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⁵This agreement may have been fortuitous. The results of Ref. 1 are changed since the wave function was not accurate enough, whereas the results of Ref. 4 are

changed because of second-order perturbation theory. We have not been able to link these two modifications. However, with the use of corrected wave functions and with the inclusion of second-order perturbation theory, the new results do agree.

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Superradiant Intensity Fluctuations*†

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 (Received 30 March 1970)

We describe a new approach to the problem of superradiant spontaneous emission, which is applicable whether the number of atoms is large or small. We use this approach to formulate intensity correlations of arbitrary order, and then explicitly evaluate the second-order normally ordered relative intensity fluctuation. Large fluctuations at the onset of emission indicate the discrete nature of the spontaneous emission process; and the very small fluctuations found later at the emission peak show that the emitting process has by then become essentially classical.

In his celebrated paper,¹ Dicke determined the angular correlations of successive photons emitted spontaneously by a collection of N excited two-level atoms. In this paper, we will present a related result, the expected intensity fluctuations of the radiation emitted by the same type of atomic system.

There are two main motivations for our work. In the first place, superradiant intensity fluctuations seem to offer a cruder measure of the statistical properties of the radiation, and so should be more readily measurable than single-photon angular correlations. In the second place, we can speak of an aspect of the superradiant emission problem that was not treated by Dicke, and is not easily treated by his method. Specifically, we are able to discuss the change with time of the intensity fluctuations as the system superradiantly decays from its excited to its ground state.

Because of our ability to follow the intensity fluctuations in time, we will be able to comment on the extent to which the radiation is "classical" or "quantum" in character at the various stages of the emission process. Thus, our approach offers a very simple method of answering questions which recently have interested several researchers.²⁻⁴

First, let us recall Dicke's method as it applies to a calculation of the radiated intensity $I(t)$ itself. The central element is the imposition of an energy

balance between the energy leaving the atomic system and the energy appearing in the radiation field. That is

$$I(t) = \hbar\omega \times (\text{single-photon emission rate}) \\ = \hbar\omega \times (\text{atomic transition rate}), \quad (1)$$

and the atomic transition rate is then calculated in first-order perturbation theory.

The interaction Hamiltonian which governs the atomic transition rate is taken to be of the usual electric dipole form

$$H_I = -\vec{E}(\vec{R}, t) \cdot \vec{D}, \quad \vec{D} = \sum_{i=1}^N \vec{D}_i, \quad (2)$$

where for simplicity all of the two-level atoms with dipole moment operators \vec{D}_i have been assumed located within a wavelength of each other at the position \vec{R} .

If we define, with Dicke, the atomic raising and lowering operators $R_{\pm} = \text{constant} \times (\vec{D}_x \pm i\vec{D}_y)$ for $\Delta m = 1$ transitions, and evaluate the transition rate for spontaneous emission, we find Dicke's result

$$I = \text{const} \times \langle R_+ R_- \rangle, \quad (3)$$

where the brackets signify expectation value taken in the appropriate emitting atomic state.

Now let us consider an alternative approach,⁵ which will be seen to offer a direct method of calculating intensity fluctuations of all orders as well as the intensity itself. We will change our point

of view from that of the emission process to that of the detector. The far-field intensity of radiation emitted from a localized current source is simply related to the Poynting flux \vec{S} in the usual way,

$$\langle \hat{I}(\vec{k}, t) \rangle d\Omega(\hat{k}) = \langle : \vec{S} \cdot \hat{k} : \rangle d(\text{area}). \quad (4)$$

The normal ordering eliminates vacuum fluctuations.

In order to evaluate relation (4), we employ the well-known retarded solution to Maxwell's wave equation

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int d\vec{r}' | \vec{r} - \vec{r}' |^{-1} \vec{J}_{\text{ret}}(\vec{r}', t) \quad (5)$$

plus the relation

$$\vec{J} = \sum \dot{\vec{D}}_i \delta^3(\vec{r} - \vec{r}_i).$$

One finds, of course, Dicke's result when (4) is evaluated in the far-field radiation zone. That is to say, Dicke's result given in (3), which was obtained by perturbation theory, gives the far-field radiated intensity without retardation corrections. We will continue to ignore the small effects due to retardation.

