944 (1964).

[18] R. B. Griffiths, J. Chem. Phys. 43, 1958 (1965).

[19] R. B. Griffiths, Phys. Rev. Letters 14, 623 (1965).

[20] G. S. Rusbrooke, J. Chem. Phys. 39, 842 (1963).

[21] M. J. Buckingham (unpublished).

[22] C. Boghosian, H. Meyer, and J. E. Rives, Phys. Rev. 146, 110 (1966).

[23] H. Van Dijk, M. Durieux, J. R. Clement, and J. K. Logan, Physica 24, S129 (1954).

[24] P. Roach, Phys. Rev. 170, 213 (1968).

[25] D. Elwell and H. Meyer, Phys. Rev. 164, 245 (1967); D. Elwell, thesis, Duke University, 1967 (unpublished).

[26] O. V. Lounasmaa, Cryogenics 1, 1 (1961).

[27] R. H. Sherman and F. J. Edeskuty, Ann. Phys. (N.Y.) 9, 522 (1960).

[28] E. C. Kerr and R. D. Taylor, Ann. Phys. (N.Y.) 20, 450 (1962).

[29] Interestingly, the ratio $\bar{\rho}(T=0)/\rho_c$ is 0.98 for He^3, while for He^4 it is $0.145/(2\times 0.069)=1.05$, where we use the value of the density ρ_L $(T=0)$ obtained by K. R. Atkins and M. H. Edwards [Phys. Rev. 97, 1429 (1955)] and ρ_c obtained by Roach, Ref. 24.

[30] B. Widom, J. Chem. Phys. 43, 3898 (1965), p. 3901.

[31] M. Coopersmith, Phys. Rev. Letters 20, 432 (1968).

[32] H. Kierstead (private communication).

PHYSICAL REVIEW A VOLUME 2, NUMBER 4 OCTOBER 1970

Kinetic Theory of a Weakly Coupled Fluid*

Dieter Forster† and P. C. Martin

Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 6 April 1970)

Virtually all measurable properties of a classical fluid may be determined from the expectation value of the phase-space density operator $f(\vec{r}\vec{p}t)=\Sigma_\alpha\,\delta(\vec{r}-\vec{r}^{\,\alpha}(t))\,\delta(\vec{p}-\vec{p}^{\,\alpha}(t))$, and the phase-space density correlation function $\langle f(\vec{r}\vec{p}t)f(\vec{r}\,'\vec{p}\,'t')\rangle-\langle f(\vec{r}\vec{p}t)\rangle\langle f(\vec{r}\,'\vec{p}\,'t')\rangle$, a matrix with indices $(\vec{r}\vec{p}t)$. Systematic procedures for approximating this matrix, unhindered by secular effects, are always based on approximations to its inverse. For a weakly coupled fluid, the inverse can be expanded in powers of λ, the ratio of potential to kinetic energy. The leading term in this expansion gives rise to a Vlasov equation for the phase-space correlation function. The next term is the first that includes collisions, and results in relaxation towards equilibrium. This paper is concerned with the detailed study of the resulting fundamental nontrivial approximation. It is not Markovian and is perfectly reversible. Although the approximation is complicated, it is tractable analytically in various limits, and numerically for all wavelengths and frequencies. In this paper, only the behavior in certain limits is evaluated. Particular attention is directed toward its contractions – the density correlation function, which is measured by inelastic neutron and light scattering, and the momentum correlation function. Calculation of the former at long wavelengths corroborates the Landau-Placzek expression for light scattering, and therefore demonstrates that the kinetic equation predicts hydrodynamic behavior at long times. Since the correlation function is correct to order λ^2, it has, in contrast to a solution to the Boltzmann equation, the correct long-wavelength velocity of sound, $c^2=(dp/dmn)_s\neq\frac{5}{3}\,k_BT/m$. It also predicts different transport coefficients than those deduced from a Boltzman equation. These include a nonvanishing bulk viscosity. The transport coefficients reduce to those derived from the Boltzmann equation at low densities. Some aspects of the short-time behavior are also discussed.

I. INTRODUCTION

The measurable properties of a system of classical spinless particles are properties which can be constructed from observables formed from the field

$$f(\vec{r}\vec{p}t)=\sum_\alpha \delta(\vec{r}-\vec{r}^{\,\alpha}(t))\,\delta(\vec{p}-\vec{p}^{\,\alpha}(t)),$$

in which α extends over all the indistinguishable particles in the system. Thus, a study of the dynamics of such a system is an investigation of $\langle f(\vec{r}\vec{p}t)\rangle$, $\langle f(\vec{r}\vec{p}t)f(\vec{r}\,'\vec{p}\,'t')\rangle$, etc. Fundamentally, the equation which governs these products is the Liouville equation, and the determination of these correlations for an arbitrary state requires a specification of the initial conditions of all products $\langle f(\vec{r}\vec{p}t)\cdots\rangle$, since they are all necessary to characterize an arbitrary state. It has been recognized for some time, however, that an important class of problems can be understood from a knowledge of $\langle f(\vec{r}\vec{p}t)f(\vec{r}\,'\vec{p}\,'t')\rangle_{\text{eq}}$ in an equilibrium ensemble. In particular, because of the connection between fluctuations and linear response, the equilibrium fluctuation function

$$S(\vec{r}-\vec{r}\,',t-t';\vec{p}\vec{p}\,')=\langle\big(f(\vec{r}\vec{p}t)-\langle f(\vec{r}\vec{p}t)\rangle_{\text{eq}}\big)\times\big(f(\vec{r}\,'\vec{p}\,'t')-\langle f(\vec{r}\,'\vec{p}\,'t')\rangle_{\text{eq}}\big)\rangle_{\text{eq}}$$

$$=\int \frac{d\vec{k}}{(2\pi)^3}\int \frac{d\omega}{2\pi}\, S'(\vec{k}\omega;\vec{p}\,\vec{p}')\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')-i\omega(t-t')} \tag{1.1}$$

gives the time dependence of $\langle f(\vec{r}\,\vec{p}\,t)\rangle_{\text{non eq}}$ for any state that is only slightly displaced from equilibrium. An important special case, namely,

$$\langle n(\vec{r}\,t)\,n(\vec{r}\,'t')\rangle_{\text{eq}} - n^2 = \int d\vec{p}\int d\vec{p}\,'\, S(\vec{r}-\vec{r}\,', t-t'; \vec{p}\,\vec{p}\,') \tag{1.2}$$

contains all the information measured in neutron and light scattering experiments. For long times and large distances, the density correlation function will give results in agreement with linearized hydrodynamics if linearized hydrodynamics is correct. It will also give the values of the thermodynamic properties and transport coefficients.

Because the fluctuation function $S(\vec{r}-\vec{r}\,', t-t'; \vec{p}\,\vec{p}\,')$ contains such a wealth of information, calculating it for arbitrary distances and times is bound to present problems. To our knowledge, this problem is so difficult that it has only been systematically attacked in the quantum-mechanical case,[1] where other properties are normally in question. We wish to describe here a study of the simplest nontrivial equation for this function – an equation which determines it in a weakly interacting classical system. By a weakly interacting system we mean a fluid in which the interparticle potential v is short ranged, nonsingular, and much smaller in magnitude than $k_B T \equiv \beta^{-1}$.

Before discussing the equation specifically, let us note some of its essential properties. At short times, the equation must be completely reversible. A Taylor series for the time development must have coefficients which depend on instantaneous equilibrium properties. Conversely, at long times, we expect it to make predictions similar to those of a Boltzmann equation. However, it should remedy the defects in that equation. In order to predict transport coefficients, it is necessary to retain terms of order λ^2 (where $\lambda = \beta v$) in a collision operator. But, if hydrodynamics is correct, the correct collision operator to order λ^2 should yield a sound velocity that agrees with the thermodynamic expression $(dp/dmn)_s = c^2$. That velocity c^2 differs from $\frac{5}{3}\, k_B T/m$ by terms of order λ and λ^2. In addition, to order λ^2, the energy density and pressure of the fluid have potential, not just kinetic contributions. These corrections are lost if the system is studied in the unnatural limit which leads to a Boltzmann equation and which has previously been discussed in the literature – the uniform ($k \to 0$) limit in which we have $\lambda \to 0$, $t \to \infty$ such that $\lambda^2 t$ remains constant. Our approximation is an approximation for the inverse correlation function. It involves only the parameter λ, and it is therefore valid for arbitrary frequency ω and wave number k. Our results reduce to the previously calculated uniform long-time weak-coupling[2] results only when the density is low. They deviate in a way which makes the Bogoliubov hypothesis about a succession of relaxation times seem quite unlikely.[3]

While our results are systematic and instructive, they have not yet been applied to a real fluid. We have not included strong repulsions. If a low-density hard-core gas were studied and the weak attraction handled by the techniques we have outlined, we would have a systematic procedure for finding the correlation functions for real fluids. At the present time, that problem has not been treated, but we believe we know how to handle it. The modifications for plasmas are straightforward.

After writing down the equation of motion for $S(\vec{r}-\vec{r}\,', t-t'; \vec{p}\,\vec{p}\,')$ which forms the basis of our discussion, and explaining our weak-coupling approximation, we shall concentrate on showing that the equation has those features which we know it must have. Thus, in Sec. II we shall discuss certain symmetries, including the symmetry under time reversal, and a positivity property which guarantees the dynamical stability of the system. We shall also find that the sum rules are consistently fulfilled so that the short-time behavior is correctly rendered. Little is known about the required behavior at intermediate times, except for the conservation laws which we shall discuss in Sec. III. At long times and for slow spatial variations, linearized hydrodynamics is expected to take over, and we will show that this "contraction of the description" is correctly given, both for transverse (Sec. V) and longitudinal (Sec. VI) modes. We believe ours to be the first consistent microscopic evaluation of the linearized hydrodynamic character of a physical distribution function in a well-defined (but admittedly simple) physical system.

The fundamental quantity in our discussion is the complex fluctuation function

$$S(\vec{k}z;\vec{\xi}\,\vec{\xi}\,') = \int \frac{d\omega}{2\pi}\,\frac{S'(\vec{k}\omega;\vec{\xi}\,\vec{\xi}\,')}{\omega - z}, \tag{1.3}$$

which is analytic for $\operatorname{Im} z \neq 0$. For z in the upper half-plane, it is the Laplace transform of $S(\vec{k}t;\vec{\xi}\,\vec{\xi}\,')$:

$$S(\vec{k}z;\vec{\xi}\,\vec{\xi}\,') = i\int_0^\infty dt\, e^{izt}\, S(\vec{k}\,t;\vec{\xi}\,\vec{\xi}\,'). \tag{1.4}$$

We shall use dimensionless momentum variables

$$\vec{\xi} = \vec{p}/mv_0 \quad\text{with}\quad v_0^2 = (m\beta)^{-1},$$

where m is the particle mass and v_0 is the thermal velocity. $S(\vec{k}z;\vec{\xi}\,\vec{\xi}\,')$ has been scaled so that the complex density-density fluctuation function is given by

$$g_{nn}(\vec{k}z) = \int d\vec{\xi}\int d\vec{\xi}\,'\, S(\vec{k}z;\vec{\xi}\,\vec{\xi}\,'), \tag{1.5}$$

and the momentum density fluctuation function is given by

$$\vec{g}_{ij}(\vec{k}z) = (mv_0)^2 \int d\vec{\xi} \int d\vec{\xi}'\ \xi_i \xi_j'\ S(\vec{k}z;\vec{\xi}\vec{\xi}'). \tag{1.6}$$

Of course, a direct expansion of $S(\vec{k}z)$ in powers of the weak-pair potential $v(\vec{k})$ is inadmissible and leads to unphysical divergences. To avoid these divergences and take proper account of the secular effects of collisions, we have introduced a quantity denoted by $\Sigma(\vec{k}z)$. In transport-theory language, $\Sigma(\vec{k}z)$ is the memory function for the one-particle distribution function; in field theory, it would be called the mass operator associated with $S(\vec{k}z)$.

The basic equation of motion is

$$(z - v_0\vec{k}\cdot\vec{\xi})\, S(\vec{k}z;\vec{\xi}\vec{\xi}') - \Sigma(\vec{k}z;\vec{\xi}\,\bar{\vec{\xi}})\, S(\vec{k}z;\bar{\vec{\xi}}\,\vec{\xi}') = -S^0(\vec{k};\vec{\xi}\,\vec{\xi}'). \tag{1.7}$$

In this equation, and throughout the paper, an integration $\int d\bar{\vec{\xi}}$ over barred variables is implied.

