, 1513 (1969); V. G. Gorshkov and B. S. Polikhanov, Zh. Eksperim. i Teor. Fiz. Piz'ma v Redaktsiyu 9, 464 (1969) [Soviet Phys. JETP Letters 9, 279 (1969)]; S. Klarsfeld, Nuovo Cimento Letters 1, 682 (1969).

[4]See, e.g., W. Heitler, *The Quantum Theory of Radiation* (Oxford U. P., New York, 1954), Chap. V, Sec. 19.1. We use natural units, such that $\hbar=c=1$.

[5]The relativistic limitations of this formula have been considered by V. G. Gorshkov, A. I. Mikhailov, V. S. Polikanov, and S. G. Sherman, Phys. Letters 30A, 455 (1969).

[6]J. Schwinger, J. Math. Phys. 5, 1606 (1964), Eq. (3′). Essentially the same result was obtained also in the papers quoted in Ref. 9 of DA. Our Green's function differs in sign from the one of Schwinger. We are denoting, in the present work, by τ the quantity λ/X which was denoted by κ in DA.

[7]The standard integral representation for the function F_1 is the one given in A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. I, p. 231 Eq. (5). (This book will be referred to in the following as HTF.) The procedure for passing from the quoted formula to our Eq. (29) is the same as for passing from HTF, p. 114, Eq. (1) to Eq. (3).

[8]The results derived in Ref. 3 agree with ours.

[9]B. Nagel, Arkiv Fysik 24, 479 (1963).

[10]HTF, p. 107, Eq. (34); p. 105, Eqs. (9) and (14).

[11]This is given by J. Fischer, Ann. Physik 8, 821 (1931), Eq. (15).

[12]See Eq. (19). The contour of integration in Eq. (14) can be deformed so that this is true to an arbitrary high degree of accuracy.

[13]R. H. Dalitz, Proc. Roy. Soc. (London) A206, 509 (1951), Eq. (A3); R. R. Lewis, Phys. Rev. 102, 537 (1956), Eqs. (18a) and Appendix 12. The integral over $\vec{p}_1$ is essentially the derivative with respect to μ of the integral $M_2(\mu,\nu)$ of Lewis.

[14]See Ref. 13, Appendix, Eqs. (1), (9), and (10). In order to apply this formula one must take $\nu>0$ and $X,\ \mu>0$ (which we already assumed).

[15]The proof of this statement is based on classical theorems regarding the parameter dependence of definite integrals and is similar to the one given in Ref. 1 after Eq. (37). Since in the present case the proof is more elaborate, we will not present it here.

[16]The condition $\text{Re}X>0$ guarantees that the X appearing in Eq. (A6) is defined in the same way as the one appearing in the Green's function [see Eq. (18)].

[17]HTF, p. 231, Eq. (5).

[18]Either sign of $(x_1x_2)^{1/2}$ may be taken.

[19]HTF, p. 59, Eq. (10).

[20]HTF, p. 105, Eq. (2).

[21]O. Perron, Sitzber. Heidelberg. Akad. Wiss. Math. Naturev. Kl. Abhandl. VIIIA, 1 (1917); and Ref. 9, Eq. (62).