However, we can now see that we can compute higher-order correlations of the type $\langle : I(\vec{k}_1, t) \cdots I(\vec{k}_p, t) : \rangle$ in terms of the atomic variables and states by making systematic use of relations (4) and (5). In this note we want to give the result of calculating the simplest of such objects, the single-time second-order total intensity fluctuation. In order to eliminate irrelevant atomic constants, as well as to exhibit the relative scale of these fluctuations, we actually have evaluated the normally ordered dispersion, normalized to the square of the intensity itself. If we denote this object by $\Delta I^{(2)}$, we find

$$\Delta I^{(2)} = \frac{\langle : I(t) I(t) : \rangle - \langle : I(t) : \rangle^2}{\langle : I(t) : \rangle^2}, \quad (6)$$

which reduces to

$$\Delta I^{(2)} = \frac{\langle R_+ R_+ R_- R_- \rangle - \langle R_+ R_- \rangle^2}{\langle R_+ R_- \rangle^2}. \quad (7)$$

Here we have defined the total intensity

$$I(t) = \int I(\vec{k}, t) d\Omega(\hat{k}).$$

In order to compute the expectation values in (7), we work with the "radiation reaction" states which we have described elsewhere.^{5,6} These states are simply products of single-atom states. The single-atom states are taken to be coherent superpositions of upper and lower states of the two-level atom

$$| \theta \rangle = \prod_{i=1}^N | \theta \rangle_i, \quad (8a)$$

$$| \theta \rangle = \sin \frac{1}{2} \theta | + \rangle_i + \cos \frac{1}{2} \theta | - \rangle_i. \quad (8b)$$

Clearly, when $\theta = \pi$, all N atoms are in their excited state; when $\theta = 0$, all atoms are in their ground state; and when $\theta = \frac{1}{2}\pi$, each atom exhibits its maximum dipole moment.

The result of evaluating the expectations in (7) is a rational fraction in N , the number of two-level atoms, which we may write

$$\Delta I^{(2)} = \frac{\alpha N^4 + \beta N^3 + \gamma N^2 + \delta N}{aN^4 + bN^3 + cN^2}, \quad (9)$$

where the coefficients are the following polynomials in $\cos \theta$:

$$\begin{aligned} \alpha &= 0, \\ \beta &= -\frac{1}{4} \cos \theta [1 - \cos \theta - \cos^2 \theta + \cos^3 \theta], \\ \gamma &= -\frac{3}{8} + \frac{3}{4} \cos \theta + \frac{1}{4} \cos^2 \theta - \frac{5}{4} \cos^3 \theta + \frac{5}{8} \cos^4 \theta, \\ \delta &= \frac{1}{8} - \frac{3}{4} \cos^2 \theta + \cos^3 \theta - \frac{3}{8} \cos^4 \theta \end{aligned} \quad (10)$$

and

$$\begin{aligned} a &= \frac{1}{16} [1 - 2 \cos^2 \theta + \cos^4 \theta], \\ b &= \frac{1}{8} - \frac{1}{4} \cos \theta + \frac{1}{4} \cos^3 \theta - \frac{1}{8} \cos^4 \theta, \\ c &= \frac{1}{16} - \frac{1}{4} \cos \theta + \frac{3}{8} \cos^2 \theta - \frac{1}{4} \cos^3 \theta + \frac{1}{16} \cos^4 \theta. \end{aligned} \quad (11)$$

In Fig. 1, we have displayed the dependence of $\Delta I^{(2)}$ on θ for the case $N = 10^6$. Figure 1 shows that during the initial stages of the superradiant emission when all N atoms are nearly in their upper states, and θ is nearly π , the fluctuations are large, and $\Delta I^{(2)} \sim 1$ as a result. Very quickly, however, the relative fluctuation decreases below 10^{-2} . By the time half of the energy has been emitted and $\theta = \frac{1}{2}\pi$, $\Delta I^{(2)}$ has become a very small negative number. The exact limiting values at $\theta = \pi$ and $\theta = \frac{1}{2}\pi$ are

$$\Delta I^{(2)}(\pi) = 1 - \frac{2}{N}, \quad \Delta I^{(2)}(\frac{1}{2}\pi) = -\frac{(6 - 2N^{-1})}{(N+1)^2}. \quad (12)$$

For the case of pure superradiant emission, the time dependence of the radiation reaction states $| \theta \rangle$ is known, and the parameter θ may be given explicitly as a function of time.⁶ The use of this functional relation allows us to plot $\Delta I^{(2)}$ as a function of time during the emission process. This plot is shown in Fig. 2. Qualitatively, it is the same as Fig. 1. The only important difference is that the interval of small dispersion is seen to occupy a relatively small portion of the time axis.