While there is at present no practical method to calculate $\Sigma(\vec{k}z)$ in general, it is not difficult to obtain it as a power series in the potential and S, or in a more primitive form, as a power series in the potential and the free-particle correlation function. In this paper, we shall be concerned with the approximate Σ that results from carrying out the latter expansion to second order in the potential. This approximation gives

$$\Sigma(\vec{k}z;\vec{\xi}\,\vec{\xi}') = \Sigma^{(s)}(\vec{k}\vec{\xi}) + \Sigma^{(c)}(\vec{k}z;\vec{\xi}\,\vec{\xi}'), \tag{1.8a}$$

$$\Sigma^{(s)}(\vec{k}\vec{\xi}) = -nc(\vec{k})v_0\vec{k}\circ\vec{\xi}\ \phi(\vec{\xi}), \tag{1.8b}$$

$$\begin{aligned}\Sigma^{(c)}(\vec{k}z;\vec{\xi}\,\vec{\xi}')\,\phi(\vec{\xi}') = -\frac{n}{(mv_0)^2} \\ \times\Bigg(v^2(\bar{\vec{k}})\,\bar{\vec{k}}\cdot\vec{\partial}\ \bar{\vec{k}}\cdot\vec{\partial}'\ \frac{\phi(\bar{\vec{\xi}})\,\phi(\vec{\xi})\,\delta(\bar{\vec{\xi}}-\vec{\xi}')}{v_0(\vec{k}-\bar{\vec{k}})\cdot\bar{\vec{\xi}} + v_0\bar{\vec{k}}\cdot\bar{\vec{\xi}} - z} \\ + v(\vec{k}-\bar{\vec{k}})\,v(\bar{\vec{k}})\,\bar{\vec{k}}\cdot\vec{\partial}\,(\vec{k}-\bar{\vec{k}})\cdot\vec{\partial}' \\ \times\frac{\phi(\vec{\xi})\,\phi(\vec{\xi}')}{v_0(\vec{k}-\bar{\vec{k}})\cdot\vec{\xi} + v_0\bar{\vec{k}}\cdot\vec{\xi}' - z}\Bigg),\end{aligned} \tag{1.8c}$$

where $\int d\bar{\vec{k}}/(2\pi)^3$ is implied, and $\vec{\partial} \equiv \vec{\nabla}_{\vec{\xi}}$; n is the particle density,

$$\phi(\vec{\xi}) = (2\pi)^{-3/2} e^{-\xi^2/2} \tag{1.9}$$

is the Maxwellian velocity distribution, and

$$v(\vec{k}) = \int d\vec{r}\, e^{-i\vec{k}\cdot\vec{r}}\, v(\vec{r}). \tag{1.10}$$

$\Sigma^{(s)}(\vec{k}\vec{\xi})$ accounts for the mean field on one particle due to all the others. It has the form of the linearized Vlasov operator, but with $-\beta v(\vec{r})$ replaced by the direct correlation function $c(\vec{r})$. To order v^2, $c(\vec{r})$ is given by

$$c(\vec{r}) = -\beta v(\vec{r}) + \tfrac{1}{2}[\beta v(\vec{r})]^2. \tag{1.11}$$

The correct initial condition, the equal-time correlation function, occurs on the right-hand side of (1.7) as

$$S^0(\vec{k};\vec{\xi}\,\vec{\xi}') = n\,[\phi(\vec{\xi})\,\delta(\vec{\xi}-\vec{\xi}') + \phi(\vec{\xi})\,nh(\vec{k})\,\phi(\vec{\xi}')]. \tag{1.12}$$

The function $h(\vec{k})$ is the spatial transform of $h(\vec{r})$ and $h(\vec{r}) + 1 = g(\vec{r})$ is the pair correlation function. The transforms of $h(\vec{r})$ and $c(\vec{r})$ are related by

$$h(\vec{k}) = c(\vec{k})/[1 - nc(\vec{k})]. \tag{1.13}$$

If we were to omit $\Sigma^{(c)}(\vec{k}z)$ and use the proper $c(\vec{k})$, we would have a collisionless equation for $S(\vec{k}z)$ that is correct at very short times. This approximation has been suggested by many authors, and used by Nelkin and Ranganathan[4] to interpret collisionless "sound" data in liquid lead.

The influence of collisions and their "memory" effects is contained in its most primitive form in $\Sigma^{(c)}(\vec{k}z)$. Essentially the same second-order result (1.8) has been independently obtained by Akcasu and Duderstadt[5] on the basis of Mori's Langevin-equation formalism.[6] Our own derivation is based on a different and more systematic scheme, which will be described elsewhere. However, the reader can simply obtain $\Sigma^{(c)}(\vec{k}z)$ by calculating $S(\vec{k}z)$ itself to second order which is trivial, and then solving Eq. (1.7) for $\Sigma(\vec{k}z)$.

Equations (1.7) and (1.8) constitute our weak-coupling approximation. The only parameter assumed small is βv. Thus, the approximation should apply for all values of k and ω (or space and time differences), provided that the potential is weak enough that terms of order $(\beta v)^3$ in $\Sigma(\vec{k}z)$ can be neglected.

In the Markovian limit of very small k and z, $\Sigma^{(c)}(\vec{k}z)$ reduces to the linearized Fokker-Planck operator $-iK$ derived in the standard weak-coupling theory[7] of many-body systems. In that limit, we have

$$\lim_{\substack{\vec{k}\to 0\\ \omega\to 0}} \Sigma^{(c)}(\vec{k},\ \omega + i\epsilon;\vec{\xi}\,\vec{\xi}') = -iK(\vec{\xi}\,\vec{\xi}'), \tag{1.14}$$

$$\begin{aligned}K(\vec{\xi}\,\vec{\xi}')\,\phi(\vec{\xi}') = \nu\,\partial_i\partial_j'[\phi(\vec{\xi})\,\delta(\vec{\xi}-\vec{\xi}') \\ \times\omega_{ij}(\vec{\xi}-\bar{\vec{\xi}})\,\phi(\bar{\vec{\xi}}) - \phi(\vec{\xi})\,\omega_{ij}(\vec{\xi}-\vec{\xi}')\,\phi(\vec{\xi}')],\end{aligned} \tag{1.15}$$

where

$$\omega_{ij}(\vec{\xi}) = \partial_i\partial_j\,|\vec{\xi}|, \tag{1.16}$$

and

$$\nu = (nv_0/8\pi)\int_0^\infty dk\ k^3\,[\beta v(\vec{k})]^2 \tag{1.17}$$

is an effective relaxation frequency. $K(\vec{\xi}\,\vec{\xi}')$ can also be obtained from the linearized Boltzmann operator if we use the collision cross section to order v^2. Note that the limit in (1.14) does not depend on the order in which it is performed.

II. SOME GENERAL PROPERTIES OF S AND Σ

We first summarize some general properties of $S(\vec{k}z;\vec{\xi}\,\vec{\xi}')$: symmetries, sum rules, and a positivity condition connected with stability. Since $S(\vec{k}z;\vec{\xi}\,\vec{\xi}')$ is uniquely determined by Eq. (1.7) once the mass operator is known, $\Sigma(\vec{k}z;\vec{\xi}\,\vec{\xi}')$ has to exhibit corresponding symmetries. We shall see that our approximate $\Sigma(\vec{k}z)$ satisfies these requirements.

A. Symmetries

For a system invariant under translations, rotations, parity, and time reversal, it is easy to show that $S'(\vec{k}\omega;\vec{\xi}\,\vec{\xi}')$ is

(a) real;
(b) even under $\vec{k}, \omega \rightarrow -\vec{k}, -\omega$;
(c) symmetric with respect to $\vec{\xi} \leftrightarrow \vec{\xi}'$; (2.1)
(d) invariant under rotations-reflections.

Rotational invariance implies that $S'(\vec{k}\omega;\vec{\xi}\,\vec{\xi}')$ depends only on the six scalar combinations of the three vectors $\vec{\xi}$, $\vec{\xi}'$, and $\vec{k}$. For $S(\vec{k}z;\vec{\xi}\,\vec{\xi}')$, the first two properties imply that

$$S(\vec{k}z;\vec{\xi}\,\vec{\xi}') = -S(-\vec{k}-z;\vec{\xi}\,\vec{\xi}') = [S(\vec{k}z^*)]^* . \qquad (2.2)$$

It is not difficult to show that the equation of motion (1.7) guarantees these symmetries whenever

$$\Sigma^{(c)}(\vec{k}z;) = -\Sigma^{(c)}(-\vec{k}, -z;) = [\Sigma^{(c)}(\vec{k}z^*;)]^* , \qquad (2.3)$$

$$\Sigma^{(c)}(\vec{k}z;\vec{\xi}\,\vec{\xi}')\,\phi(\vec{\xi}') = \Sigma^{(c)}(\vec{k}z;\vec{\xi}'\,\vec{\xi})\,\phi(\vec{\xi}) , \qquad (2.4)$$

and $\Sigma^{(c)}$ is invariant under spatial rotations-reflections. All of these conditions are met by the second-order mass operator of (1.8c). In particular, we remark that Eq. (1.7) is completely reversible with the kernel (1.8).

B. Sum Rules

Frequency moment analysis provides a useful tool in the study of correlation functions. It has been employed by many authors to determine parameters in model theories. Since the frequency moments of $S'(\vec{k}\omega)$ are the coefficients in a Taylor series for the time development, they determine the dynamics for short-time differences. The first few moments are

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,S'(\vec{k}\omega;\vec{\xi}\,\vec{\xi}') = n\,\phi(\vec{\xi})\,\delta(\vec{\xi}-\vec{\xi}') + \phi(\vec{\xi})\,n^2h(\vec{k})\,\phi(\vec{\xi}') , \qquad (2.5a)$$

$$\int\frac{d\omega}{2\pi}\,\omega S'(\vec{k}\omega;\vec{\xi}\,\vec{\xi}') = nv_0\vec{k}\cdot\vec{\xi}\,\phi(\vec{\xi})\,\delta(\vec{\xi}-\vec{\xi}') , \qquad (2.5b)$$

$$\begin{aligned}\int\frac{d\omega}{2\pi}\,\omega^2 S'(\vec{k}\omega;\vec{\xi}\,\vec{\xi}') &= n[v_0\vec{k}\cdot\vec{\xi}]^2\phi(\vec{\xi})\,\delta(\vec{\xi}-\vec{\xi}') \\ &+ (n^2/m)\int d\vec{r}\,g(\vec{r})\,\vec{\nabla}_i\vec{\nabla}_j v(\vec{r}) \\ &\times\partial_i\partial_j\phi(\vec{\xi})\,\delta(\vec{\xi}-\vec{\xi}') - (n^2/m)\int d\vec{r}\,g(\vec{r}) \\ &\times\cos(\vec{k}\cdot\vec{r})\,\vec{\nabla}_i\vec{\nabla}_j v(\vec{r})\,\partial_i\partial_j'\phi(\vec{\xi})\,\phi(\vec{\xi}').\end{aligned} \qquad (2.5c)$$

The well-known sum rule for density fluctuations is a special case. We recover it by integrating (2.5c) over momenta. The sum rules for $S'(\vec{k}\omega)$ are also the coefficients in an asymptotic expansion of $S(\vec{k}z)$ for large z, since we have

$$S(\vec{k}z;\vec{\xi}\vec{\xi}') = -\sum_{n=0}^{\infty} z^{-(n+1)}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\omega^n S'(\vec{k}\omega;\vec{\xi}\,\vec{\xi}') . \qquad (2.6)$$

Inserting this expansion into Eq. (1.7) and comparing term by term, we derive two conditions on the asymptotic behavior of $\Sigma(\vec{k}z)$; the first states

$$\lim_{z\to\infty}\Sigma(\vec{k}z;\vec{\xi}\,\vec{\xi}') = \Sigma^{(s)}(\vec{k}\vec{\xi}) , \qquad (2.7)$$

as given in (1.8b) ; the second states

$$\begin{aligned}\lim_{z\to\infty} z\Sigma^{(c)}(\vec{k}z;\vec{\xi}\,\vec{\xi}')\phi(\vec{\xi}') &= (n/m)\int d\vec{r}\,g(\vec{r})\,\vec{\nabla}_i\vec{\nabla}_j v(\vec{r}) \\ &\times\partial_i\partial_j\phi(\vec{\xi})\,\delta(\vec{\xi}-\vec{\xi}') - (n/m)\int d\vec{r}\cos(\vec{k}\cdot\vec{r}) \\ &\times[g(\vec{r})\vec{\nabla}_i\vec{\nabla}_j v(\vec{r}) + m\vec{\nabla}_i\vec{\nabla}_j c(\vec{r})]\,\partial_i\partial_j'\phi(\vec{\xi})\,\phi(\vec{\xi}').\end{aligned} \qquad (2.8)$$

The approximate $\Sigma^{(c)}$ of (1.8c) fulfills this sum rule to order v^2, and thus guarantees (2.5c) to at least the same order.

C. Positivity

The collision operator $\Sigma^{(c)}(\vec{k}z)$ of (1.8c) is analytic in z for $\mathrm{Im}\,z \neq 0$, and vanishes as $z\to\infty$. Therefore, we can write it in spectral form as

$$\Sigma^{(c)}(\vec{k}z) = -\int\frac{d\omega}{2\pi}\,\frac{\Gamma(\vec{k}\omega)}{\omega-z} , \qquad (2.9)$$

where

$$\begin{aligned}(mv_0)^2\Gamma(\vec{k}\omega;\vec{\xi}\,\vec{\xi}')\,\phi(\vec{\xi}') &= 2n\pi\,[v^2(\vec{\bar{k}})\,\vec{\bar{k}}\cdot\vec{\partial}\,\vec{\bar{k}}\cdot\vec{\partial}'\,\phi(\vec{\xi}) \\ &\times\delta(\vec{\xi}-\vec{\xi}')\,\delta(\omega-v_0(\vec{k}-\vec{\bar{k}})\cdot\vec{\xi}-v_0\vec{\bar{k}}\cdot\vec{\bar{\xi}})\,\phi(\vec{\bar{\xi}}) \\ &+ v(\vec{k}-\vec{\bar{k}})\,v(\vec{\bar{k}})\,\vec{\bar{k}}\cdot\vec{\partial}\,(\vec{k}-\vec{\bar{k}})\cdot\vec{\partial}'\,\phi(\vec{\xi}) \\ &\times\delta(\omega-v_0(\vec{k}-\vec{\bar{k}})\cdot\vec{\xi}-v_0\vec{\bar{k}}\cdot\vec{\xi}')\,\phi(\vec{\xi}')] .\end{aligned} \qquad (2.10)$$

In addition, we observe that Γ is positive (semi) definite in the sense that

$$\int d\vec{\xi}\int d\vec{\xi}'\,\psi^*(\vec{\xi})\,\Gamma(\vec{k}\omega;\vec{\xi}\,\vec{\xi}')\,\phi(\vec{\xi}')\,\psi(\vec{\xi}') = \langle\psi|\,\Gamma(\vec{k}\omega)\,|\psi\rangle \geq 0 \qquad (2.11)$$

for arbitrary functions ψ. We have used a bracket notation for integrals in momentum space which is obvious for (2.11), and explained in some detail in Sec. IV.

For arbitrary k, ω the only eigenfunction of Γ with zero eigenvalue is $\psi(\vec{\xi}) = 1$. As we let $\vec{k} \to 0$, three more "invariant" eigenfunctions appear, namely, $\vec{\xi}_i$, $i = 1$, 2, and 3. Finally, if we also let $\omega \to 0$, we obtain $\psi(\vec{\xi}) = \xi^2$ as a fifth solution. In this limit, $\Gamma(\vec{k}\omega)$ reduces to the kinetic operator K, and the five invariant eigenfunctions 1, $\vec{\xi}$, ξ^2 reflect the conservation of mass, momentum, and (kinetic) energy. All other eigenvalues of K are positive.

In the upper-half of the complex z plane, the imaginary part of $-\Sigma^{(c)}(\vec{k}z)$ is positive, that is,

$$\langle \psi | -\mathrm{Im}\, \Sigma^{(c)}(\vec{k}z) | \psi \rangle \geqslant 0, \text{ if } \mathrm{Im}\, z > 0. \tag{2.12}$$

In the lower-half z plane, it is negative.

The positivity of Γ is connected with the dynamical stability of the system. Together with Eqs. (1.7) and (1.8), Eq. (2.12) guarantees the required positivity of $S'(\vec{k}\omega)$, that is

$$\int d\vec{\xi} \int d\vec{\xi}\,' \ \psi^*(\vec{\xi})\, S(\vec{k}\omega; \vec{\xi}\vec{\xi}\,')\, \psi(\vec{\xi}\,') \geq 0. \tag{2.13}$$

One consequence of (2.13) is

$$\lim_{z \to 0} [-z g_{nn}(\vec{k}z)] = n\left(1 + nh(\vec{k})\right) = n(1 - nc(\vec{k}))^{-1} \geq 0\ . \tag{2.14}$$

The exact direct correlation function $1 - nc(\vec{k})$ is automatically positive. For our approximate theory, however, $c(r)$ is given by Eq. (1.11), and $1 - nc(\vec{k})$ does not necessarily come out positive. To resolve the apparent inconsistency, we recall the analytic properties of $S(\vec{k}z)$ used in deriving (2.14) from (2.12). Equation (1.3) can be employed only if $g_{nn}(\vec{k}z)$ is analytic apart from a branch cut on the real z axis. But the condition for this analyticity is

$$1 - nc(\vec{k}) \geqslant 0. \tag{2.15}$$

As soon as $1 - nc(\vec{k})$ becomes negative for some k, a pair of conjugate poles in $g_{nn}(\vec{k}z)$ appears on the imaginary axis, indicating the breakdown of our approximation. For temperatures so high that $\beta v \ll 1$, (2.15) must be true, and we may expect our approximation to be adequate.

The appearance of poles in $g_{nn}(\vec{k}z)$ signals an instability in the system with a clear physical meaning, although the poles may not be the leading singularities in the rigorous correlation function. We do not wish to pursue the reason for this subtle reservation here. Suffice it to say that the poles or instabilities are symptomatic of phase transitions. More specifically, the appearance of the poles signals that the approximate spatially uniform solution is certainly unstable to infinitesimal disturbances tending to separate it into a two-phase system. When the poles emerge at $k = 0$, they indicate that the system is unstable to formation of a liquid-gas two-phase system; when they appear for $k \neq 0$, they indicate an instability to formation of a spatially inhomogeneous system, i.e., to some sort of solid with lattice spacing $\sim k^{-1}$.

Moreover, we note that it is only the value of the direct correlation function $c(\vec{k})$, not the dynamical operator $\Sigma^{(c)}$ itself, which causes the possible appearance of forbidden poles in $g_{nn}(\vec{k}z)$. Indeed, we might attempt to improve the theory by using the correct static $c(\vec{k})$ instead of its approximate form (1.11). Then $1 - nc(\vec{k})$ is greater or equal to zero for all k, and poles never appear. While no longer systematic, this procedure has been successfully utilized by Nelkin and Ranganathan to explain the excitations in liquid lead.

III. CONSERVATION LAWS

The conservation laws are among the most fundamental properties of a many-particle system, severely limiting the possible modes by which disturbances decay. Indeed, heat conduction and other transport processes are only defined in conserving systems. Deviations in the conserved densities brought about by some local disturbance, cannot disappear locally; they must spread slowly throughout the system.

Let us demonstrate that the equation of motion (1.7) leads to conservation of particle number, momentum, and energy.

The conserved operator densities are given in terms of $f(\vec{r}\vec{\xi}t)$ by

$$n_{\mathrm{op}}(\vec{r}t) = \int d\vec{\xi}\, f(\vec{r}\vec{\xi}t)\ , \tag{3.1a}$$

$$\vec{g}_{\mathrm{op}}(\vec{r}t) = mv_0 \int d\vec{\xi}\ \vec{\xi} f(\vec{r}\vec{\xi}t)\ , \tag{3.1b}$$

$$\epsilon_{\mathrm{op}}(\vec{r}t) = \tfrac{1}{2} m v_0^2 \int d\vec{\xi}\ \xi^2 f(\vec{r}\vec{\xi}t) + \tfrac{1}{2} \int d\bar{\vec{r}}\, v(\vec{r} - \bar{\vec{r}})\, n_{\mathrm{op}}(\vec{r}t)\, n_{\mathrm{op}}(\bar{\vec{r}}t)\ . \tag{3.1c}$$

They satisfy the equations

$$\frac{\partial}{\partial t} n_{\mathrm{op}}(\vec{r}t) + \vec{\nabla} \cdot \frac{\vec{g}_{\mathrm{op}}(\vec{r}t)}{m} = 0\ , \tag{3.2a}$$

$$\frac{\partial}{\partial t} \vec{g}_{\mathrm{op}}(\vec{r}t) + \vec{\nabla} \cdot \vec{\tau}_{\mathrm{op}}(\vec{r}t) = 0\ , \tag{3.2b}$$

$$\frac{\partial}{\partial t} \epsilon_{\mathrm{op}}(\vec{r}t) + \vec{\nabla} \cdot \vec{j}^{\,\epsilon}_{\mathrm{op}}(\vec{r}t) = 0\ , \tag{3.2c}$$

In terms of $f(\vec{r}\xi t)$, the stress tensor $\vec{\tau}_{\mathrm{op}}(\vec{r}t)$ is

$$\tau_{\mathrm{op}}(\vec{r}t)_{ij} = mv_0^2 \int d\vec{\xi}\, \vec{\xi}_i \vec{\xi}_j f(\vec{r}\xi t) - \frac{1}{2} \int d\vec{\bar{r}}\, \frac{\bar{\vec{r}}_i \bar{\vec{r}}_j}{\bar{r}} \frac{dv(\bar{\vec{r}})}{d\bar{r}}$$

$$\times \int_0^1 ds\, n_{\mathrm{op}}\big(\vec{r} - \tfrac{1}{2}\vec{\bar{r}}(s-1),\, t\big) n_{\mathrm{op}}\big(\vec{r} - \tfrac{1}{2}\vec{\bar{r}}(s+1), t\big). \tag{3.2d}$$

The energy current $\vec{j}^{\,\epsilon}_{\mathrm{op}}(\vec{r}t)$ may be expressed in a similar fashion. The microscopic equations (3.2) imply conservation laws for correlation functions like

$$n(\vec{r}-\vec{r}\,',\, t-t';\vec{\xi}\,') \equiv \langle \big(n_{\mathrm{op}}(\vec{r}t) - \langle n_{\mathrm{op}}(\vec{r}t)\rangle_{\mathrm{eq}}\big) \times \big(f(\vec{r}\,'\vec{\xi}\,'t') - \langle f(\vec{r}\,'\vec{\xi}\,'t')\rangle_{\mathrm{eq}}\big)\rangle_{\mathrm{eq}}\,. \tag{3.3}$$

For simplicity, we shall call contracted correlation functions like $n(\vec{r}t;\vec{\xi}\,')$ the "density," etc. The Laplace transform of these functions [with the convention (1.4)] yields conservation laws in the form

$$zn(\vec{k}z;\vec{\xi}\,') - (1/m)\vec{k}\cdot\vec{g}(\vec{k}z;\vec{\xi}\,') = -n(\vec{k}\vec{\xi}\,')\,, \tag{3.4a}$$

$$zg_i(\vec{k}z;\vec{\xi}\,') - k_j\tau_{ij}(\vec{k}z;\vec{\xi}\,') = -g_i(\vec{k}\vec{\xi}\,'), \tag{3.4b}$$

$$z\epsilon(\vec{k}z;\vec{\xi}\,') - \vec{k}\cdot\vec{j}^{\,\epsilon}(\vec{k}z;\vec{\xi}\,') = -\epsilon(\vec{k}\vec{\xi}\,'). \tag{3.4c}$$

The equal-time (thermodynamic) correlation functions occur on the right-hand side of these equations as initial conditions.

We first consider the "kinetic" approximation, replacing $\Sigma(\vec{k}z;\vec{\xi}\vec{\xi}\,')$ in (1.7) by $-iK(\vec{\xi}\vec{\xi}\,')$, and omitting the interaction term on the right-hand side:

$$(z - v_0\vec{k}\cdot\vec{\xi})S(\vec{k}z;\vec{\xi}\vec{\xi}\,') + iK(\vec{\xi}\vec{\bar{\xi}})S(\vec{k}z;\vec{\bar{\xi}}\vec{\xi}\,') = -n\phi(\vec{\xi})\delta(\vec{\xi}-\vec{\xi}\,')\,. \tag{3.5}$$

Conservation of particle number momentum, and (kinetic) energy then follow from the identities

$$\int d\vec{\xi}\,\{1,\ \vec{\xi},\ \xi^2\}K(\vec{\xi}\vec{\xi}\,') = 0\,. \tag{3.6}$$

These identities, previously noted after Eq. (2.11), are also easily verified directly.