There are several comments that should be made about the relative fluctuation $\Delta I^{(2)}$. The first is that the negative values (which are plotted with a dashed line as if they were positive) are

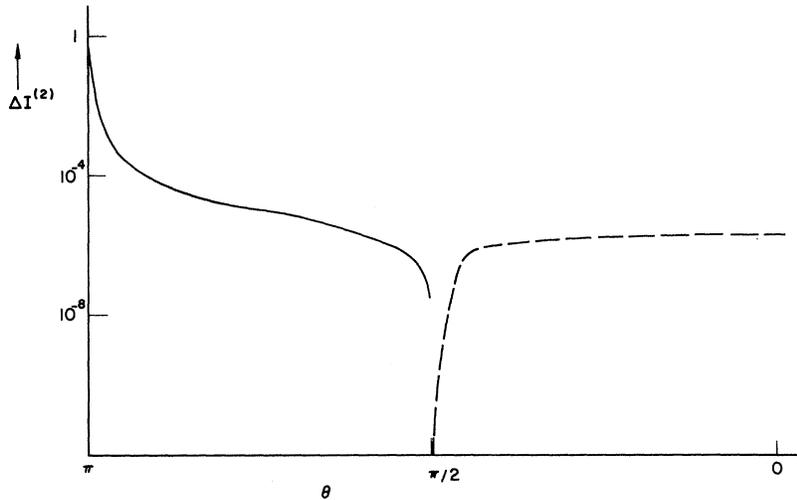


FIG. 1. Relative intensity fluctuation versus superradiant decay angle θ . The negative portion of the curve is plotted with a dashed line as if it were positive.

never large, being at most of order $1/N$. The very existence of negative values is due to the normally ordered form⁷ employed in (6), and reflects the fact that the number of photons is limited to N , the number of atoms which could have been excited initially.

Note that if the state of the field were a coherent state, $\Delta I^{(2)}$ would vanish identically. For this reason, we can say that superradiant emission, near its peak around $\theta = \frac{1}{2}\pi$, is nearly classical. That is, it has the characteristic small $\Delta I^{(2)}$ of radiation emitted by a classical deterministic current. Bialynicka-Birula has indeed shown in a certain approximation⁴ that the density matrix of the field in the diagonal coherent state representation is just that of a pure coherent state.

Within the context of the above approach it is reasonable to define a photon number operator n

by the relation

$$n = \frac{1}{2}N - R_3, \quad (13)$$

where R_3 is the atomic energy operator in units of $\hbar\omega$. Since we choose the zero of energy such that $\langle R_3 \rangle = \frac{1}{2}N$ when all the atoms are excited, (13) is consistent with the initial condition - all atoms excited, no photons. At the time of peak emission, we find, using (13), the variance and mean photon number are related by

$$\sigma^2 = \langle n \rangle (1 - \langle n \rangle / N). \quad (14)$$

This is the result for a binomial distribution, and is to be expected for independent emitters. Bonifacio and Preparata³ have found an approximate version of this result.

Unfortunately, it does not seem useful to make further comparisons. The other treatments of

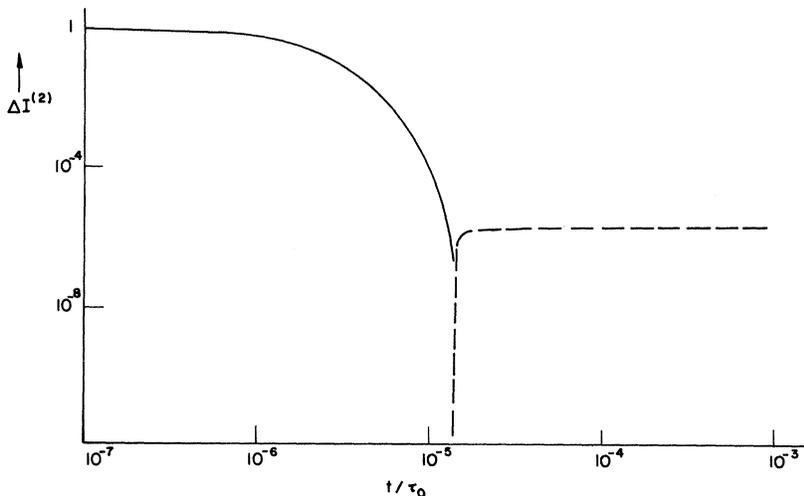


FIG. 2. Relative intensity fluctuation versus superradiant decay time, in units of the single-atom spontaneous lifetime. Dashed portion as in Fig. 1.