The conserved "densities" for the kinetic equation are

$$n(\vec{k}z;\vec{\xi}\,') = \int d\vec{\xi}\, S(\vec{k}z;\vec{\xi}\vec{\xi}\,')\,, \tag{3.7a}$$

$$\vec{g}(\vec{k}z;\vec{\xi}\,') = mv_0\int d\vec{\xi}\,\vec{\xi}S(\vec{k}z;\vec{\xi}\vec{\xi}\,')\,, \tag{3.7b}$$

$$\epsilon^{\mathrm{kin}}(\vec{k}z;\vec{\xi}\,') = \tfrac{1}{2}m\,v_0^2\int d\vec{\xi}\,\xi^2 S(\vec{k}z;\vec{\xi}\vec{\xi}\,')\,, \tag{3.7c}$$

and the (kinetic) "stress tensor" and "energy" currents are

$$\tau_{ij}^{\mathrm{kin}}(\vec{k}z;\vec{\xi}\,') = mv_0^2\int d\vec{\xi}\,\xi_i\xi_j S(\vec{k}z;\vec{\xi}\vec{\xi}\,')\,, \tag{3.7d}$$

$$\vec{j}^{\,\epsilon,\mathrm{kin}}(\vec{k}z;\vec{\xi}\,') = \tfrac{1}{2}mv_0^2 v_0\int d\vec{\xi}\,\vec{\xi}\,\xi^2 S(\vec{k}z;\vec{\xi}\vec{\xi}\,'). \tag{3.7e}$$

For the full-mass operator $\Sigma(\vec{k}z;\vec{\xi}\vec{\xi}\,')$, the verification of the conservation laws is more complicated. To prove them, we return to the equation of motion (1.8), or

$$\begin{aligned}(z - v_0\vec{k}\cdot\vec{\xi})S(\vec{k}z;\vec{\xi}\vec{\xi}\,') &- \Sigma^{(s)}(\vec{k}\vec{\xi})n(\vec{k}z;\vec{\xi}\,') \\ &- \Sigma^{(c)}(\vec{k}z;\vec{\xi}\vec{\bar{\xi}})S(\vec{k}z;\vec{\bar{\xi}}\vec{\xi}\,') \\ &= -n[\phi(\vec{\xi})\delta(\vec{\xi}-\vec{\xi}\,') + nh(\vec{k})\phi(\vec{\xi})\phi(\vec{\xi}\,')]\,.\end{aligned} \tag{3.8}$$

A. Conservation of Particle Number

Particle conservation is still easily verified. From the explicit form of $\Sigma(\vec{k}z;\vec{\xi}\vec{\xi}\,')$ in Eqs. (1.8b) and (1.8c), we see that

$$\int d\vec{\xi}\,\Sigma^{(s)}(\vec{k}\vec{\xi}) = \int d\vec{\xi}\,\Sigma^{(c)}(\vec{k}z;\vec{\xi}\vec{\xi}\,') = 0\,. \tag{3.9}$$

Integrating Eq. (3.8) over $\vec{\xi}$, we obtain the conservation law (3.4a), with

$$n(\vec{k}\vec{\xi}\,') = n\big(1 + nh(\vec{k})\big)\,\phi(\vec{\xi}\,')\,. \tag{3.10}$$

This is the correct initial value. Of course, in our approximation, $h(\vec{k})$ is given by (1.13), with $c(\vec{k})$ to second order in v.

B. Conservation of Momentum

From Eqs. (1.8b) and (1.8c), we find

$$\int d\vec{\xi}\,\xi_i\Sigma^{(s)}(\vec{k}\vec{\xi}) = -nc(\vec{k})v_0k_i\,, \tag{3.11}$$

$$\begin{aligned}(mv_0)^2\int d\vec{\xi}\,\xi_i\Sigma^{(c)}(\vec{k}z;\vec{\xi}\vec{\xi}\,')\phi(\vec{\xi}\,') &= -nv(\vec{\bar{k}})[v(\vec{\bar{k}}-\vec{k})(\bar{k}_i - k_i) - v(\vec{\bar{k}})\bar{k}_i] \\ &\times \vec{\bar{k}}\cdot\vec{\partial}\,'\,\frac{\phi(\vec{\bar{\xi}})\phi(\vec{\xi}\,')}{v_0(\vec{k}-\vec{\bar{k}})\cdot\vec{\xi}\,' + v_0\vec{\bar{k}}\cdot\vec{\bar{\xi}} - z}\,.\end{aligned} \tag{3.12}$$

Because the total momentum is conserved, both of these expressions vanish as $k \to 0$. Assuming $v(\vec{\bar{k}})$ is differentiable, we can extract an explicit factor of k_j from (3.12) and write

$$v(\vec{\bar{k}}-\vec{k})(\bar{k}-k)_i - v(\bar{k})\bar{k}_i \equiv \tilde{v}_{ij}(\vec{k}\vec{\bar{k}})k_j\,, \tag{3.13}$$

with

$$\tilde{v}_{ij}(\vec{k}\vec{\bar{k}}) \equiv -\frac{\partial}{\partial\bar{k}_j}\int_0^1 ds\; v(\vec{\bar{k}} - s\vec{k})(\bar{k}_i - sk_i).$$

Note that $\tilde{v}_{ij}$ is symmetric in $i \to j$. Multiplying Eq. (3.8) by $mv_0\xi_i$ and integrating, we obtain the momentum conservation law in the form (3.4b), with the correct initial condition

$$g_i(\vec{k}\vec{\xi}\,') = nmv_0\xi'_i\phi(\vec{\xi}\,')\,. \tag{3.14}$$

The "stress tensor" can be written

$$\begin{aligned}\tau_{ij}(\vec{k}z;\vec{\xi}\,') = mv_0^2\int d\vec{\xi}\,[\xi_i\xi_j &- nc(\vec{k})\delta_{ij} \\ &+ T_{ij}(\vec{k}z;\vec{\xi})]S(\vec{k}z;\vec{\xi}\vec{\xi}\,')\,,\end{aligned} \tag{3.15}$$

with

$$T_{ij}(\vec{k}z;\vec{\xi})\phi(\vec{\xi}) = -(n/m^2v_0^3)v(\vec{\bar{k}})\tilde{v}_{ij}(\vec{k}\vec{\bar{k}})$$

$$\times\,\vec{\bar{\mathrm{k}}}\cdot\vec{\partial}\,\frac{\phi(\vec{\bar{\xi}})\,\phi(\vec{\xi})}{v_0(\vec{\mathrm{k}}-\vec{\bar{\mathrm{k}}})\cdot\vec{\xi}+v_0\vec{\bar{\mathrm{k}}}\cdot\vec{\bar{\xi}}-z}\;. \tag{3.16}$$

From (3.1d), it is clear that in order to calculate the "stress tensor," we have to know, not only the two-particle function $S(\vec{\mathrm{k}}z;\vec{\xi}\vec{\xi}')$, but the corresponding three-particle function. In the context of a closed equation for $S(\vec{\mathrm{k}}z;\vec{\xi}\vec{\xi}')$, this dependence is incorporated in the operator $T_{ij}(\vec{\mathrm{k}}z;\vec{\xi})$, which depends explicitly on k and z. In agreement with general principles, it transforms as a symmetric tensor under spatial rotations. If we replace $S(\vec{\mathrm{k}}z)$ in (3.15) by its free-particle limit $S^0(\vec{\mathrm{k}}z)$, we obtain the "stress tensor" correct to order v^2.

The function $T_{ij}(\vec{\mathrm{k}}z;\vec{\xi})$ remains finite (and is real) as $k\to 0$, $z\to i\epsilon$:

$$\lim_{k\to 0} T_{ij}(\vec{\mathrm{k}}i\epsilon;\vec{\xi})=T_{ij}(\vec{\xi})$$

$$=\tfrac{3}{2}\alpha\,[(\vec{\partial}-\vec{\xi})\cdot\vec{\partial}]\,\{(\vec{\xi}-\vec{\bar{\xi}})_i(\vec{\xi}-\vec{\bar{\xi}})_j/|\vec{\xi}-\vec{\bar{\xi}}|^2$$

$$-\delta_{ij}\ln|\vec{\xi}-\vec{\bar{\xi}}|\}\,\phi(\vec{\bar{\xi}})\;, \tag{3.17}$$

where

$$\alpha=\tfrac{1}{3}n\int d\vec{\mathrm{r}}\,[\beta v(\vec{\mathrm{r}})]^2\;\cdot \tag{3.18}$$

This term and the second term in (3.15) give correlational contributions to the limiting value of τ_{ij}. Their absence from the kinetic equation (3.5) indicates that the latter is deficient even in the long-time large-distance limit. If we began by replacing $\Sigma(\vec{\mathrm{k}}z)$ by its limit $-iK$, we would omit terms of the same order in k, z as the streaming term $z-v_0\vec{\mathrm{k}}\cdot\vec{\xi}$ – terms which must be retained in the hydrodynamic limit. We shall see later that even those terms of second order in k, z in $\Sigma^{(c)}(\vec{\mathrm{k}}z)$ contribute to the transport coefficients.

C. Conservation of Energy

The "kinetic energy" is given by Eq. (3.7) in terms of the two-particle function $S(\vec{\mathrm{k}}z)$. In general, the "potential energy" is obtained from the corresponding three-particle function for which we have no equation. Nevertheless, we can use Eq. (1.7) to derive a conservation law, and to infer an expression for the "potential energy" in terms of $S(\vec{\mathrm{k}}z)$ itself.

From (3.6) and (1.14), we expect that

$$\int d\vec{\xi}\;\xi^2\Sigma^{(c)}(\vec{\mathrm{k}}z;\vec{\xi}\vec{\xi}')=-zE(\vec{\mathrm{k}}z;\vec{\xi}')+v_0\vec{\mathrm{k}}\cdot\vec{\mathrm{J}}^{\epsilon}(\vec{\mathrm{k}}z;\vec{\xi}'), \tag{3.19}$$

where the functions $E(\vec{\mathrm{k}}z;\vec{\xi}')$ and $\vec{\mathrm{J}}^{\epsilon}(\vec{\mathrm{k}}z;\vec{\xi}')$ have finite limits as $k\to 0$ and $z\to i\epsilon$. Indeed, with the approximate $\Sigma^{(c)}$ of (1.8c), the limits are readily calculated:

$$-E(\vec{\xi})=6\,\alpha\,(\vec{\partial}-\vec{\xi})\cdot\vec{\partial}\ln(|\vec{\xi}-\vec{\bar{\xi}}|)\,\phi(\vec{\bar{\xi}})\;, \tag{3.20}$$

$$-J_i^{\epsilon}(\vec{\xi})=3\,\alpha(\vec{\partial}-\vec{\xi})\circ\vec{\partial}\,\partial_i\ln(|\vec{\xi}-\vec{\bar{\xi}}|)\,\phi(\vec{\bar{\xi}})\;, \tag{3.21}$$

with α given by (3.18). Equation (3.19) implies a conservation law of the form

$$z\tilde{\epsilon}(\vec{\mathrm{k}}z;\vec{\xi}')-\vec{\mathrm{k}}\cdot\vec{\tilde{\mathrm{j}}}^{\epsilon}(\vec{\mathrm{k}}z;\vec{\xi}')$$

$$=-\tfrac{1}{2}nmv_0^2\big(\xi'^2+3nh(\vec{\mathrm{k}})\big)\,\phi(\vec{\xi}')\;, \tag{3.22}$$

with

$$\tilde{\epsilon}(\vec{\mathrm{k}}z;\vec{\xi}')=\tfrac{1}{2}mv_0^2\int d\vec{\xi}\,[\xi^2+E(\vec{\mathrm{k}}z;\vec{\xi})]\,S(\vec{\mathrm{k}}z;\vec{\xi}\vec{\xi}')\;, \tag{3.23a}$$

$$\vec{\tilde{\mathrm{j}}}^{\epsilon}(\vec{\mathrm{k}}z;\vec{\xi}')=\tfrac{1}{2}mv_0^3\int d\vec{\xi}\,[\vec{\xi}\xi^2+\vec{\mathrm{J}}^{\epsilon}(\vec{\mathrm{k}}z;\vec{\xi}')]\,S(\vec{\mathrm{k}}z;\vec{\xi}')\;. \tag{3.23b}$$

For our purposes, it will be sufficient to show that Eq. (3.19) is correct and implies a fifth conservation law (3.22) whether or not the quantity ϵ is the "energy density." Indeed, the right-hand side of (3.22) is the equal-time correlation function of the kinetic, not the total, "energy density." However, at equal times, the potential "energy density" takes the form

$$\epsilon^{\mathrm{pot}}(\vec{\mathrm{k}}\vec{\xi})=\tilde{E}(\vec{\mathrm{k}})\,\phi(\vec{\xi})\;\;, \tag{3.24}$$

and to order v^2 is

$$\tilde{E}(\vec{\mathrm{k}})=\tfrac{1}{2}nmv_0^2\big[\big(\beta v(\vec{\mathrm{k}})-\beta^2v(\vec{\mathrm{k}}-\vec{\bar{\mathrm{k}}})\,v(\vec{\bar{\mathrm{k}}})\big)$$

$$+(\text{same at }k=0)\big]\,. \tag{3.25}$$

Therefore, with the aid of (3.4a), we see that

$$\epsilon(\vec{\mathrm{k}}z;\vec{\xi}')=\tfrac{1}{2}mv_0^2\int d\vec{\xi}\,[\xi^2+\tilde{E}(\vec{\mathrm{k}})+E(\vec{\mathrm{k}}z;\vec{\xi})]$$

$$\times S(\vec{\mathrm{k}}z;\vec{\xi}\vec{\xi}') \tag{3.26}$$

fulfills the conservation law (3.4c) and has the correct initial condition to order v^2. Equation (3.19) defines $E(\vec{\mathrm{k}}z;\vec{\xi}')$ uniquely only at $k=0$. However, if we choose

$$E(\vec{\mathrm{k}}z;\vec{\xi})\,\phi(\vec{\xi})=\frac{n}{m^2v_0^3}\,v(\vec{\bar{\mathrm{k}}})\;v(\vec{\mathrm{k}}-\vec{\bar{\mathrm{k}}})+v(\vec{\bar{\mathrm{k}}})$$

$$\times\,\vec{\bar{\mathrm{k}}}\cdot\vec{\partial}\,\frac{\phi(\vec{\xi})\,\phi(\vec{\bar{\xi}})}{v_0(\vec{\mathrm{k}}-\vec{\bar{\mathrm{k}}})\circ\vec{\xi}+v_0\vec{\bar{\mathrm{k}}}\cdot\vec{\bar{\xi}}-z}\;, \tag{3.27}$$

which is compatible with (1.8c) and (3.19), we can show that $\epsilon(\vec{\mathrm{k}}z;\vec{\xi})$ is the total "energy density" to order v^2.

This concludes our discussion of the conservation laws. To summarize, we have shown that the approximations (1.8b) and (1.8c) for the mass operator conserve particle number, momentum, and energy. Moreover, the conservation laws have been written in differential form, in terms of well-defined currents defined in terms of $S(\vec{\mathrm{k}}z)$. We have found that the stress tensor, the energy, and the energy current each contained terms not present in the kinetic equation, but nonetheless persistent in the long-time large-distance limit.

IV. HYDRODYNAMICAL LIMIT

In the remainder of this paper, we will examine the behavior of the system for long times and large distances, or equivalently for small wave numbers k and frequencies z. In this limit, we expect the familiar hydrodynamical "contraction" of the description to occur. The dynamics should be dominated by the five conservation laws, and the complex equations of motion (1.7) and (1.8) should reduce to a much simpler set of five partial differential equations, the Navier-Stokes equations of fluid dynamics. Let us examine whether this expectation is borne out. The connection between the linearized Navier-Stokes equations and the correlation functions in the hydrodynamical limit has been investigated[8] in great detail. The simplest function is the one that describes transverse momentum density fluctuations. It is defined by separating g_{ij} into longitudinal and transverse parts:

$$g_{ij}(\vec{k}z)=\frac{k_ik_j}{k^2}\,g_l(\vec{k}z)+\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)g_t(\vec{k}z)\ . \tag{4.1}$$

The diffusive behavior of shear is exhibited in

$$g_t(\vec{k}z)=\frac{-n\,(mv_0)^2}{z+ik^2\eta/mn}\ ;\quad \mathrm{Im}z>0 \tag{4.2}$$

by a pole in the lower-half of the complex z plane, whose imaginary part is proportional to the shear viscosity η.

The longitudinal part is more complicated. It is convenient to write g_l in terms of the density-density fluctuation function to which it is simply related by particle conservation. The result is[8]

$$g_{nn}(\vec{k}z)=-mn\,v_0^2\left(\frac{\partial n}{\partial p}\right)_T \times\left(\frac{1-c_v/c_p}{z+ik^2D_T}+\frac{c_v}{c_p}\,\frac{z+ik^2[\,\Gamma+D_T(c_p-c_v)/c_v]}{z^2-c^2k^2+izk^2\,\Gamma}\right)\ , \tag{4.3}$$

where c_p and c_v are the specific heats at constant pressure and volume,

$$mnc_p=\frac{T}{V}\left(\frac{\partial S}{\partial T}\right)_p\ ,\quad mnc_v=\frac{T}{V}\left(\frac{\partial S}{\partial T}\right)_n\ ; \tag{4.4}$$

c is the isentropic sound velocity,

$$c^2=\left(\frac{\partial p}{\partial mn}\right)_S=\frac{c_p}{c_v}\left(\frac{\partial p}{\partial mn}\right)_T\ ; \tag{4.5}$$

and the damping constants are given by

$$D_T=\frac{\kappa}{mnc_p}\ ;\quad \Gamma=\frac{(\frac{4}{3}\eta+\zeta)}{mn}+D_T\left(\frac{c_p}{c_v}-1\right) \tag{4.6}$$

in terms of the heat conductivity κ, and the shear and bulk viscosities η and ζ.

Similar expressions can be written for the other correlation functions. We shall be satisfied with calculating g_{ij}, or g_t and g_{nn}, since all relevant transport and thermodynamic coefficients can be identified from them.

We shall examine the functions g_{nn} and g_t as they emerge from the general equations of motion (1.7) and (1.8), and show that in the hydrodynamical limit, they reduce to the form given above. The comparison will serve to identify the thermodynamic and transport coefficients in terms of the interparticle potential. This calculation is tantamount to the computation of transport coefficients from Kubo-type formulas.

Finally, let us remark on the implications of this derivation. In addition to the conservation laws, there are two basic assumptions that enter the equations of fluid dynamics. (a) The hypothesis of local equilibrium: The system is assumed to be completely described by local values of the thermodynamic variables, and the various local variables are related by thermodynamic identities. (b) The constitutive relations: The fluxes of conserved quantities are proportional to the gradients of the local variables.

The first of these is a statement about the unimportance of other variables at sufficiently long wavelengths and times. It has previously been verified by general arguments which assume no hidden constants of the motion. The second is a statement about the differentiability of the fluxes with wave number. Our derivation shows how both of these conditions are met for the collision operator given by Eq. (1.8c). Whether the assumptions we have made add credence to the arguments of ergodicity and differentiability is a matter for the reader to decide. Clearly, our arguments are not rigorous, but it is difficult to see what would cause them to fail. Of course the same might be said for the general arguments[8,9] given earlier.

Before proceeding, let us introduce a few definitions. It will be convenient to use a bracket notation for integrals in momentum space. We denote the scalar product of two functions $\varphi(\vec{\xi})$, $\psi(\vec{\xi})$, by

$$\langle\varphi|\psi\rangle\equiv\int d\vec{\xi}\ \varphi^*(\vec{\xi})\,\phi(\vec{\xi})\,\psi(\vec{\xi})\ . \tag{4.7a}$$

This product has the usual properties. We also define the "matrix elements" of $M(\vec{\xi}\vec{\xi}\,')$ by

$$\langle\varphi|M|\psi\rangle\equiv\int d\vec{\xi}\int d\vec{\xi}'\ \varphi^*(\vec{\xi})\,M(\vec{\xi}\,\vec{\xi}\,')\,\phi(\vec{\xi}\,')\,\psi(\vec{\xi}\,')\ . \tag{4.7b}$$

Since the Fokker-Planck operator K of Eq. (1.15) is real and symmetric in the sense of Eq. (2.4), it is Hermitian, and its left- and right-hand side eigenfunctions are identical.

The subspace on which K vanishes is of particular importance in hydrodynamics. It is spanned by the five orthonormal states $\varphi_\nu(\vec{\xi})$, $\nu = 1 \cdots 5$, where

$$\varphi_1(\vec{\xi}) \equiv n(\vec{\xi}) = 1, \qquad \varphi_2(\vec{\xi}) \equiv g_l(\vec{\xi}) = \xi_3,$$
$$\varphi_3(\vec{\xi}) \equiv \epsilon(\vec{\xi}) = (\xi^2 - 3)/\sqrt{6}, \tag{4.8}$$
$$\varphi_4(\vec{\xi}) \equiv g_t(\vec{\xi}) = \xi_1, \qquad \varphi_5(\vec{\xi}) \equiv g_t'(\vec{\xi}) = \xi_2 .$$

We have taken $\vec{k}$ in the 3 direction so that φ_ν, $\nu = 1 \cdots 3$ are longitudinal modes and φ_ν, $\nu = 4, 5$ are transverse modes. These five states can be understood as part of a complete orthonormal set $\{\varphi_\nu(\vec{\xi})\}$ such that

$$\langle \varphi_\nu | \varphi_\mu \rangle = \delta_{\mu\nu} \tag{4.9a}$$

$$\Sigma_\nu \, | \varphi_\nu \rangle \langle \varphi_\nu | = 1 . \tag{4.9b}$$

As an example, we could use the generalized Hermite polynomial tensors whose properties have been discussed by Grad[10]; for our present purposes, however, it is not necessary to specify the φ_i beyond the first five functions.

We shall make extensive use of the projector

$$P = \sum_{\nu=1}^{5} | \varphi_\nu \rangle \langle \varphi_\nu | = 1 - Q , \tag{4.10}$$

which projects onto the subspace where K vanishes;

$$KP = PK = 0 . \tag{4.11}$$

Finally, we remark that the conservation laws (3.12) and (3.19) for $\Sigma^{(c)}(\vec{k}z)$ may be expressed in the form

$$\Sigma^{(c)}(\vec{k}z) | n \rangle = 0 , \tag{4.12a}$$

$$\Sigma^{(c)}(\vec{k}z) | \xi_i \rangle = v_0 k_j | T_{ij}(\vec{k}z) \rangle\!\rangle , \tag{4.12b}$$

$$\Sigma^{(c)}(\vec{k}z) | \epsilon \rangle \sqrt{6} = -z | E(\vec{k}z) \rangle\!\rangle + v_0 \vec{k} \cdot | \vec{j}^{\epsilon}(\vec{k}z) \rangle\!\rangle , \tag{4.12c}$$

together with corresponding equations in which $\Sigma^{(c)}$ operates to the left. Note that at least in the limit as $k \to 0$, $z \to i\epsilon$, the functions T_{ij}, E, and $\vec{j}^{\epsilon}$ are all real.

V. TRANSVERSE EXCITATIONS

In this section, we shall discuss the simpler of the two functions we require, the transverse momentum density correlation function $g_t(\vec{k}z)$. To obtain it, we must evaluate the expression given formally by

$$g_t(\vec{k}z) = -n(mv_0)^2 \langle g_t | [z - \omega^0(\vec{k}) - \Sigma(\vec{k}z)]^{-1} | g_t \rangle , \tag{5.1}$$

with

$$\omega^0(\vec{k}; \vec{\xi}\vec{\xi}') \equiv v_0 \vec{k} \cdot \vec{\xi} \, \delta(\vec{\xi} - \vec{\xi}') . \tag{5.2}$$

The symmetries of $\Sigma^{(s)}$ permit us to drop it from $\Sigma(\vec{k}z)$ in (5.1). As $k \to 0$ and $z = i\epsilon \to 0$, the remaining operator $z - \omega^0(\vec{k}) - \Sigma^{(c)}(\vec{k}z)$ reduces to iK which has zero eigenvalues. Matrix elements of its inverse, like $g_t(\vec{k}z)$, may therefore be singular at $k = 0$ and $z = 0$ even though they are analytic in the upper half z plane. We shall isolate and display these singularities by projecting out the subspace on which K vanishes. This rearrangement is closely related to a commonly used[9,11] spectral representation of $g_t(\vec{k}z)$ in terms of a complex shear viscosity $\eta(\vec{k}z)$:

$$g_t(\vec{k}z) = -n(mv_0)^2 [z + ik^2 \eta(\vec{k}z)/mn]^{-1} . \tag{5.3}$$

The function $\eta(\vec{k}z)$ vanishes as $z \to \infty$. It is also an analytic function in the upper half-plane, since $\mathrm{Im}\, g_t(\vec{k}z)$ is positive. The hydrodynamic result (4.2) suggests that $\eta(\vec{k}z)$ is finite and real as $k \to 0$ and $z = i\epsilon \to 0$.