fluctuations in the superradiating process known to us are restricted to the cases of few atoms or a single mode of the radiation field. The interesting work of Bonifacio and Preparata³ is of the latter type, and one sees in it the periodic exchange of energy between atoms and field which should be expected in a cavity. The principal

drawback of the Bialynicka-Birula⁴ treatment is the restriction to times sufficiently close to the instant of peak emission, while Dillard and Robl² have concentrated on small numbers of atoms.

We would like to thank our colleagues, especially G. S. Agarwal and L. Mandel, for several discussions.

*Research partially supported by the National Science Foundation.

†A preliminary report of this research was made at the Rochester Symposium on the Electromagnetic Interactions of Two-Level Atoms, 1970 (unpublished).

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Critical Isotherm of He³

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The critical isotherm of He³ has been measured over a density range $|\Delta\rho| = |\rho - \rho_c|/\rho_c$ up to 0.6, and the chemical potential change $\mu - \mu_c$ was obtained as a function of the density change $\rho - \rho_c$. The departure from perfect antisymmetry about ρ_c , which can be represented by an expression of the form $|\Delta\rho|^\epsilon$ with $\epsilon = 6.2$, is compared with theoretical suggestions.

Recently Missoni, Sengers, and Green¹ have reported success in scaling the thermodynamic properties near the critical point for a number of fluids and magnets. They found that the change in chemical potential

$$\Delta\mu = [\mu(\rho, T) - \mu(\rho_c, T)]/P_c V_c$$

was antisymmetric in the change in density $\Delta\rho = (\rho - \rho_c)/\rho_c$ for $|\Delta\rho| < 0.30$ for the fluids He⁴, Xe, and CO₂ when $-0.01 < t = (T - T_c)/T_c < 0.03$. Along the critical isotherm, $\Delta\mu \propto \Delta\rho |\Delta\rho|^{\delta-1}$, where $\delta \approx 4.0 - 4.6$ for all fluids. We have reported² agreement for He³ with their scaled equation of state for $|\Delta\rho| < 0.25$, and in the present note we wish to describe the departure from antisymmetry for $0.25 < |\Delta\rho| < 0.6$.

Cooper, Sengers, and Green³ (CSG) have proposed a modification of the scaling ideas to extend the interpretation over an enlarged region about the critical point and also to include systems lacking an intrinsic or known symmetry. Cooper and Green⁴ found for an ideal Bose gas that the simple scaling form appears as the first term in an expansion about the critical point for a nonclassical

equation of state. The corresponding Helmholtz energy F was expressed as a function of t and reduced variable $x = t |\Delta\rho|^{-1/\beta}$. Generalizing this result to physical systems of no known symmetry, CSG assert that the free energy in the critical region is given by an expression of the form

$$F(\rho, t) = F_0(\rho, t) + |\Delta\rho|^\delta \sum_{s=1}^{\infty} \Delta\rho^{s-1} f_s(x), \quad (1)$$

where the $f_s(x)$ are regular functions. Since one has

$$\mu(\rho, t) = \left(1 + \rho \frac{\partial}{\partial \rho}\right) F(\rho, t), \quad (2)$$

we immediately see that along the critical isotherm where the asymmetric $\Delta\mu$ term varies as $\Delta\rho \times |\Delta\rho|^{\delta-1}$, the first correction which gives a symmetric term is of the order $\Delta\rho |\Delta\rho|^{\delta-1}$ with $\epsilon = \delta + 1$.

Griffiths⁵ has discussed two asymmetric lattice-gas models. For his first model, the leading symmetric term for the chemical potential change $\Delta\mu$ as a function of $\Delta\rho$ along the critical isotherm has an exponent $\epsilon = 2\delta$. A more recent model, which is a "decorated" system, gives $\epsilon = 2\delta - \beta^{-1}$.