From (5.1), we can obtain a formal expression for $\eta(\vec{k}z)$ in terms of $\Sigma(\vec{k}z)$. To save writing, we let

$$G_t(\vec{k}z) \equiv -g_t(\vec{k}z)/n(mv_0)^2 , \tag{5.4}$$

$$\tilde{\Sigma}(\vec{k}z) \equiv \omega^0(\vec{k}) + \Sigma(\vec{k}z) , \tag{5.5}$$

and omit the k, z arguments wherever possible. Inserting $P + Q = 1$ in (5.1), we have

$$G_t = \langle g_t | (z - \tilde{\Sigma} Q - \tilde{\Sigma} P)^{-1} | g_t \rangle$$
$$= \langle g_t | (z - \tilde{\Sigma} Q)^{-1} + (z - \tilde{\Sigma} Q)^{-1} \tilde{\Sigma} P (z - \tilde{\Sigma})^{-1} | g_t \rangle .$$

Since $Q | g_t \rangle = 0$, the first term reduces to $1/z$. In addition, we have

$$P(z - \tilde{\Sigma})^{-1} | g_t \rangle = | g_t \rangle \langle g_t | (z - \tilde{\Sigma})^{-1} | g_t \rangle = | g_t \rangle G_t ,$$

since the other four states in P are even under $\xi_1 \to -\xi_1$, whereas, $| g_t \rangle$ is odd and $\tilde{\Sigma}$ is even. Therefore, we obtain

$$G_t(\vec{k}z) = \left(z - \Omega_t(\vec{k}z) \right)^{-1} , \tag{5.6}$$

where $\Omega_t(\vec{k}z)$, the "memory function" of the transverse momentum, can be written in the symmetric form

$$\Omega_t(\vec{k}z) = -ik^2 \eta(\vec{k}z)/mn = \langle g_t | \tilde{\Sigma}(\vec{k}z) | g_t \rangle$$
$$+ \langle g_t | \tilde{\Sigma}(\vec{k}z) Q [z - Q \tilde{\Sigma}(\vec{k}z) Q]^{-1} Q \tilde{\Sigma}(\vec{k}z) | g_t \rangle . \tag{5.7}$$

The shear viscosity η is given by

$$\eta = \lim_{z = i\epsilon \to 0} \; \lim_{k \to 0} \; \eta(\vec{k}z) \tag{5.8}$$

when the limit exists, i.e., when there are no propagating shear waves at zero wave number. We can see that the limit exists in our model by noting that neither $\Sigma^{(c)}$ nor $\omega^0(\vec{k})$ contributes in the first term of (5.7). Using Eq. (4.12b) in the second term of

(5.7), we arrive at

$$\eta'(\vec{k}z)/mnv_0^2 = i\langle \xi_1\xi_3 + T_{13}(\vec{k}z^*)\,|Q \times [z - Q\tilde{\Sigma}(\vec{k}z)Q]^{-1}Q\,|\,\xi_1\xi_3 + T_{13}(\vec{k}z)\rangle, \qquad (5.9a)$$

$$\eta''(\vec{k}z)/mnv_0^2 = (v_0 k)^{-2}\langle g_t|\,\Sigma^{(c)}(\vec{k}z)\,|g_t\rangle. \qquad (5.9b)$$

In the limit as $k \to 0$, $z = i\epsilon \to 0$, the function $T_{13}(\vec{\xi})$ is given by (3.17), and is finite and real. The inverse operator in (5.9a) reduces to K^{-1} which is real, positive, and nonsingular, since it is now projected onto the subspace where K is not zero. (If the eigenfunctions φ_ν of K were known, we could write the limit as

$$QK^{-1}Q = \sum\nolimits_\nu' \,|\varphi_\nu\rangle K_\nu^{-1}\langle\varphi_\nu|,$$

with the sum extending only over the nonvanishing eigenvalues K_ν of K.) Taking the limit (5.8), we have

$$\eta'/mnv_0^2 = \langle \xi_1\xi_3 + T_{13}|QK^{-1}Q|\,\xi_1\xi_3 + T_{13}\rangle, \qquad (5.10a)$$

which is real and positive.

From the explicit $\Sigma^{(c)}$ of Eq. (1.7c), the integral in (5.9b) can be explicitly calculated, with the result

$$\gamma_\eta'' \equiv \frac{\eta''}{mnv_0^2} = \frac{n\sqrt{\pi}}{24 v_0}\int \frac{d\vec{k}}{(2\pi)^3}\,\frac{2}{5}k\left(\frac{d\beta v(\vec{k})}{dk}\right)^2 . \qquad (5.10b)$$

The total shear viscosity

$$\eta = \eta' + \eta'' \qquad (5.10c)$$

is real and positive as it must be.[12]

If we had replaced Eq. (1.8) by the associated Fokker-Planck equation (3.5), i.e., replaced $\Sigma(\vec{k}z)$ by $-iK$ from the start, we would have obtained a shear viscosity

$$\eta^{\text{kin}}/mnv_0^2 = \langle \xi_1\xi_3|QK^{-1}Q|\,\xi_1\xi_3\rangle. \qquad (5.11)$$

Our equations clearly display the corrections. We remarked earlier that T_{13} describes a potential contribution to the stress tensor which influences the transport behavior even in the limit of long times and large distances. It originates from those terms in $\Sigma^{(c)}(\vec{k}z)$ that are of first order in k. Second-order terms determine η'', which represents a purely correlational correction. Thus, even when we are only interested in the transport behavior, it is premature to perform the "Markovian" limit $k \to 0$, $z = i\epsilon \to 0$ in the collision operator.

From Eq. (1.17), we see that the relaxation frequency ν is proportional to the density, and varies with temperature T as $T^{-3/2}$. The kinetic viscosity η^{kin} is therefore independent of the density, and varies with temperature as $T^{5/2}$. The corrections from (4.23) are of relative order nT^{-2}.

Thus, we may write (5.10c) in the form

$$\eta^{\text{kin}}(T) = A_\eta T^{5/2},$$

$$\eta(n, T) = \eta^{\text{kin}}(T)\,[1 + B_\eta n T^{-2} + C_\eta n^2 T^{-4}], \qquad (5.12)$$

where A_η is a positive constant that determines the Fokker-Planck viscosity, B_η is due to the cross terms in (5.10a), and C_η is positive and has contributions from both η' and η''. In the limit of high temperature (and/or low density), the Fokker-Planck term $\eta^{\text{kin}} \sim T^{5/2}$ prevails.

We recall that elementary arguments give a viscosity of the form

$$\eta_{mfp} = \tfrac{1}{3} mnlv_0, \qquad (5.13)$$

where l is a mean free path. If l^{-1} is proportional to the density times a temperature-independent cross section, we find $\eta_{mfp} \sim T^{1/2}$. Our expressions have a different temperature dependence because our cross section is proportional to $(\beta v)^2$.

VI. LONGITUDINAL EXCITATIONS

Now, let us turn attention to the density-density fluctuation function $g_{nn}(\vec{k}z)$ which, in view of Eq. (1.7), is formally given by

$$g_{nn}(\vec{k}z) = -n[1 - nc(\vec{k})]^{-1}\langle n|\,[z - \omega^0(\vec{k}) - \Sigma(\vec{k}z)]^{-1}\,|n\rangle. \qquad (6.1)$$

As we noted, $g_{nn}(\vec{k}z)$ is a more complicated function than its transverse counterpart $g_t(\vec{k}z)$. In principle, a rigorous dispersion relation analogous to (5.3) can be written down, with a slight adjustment to account for a propagating mode structure. However, in the longitudinal case this dispersion relation is less useful. The reason for the complexity is apparent from the hydrodynamic form of $g_{nn}(\vec{k}z)$ [Eq. (4.3)]. There are three poles, two describing sound propagation and the third describing heat diffusion. In the longitudinal analog of $\Omega_t(\vec{k}z)$ these modes would appear coupled, introducing terms that are not slowly varying for small k.

The situation can be remedied by considering a matrix generalization of $\Omega_l(\vec{k}z)$ which explicitly couples the density, longitudinal momentum, and energy (or entropy) correlation functions. From the hydrodynamic results, we expect that such a matrix will be well behaved at small k and z. We will not write down or prove the pertinent rigorous formulas. However, our subsequent analysis proceeds essentially along these lines.

We begin by defining the matrix

$$G_{\mu\nu}(\vec{k}z) = \langle\mu|\,[z - \omega^0(\vec{k}) - \Sigma(\vec{k}z)]^{-1}\,|\nu\rangle, \qquad (6.2)$$

where $\mu, \nu = 1, 2, 3$ counts the three longitudinal states n, g_l, and ϵ of (4.8). $G_{11}(\vec{k}z)$ is the density-density fluctuation function, apart from a constant factor.

$G_{\mu\nu}(\vec{k}z)$ has the same type of singularity near $k = 0$, $z = i\epsilon \to 0$, as did $G_t(\vec{k}z)$ in Sec. V, and we will again use the projector formalism to exhibit it. In complete analogy to Eqs. (5.6) and (5.7), we can write

$$[z\delta_{\mu\sigma} - \Omega_{\mu\sigma}(\vec{\mathrm{k}}z)]G_{\sigma\mu}(\vec{\mathrm{k}}z) = \delta_{\mu\nu}, \tag{6.3}$$

where

$$\Omega_{\mu\nu}(\vec{\mathrm{k}}z) = \langle\mu|\omega^0(\vec{\mathrm{k}}) + \Sigma^{(s)}(\vec{\mathrm{k}}) + \Sigma^{(c)}(\vec{\mathrm{k}}z)|\nu\rangle + \langle\mu|[\omega^0(\vec{\mathrm{k}}) + \Sigma^{(c)}(\vec{\mathrm{k}}z)]Q\{z - Q[\omega^0(\vec{\mathrm{k}}) + \Sigma^{(c)}(\vec{\mathrm{k}}z)]Q\}^{-1} \times Q[\omega^0(\vec{\mathrm{k}}) + \Sigma^{(c)}(\vec{\mathrm{k}}z)]|\nu\rangle. \tag{6.4}$$

As in the transverse case, $\Omega_{\mu\nu}(\vec{\mathrm{k}}z)$ is analytic in z, apart from a branch cut along the real axis. $\Omega_{\mu\nu}$ is symmetric under interchange of μ and ν except for the trival static term which is easily calculated from (1.8b):

$$\langle\mu|\Sigma^{(s)}(\vec{\mathrm{k}})|\nu\rangle = -nc(\vec{\mathrm{k}})v_0 k\,\delta_{\mu,2}\delta_{\nu,1}. \tag{6.5}$$

Clearly, the first term of Eq. (6.4) represents the projection of the operator $\omega^0 + \Sigma$ onto the P subspace where K vanishes. In the limit $k \to 0$ and $z = i\epsilon \to 0$, the second term involves only the operator $QK^{-1}Q$ and is finite. Therefore, apart from the branch line on the real axis, $\Omega_{\mu\nu}(\vec{\mathrm{k}}z)$ is well behaved for small k and z.

The Fokker-Planck equation leads to a corresponding matrix $\Omega^{\mathrm{kin}}_{\mu\nu}(\vec{\mathrm{k}}z)$:

$$\Omega^{\mathrm{kin}}_{\mu\nu}(\vec{\mathrm{k}}z) = \langle\mu|\omega^0(\vec{\mathrm{k}})|\nu\rangle + \langle\mu|\omega^0(\vec{\mathrm{k}})Q \times [z - Q\omega^0(\vec{\mathrm{k}})Q + iK]^{-1}Q\omega^0(\vec{\mathrm{k}})|\nu\rangle. \tag{6.6}$$

To find the poles of $g_{nn}(\vec{\mathrm{k}}z) \sim G_{11}(\vec{\mathrm{k}}z)$, we must examine the zeros of the 3×3 matrix $[z - \Omega(\vec{\mathrm{k}}z)]$, that is, we must solve

$$\det|z - \Omega(\vec{\mathrm{k}}z)| = 0. \tag{6.7}$$

We know from Sec. II that the "causal" branch of $g_{nn}(\vec{\mathrm{k}}z)$ has no poles in the upper half of the complex z plane, as long as $1 - nc(\vec{\mathrm{k}}) > 0$. Henceforth, by $\Omega(\vec{\mathrm{k}}z)$ in Eqs. (6.3) and (6.7), we will mean the function which satisfies (6.4) for $\mathrm{Im}z > 0$, and is analytically continued past the branch cut onto a second sheet in the lower half z plane where Eq. (6.7) does have solutions.

In subsequent sections, we shall examine the roots $z = z(\vec{\mathrm{k}})$ of the determinantal equation continued in this manner. We shall show that for small wave numbers k these roots agree with the hydrodynamic modes (4.3).

A. Pole Structure at $k = 0$

To corroborate hydrodynamics, we must first show that $g_{nn}(\vec{\mathrm{k}}z)$ has poles that move towards $z = 0$ as $k \to 0$. For the Fokker-Planck equation, this is trivial. Clearly, $\Omega^{\mathrm{kin}}_{\mu\nu}(\vec{\mathrm{k}}z) \to 0$ as $k \to 0$ so that $\det|z - \Omega^{\mathrm{kin}}(0z)| = (z)^3 = 0$.

Now consider $\Omega_{\mu\nu}(\vec{\mathrm{k}}z)$ as $k \to 0$. We know from Eqs. (4.12) that

$$\langle n|\Sigma^{(c)}(0z) = \langle g_l|\Sigma^{(c)}(0z) = 0.$$

Therefore, Ω_{33} is the only component of $\Omega_{\mu\nu}(\vec{\mathrm{k}}z)$ that survives as $k \to 0$, and

$$\det|z - \Omega(0z)| = z^2(z - \Omega_{33}(0z)) = 0. \tag{6.8}$$

But we see from (4.12c) that $\Omega_{33}(0z)$ also vanishes as $z \to i\epsilon \to 0$. Hence, there are still three roots at $z = 0$. While there are no solutions to (6.8) in the upper half z plane, there probably are additional solutions on the second sheet in the lower half z plane on which $\Omega_{33}(0z)$ is continued. We shall not be concerned with these poles. Presumably they have nonvanishing imaginary parts for small k and describe modes that die out faster with time than the gapless or hydrodynamic modes. In addition, since $\Omega_{\mu\nu}(0z)$ is diagonal, they must appear in $g_{nn}(\vec{\mathrm{k}}z)$ with a strength that vanishes in the limit of small wave number k.

Two additional points deserve mention. (a) Throughout this work we have assumed that the pair potential is of finite range and nonsingular. Hence, we do not expect a dispersion relation with a real gap, that is, a plasmalike dispersion relation. For the weak-coupling mass operator, we can show there is none. If some real $\omega_p \neq 0$ were a solution to (6.8), we would have $\mathrm{Im}\Omega_{33}(0\omega_p) = 0$, and therefore,

$$\langle\epsilon|\Gamma(0, \omega_p)|\epsilon\rangle = 0, \tag{6.9}$$

since the imaginary parts of both terms in (6.4) are not positive. However, in Sec. II C, we pointed out that (6.9) is only satisfied when $\omega_p = 0$. (b) For Coulomb forces, much of the present analysis must be altered. Plasma oscillations occur since $c(\vec{\mathrm{k}})$ diverges as k^{-2}. Moreover, the kinetic operator K in (1.14) does not exist if $v(\vec{\mathrm{k}}) \sim k^{-2}$ since the friction constant ν in (1.17) diverges logarithmically. Screening corrections remedy the long-range divergence and other correlations must be included to modify the short-distance divergence.

B. Expansion for Small k and z

In order to compute the hydrodynamic poles, we need a manageable expression for $\Omega(\vec{\mathrm{k}}z)$ which is correct in the region of small k and z. For the Fokker-Planck equation, such an expression can be written down by inspection. $\langle\mu|\omega^0(\vec{\mathrm{k}})|\nu\rangle$ is of first order in k and is easily computed. The second term in (6.6) is of order k^2. We can therefore set $z = k = 0$ in the inverse operator, and obtain

$$\Omega^{\mathrm{kin}}_{ij}(\vec{\mathrm{k}}z) \approx v_0 k \begin{pmatrix} 0 & 1 & 0 \\ 1 & -iv_0 k\gamma^{\mathrm{kin}}_{22} & \sqrt{\tfrac{2}{3}} \\ 0 & \sqrt{\tfrac{2}{3}} & -iv_0 k\gamma^{\mathrm{kin}}_{33} \end{pmatrix}, \tag{6.10}$$

which is correct to second order in k and z. The coefficients are given by the matrix elements

$$\gamma_{22}^{\mathrm{kin}} = \langle \xi_3^2 | Q K^{-1} Q | \xi_3^2 \rangle , \tag{6.10$'$a}$$

$$\gamma_{33}^{\mathrm{kin}} = \tfrac{1}{6} \langle \xi_3 (\xi^2 - 3) | Q K^{-1} Q | \xi_3 (\xi^2 - 3) \rangle . \tag{6.10$'$b}$$

For the full $\Omega_{\mu\nu}(\vec{k} z)$ of Eq. (6.4), our task is more complicated. There are corrections to (6.10) in both the first- and second-order terms, and $\Omega_{\mu\nu}(\vec{k} z)$ depends on z, even to first order. Let us write down the result. Omitting only terms of third order in k and z (that is, terms that vanish as k^3, $k^2 z$, or z^3), we find that $\Omega(\vec{k} z)$ can be written in the following form:

$$\Omega_{\mu\nu}(\vec{k} z) = v_0 k \, \Omega_{\mu\nu}^{[1]}(x) + (v_0 k)^2 \Omega_{\mu\nu}^{[2]}(x) , \tag{6.11}$$

where $x = z / v_0 k$, and the first- and second-order matrices are

$$\Omega_{\mu\nu}^{[1]}(x) = \begin{pmatrix} 0 & 1 & 0 \\ 1 - nc(0) & 0 & \sqrt{\tfrac{2}{3}}(1 + \tfrac{3}{4}\alpha) \\ 0 & \sqrt{\tfrac{2}{3}}(1 + \tfrac{3}{4}\alpha) & -x\alpha \end{pmatrix} , \tag{6.12}$$

$$\Omega_{\mu\nu}^{[2]}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i\gamma_{22} & ix\gamma_{23}\sqrt{\tfrac{2}{3}} \\ 0 & ix\gamma_{23}\sqrt{\tfrac{2}{3}} & -\tfrac{2}{3} i \gamma_{33} - i x^2 \bar{\gamma}_{33} \end{pmatrix} . \tag{6.13}$$

Some details of the calculation of $\Omega^{[1]}$ and $\Omega^{[2]}$ are given in the Appendix. The linear term $\Omega^{[1]}$ comes entirely from the first line in (6.4). It contains first-order corrections to Ω^{kin} which reflect the properties (4.12) of $\Sigma^{(c)}$. The positive constant α is given by

$$\alpha = \frac{n}{3} \int \frac{d\vec{k}}{(2\pi)^3} (\beta v(\vec{k}))^2 . \tag{6.12$'$}$$

The collision term in the first line of (6.4) also contributes to the second order $\Omega^{[2]}$. This part can be expressed in terms of the positive constants γ_η'' and γ_κ'' defined by

$$\gamma_\kappa'' = \frac{n\sqrt{\pi}}{24 v_0} \int \frac{d\vec{k}}{(2\pi)^3} \, k \left[\left(\frac{d\beta v(\vec{k})}{dk} \right)^2 + 8 \left(\frac{\beta v(\vec{k})}{k} \right)^2 \right] . \tag{6.14}$$

Then the coefficients in $\Omega^{[2]}$ are of the form

$$\gamma_{22} = \tfrac{4}{3} \gamma_\eta' + \gamma' + \tfrac{1}{4} \gamma_\kappa'' + \tfrac{19}{8} \gamma_\eta'' , \tag{6.13$'$a}$$

$$\gamma_{23} = 2\gamma' + \tfrac{1}{2} \gamma_\kappa'' - \tfrac{5}{4} \gamma_\eta'' , \tag{6.13$'$b}$$

$$\gamma_{33} = \gamma_\kappa' + \gamma_\kappa'' , \tag{6.13$'$c}$$

$$\bar{\gamma}_{33} = \tfrac{8}{3} \gamma' + \tfrac{1}{2} \gamma_\kappa'' - \tfrac{5}{4} \gamma_\eta'' . \tag{6.13$'$d}$$

The primed coefficients represent the contributions of the second line of (6.4). They are all given as diagonal matrix elements of the inverse kinetic collision operator K^{-1}, and thus, involve essentially the same computational problems as those that arise from kinetic theory. They can be written in the form

$$\tfrac{4}{3} \gamma_\eta' = \langle \xi_3^2 + \tilde{T}_{33} | Q K^{-1} Q | \xi_3^2 + \tilde{T}_{33} \rangle , \tag{6.15a}$$

$$4\gamma_\kappa' = \langle \xi_3(\xi^2 - 3) + J_3^\epsilon | Q K^{-1} Q | \xi_3(\xi^2 - 3) + J_3^\epsilon \rangle , \tag{6.15b}$$

$$16\gamma' = \langle E | Q K^{-1} Q | E \rangle . \tag{6.15c}$$

All three coefficients are positive, since K^{-1} is a positive-definite operator, and finite because of the Q projection. The real functions $E(\vec{\xi})$ and $J_3^\epsilon(\vec{\xi})$ are the $k \to 0$, $z = i\epsilon \to 0$ limits of the potential energy and energy flux functions which we have defined in Eqs. (3.19) or (4.12c). Explicitly, they are given by

$$-E(\vec{\xi}) \phi(\vec{\xi}) = 6\alpha \, \partial_l \phi(\vec{\xi}) (\partial_l \ln |\vec{\xi} - \vec{\bar{\xi}}|) \phi(\vec{\bar{\xi}}) , \tag{6.16a}$$

$$-J_3^\epsilon(\vec{\xi}) \phi(\vec{\xi}) = 3\alpha \, \partial_l \phi(\vec{\xi}) (\partial_l \partial_3 \ln |\vec{\xi} - \vec{\bar{\xi}}|) \phi(\vec{\bar{\xi}}) . \tag{6.16b}$$

$\tilde{T}_{33}(\vec{\xi})$ is a component of the symmetrical and traceless tensor $\tilde{T}_{ij}(\vec{\xi})$ defined by

$$\tilde{T}_{ij}(\vec{\xi}) \phi(\vec{\xi}) = \tfrac{3}{2} \alpha \, \partial_l \phi(\vec{\xi}) \partial_l [(\xi - \bar{\xi})_i \times (\xi - \bar{\xi})_j |\vec{\xi} - \vec{\bar{\xi}}|^{-2}] \phi(\vec{\bar{\xi}}) . \tag{6.16c}$$

The tensor $T_{ij}(\vec{\xi})$ is related to the "stress tensor" function $T_{ij}(\vec{k} z ; \vec{\xi})$ of (3.16) and (3.17) by

$$\lim T_{ij}(0 i\epsilon ; \vec{\xi}) = \tilde{T}_{ij}(\vec{\xi}) + \tfrac{1}{4} \delta_{ij} E(\vec{\xi}) . \tag{6.16d}$$

The relation is easily verified by direct calculation.

If we compare $\Omega(\vec{k} z)$ of (6.11) to its kinetic equivalent, we find corrections to every matrix element. Correspondingly, we shall find corrections to all transport coefficients; in particular, the bulk viscosity is entirely due to them.

In Sec. VI C, we will use (6.11) to calculate the hydrodynamic poles, and identify the thermodynamic and transport coefficients.

C. Velocity of Sound

In obtaining the lowest-order contributions to $z(k)$, we may omit the second-order matrix $\Omega_{ij}^{[2]}$. The determinantal equation (6.7) is then easily solved

$$x_1 = 0 , \tag{6.17a}$$

$$x_{2,3}^2 = \left(\frac{c}{v_0} \right)^2 = \left(\frac{2}{3} \frac{(1 + \tfrac{3}{4}\alpha)^2}{1 + \alpha} + [1 - nc(0)] \right) . \tag{6.17b}$$

The first root $x = 0$ is consistent with the heat diffusion pole of $g_{nn}(\vec{k} z)$ [cf., Eq. (4.3)]. The double root, (6.17b) at $z = \pm ck$, is interpreted as a sound pole. The sound velocity c is real since $1 - nc(0) > 0$ whenever the system is thermodynamically stable.

If Eq. (6.17b) is the isentropic sound velocity, we should be able to calculate the specific-heat ratio c_p/c_v on the basis of (4.5). Specifically, since $(\partial p/\partial mn)_T = v_0^2[1-nc(0)]$, substitution of (6.17b) in (4.5) should give the proper value for

$$\frac{c_p}{c_v} = 1 + \frac{2}{3}\,\frac{(1+\frac{3}{4}\alpha)^2}{(1+\alpha)\,[1-nc(0)]} \quad . \tag{6.18}$$

We will verify this result thermodynamically and from the pole strength of the heat diffusion pole.

The corresponding kinetic results are simply obtained by omitting α and $nc(0)$. Then the kinetic sound velocity is

$$c^{\mathrm{kin}} = v_0\sqrt{\tfrac{5}{3}} \tag{6.19a}$$

corresponding to a specific-heat ratio of

$$(c_p/c_v)^{\mathrm{kin}} = \tfrac{5}{3} \quad . \tag{6.19b}$$

This is the free-gas result; it expresses the well-known fact that a kinetic equation of the Boltzmann type is only compatible with free-gas thermodynamics.

By contrast, the dynamic result (6.17b) for the sound velocity is identical to the thermodynamic derivative $(\partial p/\partial mn)_S$. To verify this fact, we write the derivative in terms of the energy density ϵ and the pressure p:

$$\left(\frac{\partial p}{\partial mn}\right)_S = \left(\frac{\partial p}{\partial mn}\right)_T + \frac{T}{mn}\left(\frac{\partial p}{\partial \epsilon}\right)_n^2 \quad . \tag{6.20}$$

Both ϵ and p can be determined from the direct correlation function since

$$\beta\left(\frac{\partial p}{\partial n}\right)_\beta = [1-nc(0)]\,, \qquad p(n=0)=0 \tag{6.21}$$

$$\left(\frac{\partial(\epsilon/n)}{\partial n}\right)_\beta = \frac{1}{n^2}\left(\frac{\partial(p\beta)}{\partial \beta}\right)_n, \quad \left(\frac{\epsilon}{n}\right)_{n=0} = \frac{3}{2\beta}\,. \tag{6.22}$$

With $c(\vec{k})$ given in our weak-coupling approximation by (1.11), we obtain

$$p = n/\beta + \tfrac{1}{2}n^2 v(\vec{k}=0) - \tfrac{1}{4}n^2\beta[v^2(\vec{k}=0)], \tag{6.23}$$

$$\epsilon = \tfrac{3}{2}(n/\beta) + \tfrac{1}{2}n^2 v(\vec{k}=0) - \tfrac{1}{2}n^2\beta[v^2(\vec{k}=0)], \tag{6.24}$$

where

$$[v^2(\vec{k}=0)] = \int \frac{d\vec{k}}{(2\pi)^3}\, v^2(\vec{k}) = \int d\vec{r}\, v^2(\vec{r})\,.$$

We therefore see that (6.17b) is just the statement

$$c^2 = \left(\frac{\partial p}{\partial mn}\right)_S .$$

We have therefore shown that when potential corrections are consistently treated in a microscopic dynamic calculation they lead to the same velocity as that predicted by hydrodynamics arguments for the same system.

D. Heat Diffusion

Next, let us solve the determinant equation (6.7) to second order, inserting $x = z/v_0k = 0$ for the first-order solution. Omitting only terms of order k^3, we find

$$z = -iD_T k^2, \text{ with } D_T = \tfrac{2}{3}v_0^2\,\frac{\gamma_{33}}{1+\alpha}\,\frac{c_v}{c_p}\,. \tag{6.25}$$

The thermal diffusivity D_T has been identified by comparison with (4.3). It is important to note that D_T is real and positive. The corresponding Fokker-Planck result is obtained by setting $\alpha = 0$, $c_v/c_p = \frac{3}{5}$ and replacing γ_{33} by $\gamma_{33}^{\mathrm{kin}}$.

According to hydrodynamics, a fraction $(1-c_v/c_p)$ of the total excitation strength in $g_{nn}(\vec{k}z)$ is carried by the heat diffusion pole, while the sound poles have the remaining fraction of the strength. In (6.25), we used c_v/c_p as obtained from the velocity of sound [Eq. (6.18)]. To verify this assignment, we calculate from (6.3) and (6.11) the residue of the pole (6.25). After some algebra, we obtain

$$G_{11}^{\mathrm{heat}}(\vec{k}z) = \frac{1-(c_v/c_p)}{z+ik^2D_T} \quad , \tag{6.26}$$

with the same specific-heat ratio as in (6.18). Thus our approximation also gives the proper coupling, i.e., the hydrodynamically predicted coupling between sound propagation and heat diffusion.

We obtain the thermal conductivity κ from the diffusivity D_T by using the relation $\kappa = mnc_pD_T$. If we calculate c_v from (6.24) as

$$mnc_v = \left(\frac{\partial\epsilon}{\partial T}\right)_n = \tfrac{3}{2}k_B n(1+\alpha)\,, \tag{6.27}$$

we find that the heat conductivity is

$$\kappa = k_B v_0^2 n \gamma_{33}\,, \tag{6.28}$$

where k_B is Boltzmann's constant. From (6.13′c), we therefore obtain

$$\kappa' = \gamma'_\kappa\, k_B n v_0^2\,, \tag{6.29a}$$

$$\kappa'' = \gamma''\, k_B n v_0^2\,. \tag{6.29b}$$

The two parts of $\kappa = \kappa' + \kappa''$ have the same physical origin as η' and η'' in Sec. V, and the same dependence on density and temperature,

$$\kappa^{\mathrm{kin}}(T) = A_\kappa T^{5/2}\,,$$

$$\kappa(n,T) = \kappa^{\mathrm{kin}}(T)\left(1 + B_\kappa\frac{n}{T^2} + C_\kappa\frac{n^2}{T^4}\right) \quad . \tag{6.30}$$

The positive constants A_κ and C_κ and the real constant B_κ can be found from (6.19) and (6.15b). The Fokker-Planck result is obtained by omitting κ'', and the correlational J_3^ϵ in (6.15b).

E. Damping of Sound and the Bulk Viscosity

Let us now use $z = \pm ck$ to first order, and solve the characteristic equation (6.7) to order k^2. We find

$$z = \pm\, ck - \tfrac{1}{2} i\, \Gamma k^2 . \tag{6.31}$$

The sound wave damping constant Γ can be rewritten

$$\Gamma = D_T [(c_p/c_v) - 1] + D_l , \tag{6.32a}$$

$$D_l = v_0^2 (\gamma_{22} - 2y\gamma_{23} + y^2 \gamma_{33}) , \tag{6.32b}$$

$$y = (1 + \tfrac{3}{4}\alpha)(1+\alpha)^{-1} = \frac{3}{2}\left(\frac{dp}{dT}\right)_n (mnc_v)^{-1} . \tag{6.32c}$$

By substituting for D_T and using the thermodynamic evaluations of the previous sections, we see that Γ is real, and we shall see below that it is positive.

From Eqs. (6.3) and (6.11), we can calculate the residues of the two sound poles. Omitting only terms of order k^2, we find

$$G_{11}^{\text{sound},\pm}(\vec{k}\, z) = \frac{1}{2}\,\frac{c_v}{c_p}\,\frac{1 \pm i(k/c)\left[\frac{1}{2}\Gamma + D_T(c_p/c_v - 1)\right]}{z \mp ck + \frac{1}{2} i\, \Gamma k^2} , \tag{6.33}$$

or adding the two terms,

$$G_{11}^{\text{sound}}(\vec{k}\, z) = \frac{c_v}{c_p}\,\frac{z + i k^2 \left[\Gamma + D_T(c_p/c_v - 1)\right]}{z^2 - c^2 k^2 + i z k^2\, \Gamma} , \tag{6.34}$$

in full agreement with the hydrodynamic result (4.3). From Γ, we determine the bulk viscosity by using the relation

$$mnD_l = (\tfrac{4}{3}\eta + \zeta) . \tag{6.35}$$

The shear viscosity was calculated in Sec. V. Subtracting it, we obtain the following two contributions for the bulk viscosity:

$$\zeta' = \frac{mnv_0^2}{[12(1+\alpha)]^2}\, \langle E | Q K^{-1} Q | E \rangle \equiv \frac{mnv_0^2 \gamma'}{9(1+\alpha)^2} , \tag{6.36a}$$

$$\zeta'' = \frac{mnv_0^2}{24(1+\alpha)^2}\, \left[(-2\gamma_\kappa'' + 45\gamma_\eta'')(1+\alpha)^2 + \tfrac{1}{4}\alpha^2 (2\gamma_\kappa'' - 5\gamma_\eta'')\right] . \tag{6.36b}$$

In view of the easily proven inequalities

$$\tfrac{45}{2}\gamma_\eta'' \geqslant \gamma_\kappa'' \geqslant \tfrac{5}{2}\gamma_\eta'' , \tag{6.37}$$

the approximate bulk viscosity $\zeta = \zeta' + \zeta''$ and Γ are positive definite.

It is well known that the Fokker-Planck equation predicts no bulk viscosity: The bulk viscosity in a simple fluid results from collisional transfer of kinetic into potential energy and the Boltzmann equation omits this feature.

Since such processes involve pairs of molecules, we expect $\zeta \sim n^2$. Indeed, the density and temperature dependence takes the form

$$\zeta = A n^2 T^{-3/2} [1 + O(nT^{-2})] . \tag{6.38}$$

VII. SUMMARY AND DISCUSSION

Although the basic weak-coupling approximation (1.7) and (1.8) is relatively simple, we have shown that its solution is correct at least in those regions of wave number and frequency about which we have some knowledge. For short times, this knowledge stems from the frequency sum rules, and the first few of these are relatively easy to satisfy. Indeed, several models that have been recently proposed are designed to satisfy the first few sum rules exactly. The simplest of these is the collisionless Vlasov equation that Nelkin and Ranganathan and others have studied. It is obtained by omitting $\Sigma^{(c)}(\vec{k}\, z)$ altogether, and fulfills the first two sum rules (2.5a) and (2.5b) exactly. An improved equation can be obtained by setting $\Sigma^{(c)}(\vec{k}t)$ equal to $\Sigma^{(c)}(\vec{k}0)W(t)$, where $W(t)$ is some numerical function. Then, the next sum rule (2.5c) is also fulfilled. Such a model has been discussed by Lebowitz Percus, and Sykes.[13] This model can still be exactly solved. However, since it is difficult to conserve energy it is probably more appropriate for the description of self-diffusion. For short times, it is certainly correct. Nelkin and Ortoleva[14] have suggested a somewhat different scheme based on Zwanzig's[15] high-frequency collective modes. This scheme can also be phrased as the beginning of a Taylor expansion in time, and is again valid for short times. Our own approximation makes no reference to space and time. It therefore permits us to study the attenuation of such high-frequency modes. In particular, because of the analyticity of $\Sigma^{(c)}(\vec{k}\, z)$, all frequency moments of $S'(\vec{k}\, \omega; \vec{\xi}\, \vec{\xi}')$ exist and are correct at least to order λ^2. For Markovian kinetic models, higher moments generally diverge.

Our equation is also correct in the other limit, that of long times and slow spatial variation. In this limit, our knowledge stems from the phenomenological equations of hydrodynamics, and is much more difficult to incorporate in a fundamental theory. By explicitly deriving $S_{nn}(\vec{k}\, \omega)$ and $g_t(\vec{k}\, \omega)$ from Eq. (1.8) in the limit of small k and ω, we have been able to show that Eq. (1.8) is in full agreement with the predictions of the Navier-Stokes equations. To our knowledge, this constitutes the first explicit derivation of linearized hydrodynamics with determined transport coefficients, for a physcially well-defined albeit extremely simple fluid system.

The thermodynamic derivatives and transport coefficients can all be obtained from the density and momentum density correlation functions, and we have restricted our discussion primarily to these. We have also derived, by the same technique, the hydrodynamic limit of the entropy density-density correlation function.

In our results, the transport coefficients are given in terms of matrix elements of the inverse Fokker-Planck operator K^{-1}, and their numerical computation would pose no greater problem than a corresponding calculation that starts from the Fokker-Planck equation. By contrast, however, we find correlational corrections to all transport coefficients which are of first and second order in the density, and therefore vanish only in the low-density limit when the Fokker-Planck equation seems to be appropriate. In particular, our theory gives a nonzero value for the bulk viscosity whose leading term is of the order n^2 suggested by physical considerations. We do not find any divergence difficulties of the type encountered in calculations of transport coefficients by Kawasaki and Oppenheim[16] and others.[17] However, we do not feel that our calculation has a bearing on the divergence of the density expansion since that divergence arises from terms of high order in the expansion of $\Sigma(\vec{k}z)$.

In this context, it may be well to stress that our calculation does not represent an expansion of the transport coefficients to second order in the parameter λ. Indeed, it is not difficult to see that there are contributions of orders 1 and λ to the transport coefficients (of dominant order $1/\lambda^2$) which we did not retain. Our primary aim has been to obtain consistent power series expansions for all dynamic properties, not expansions especially suited to transport coefficients. Instead, we made approximations which treat the dynamics consistently. Consistent approximations are best carried out by expressing the mass operator $\Sigma(\vec{k}z)$ in terms of $S(\vec{k}z)$. There is no difficulty, in principle, in calculating transport coefficients to order λ^n. However, to do so and maintain consistent dynamics, it would be necessary to include all terms in an expansion of Σ in powers of S to order λ^{n+4}, since the transport coefficients from the term in $\Sigma \sim \lambda^2$ begin at order λ^{-2}.

Apart from the conservation laws, hydrodynamic behavior ensues from a rather delicate balance of collision processes which ensures that local equilibrium is dynamically established and maintained. In our representation, thermodynamics can be derived from $\Sigma^{(s)}(\vec{k})$ which is given in terms of the static direct correlation functions, while the dynamics is chiefly determined by the collision operator $\Sigma^{(c)}(\vec{k}z)$. The thermodynamic derivatives which are parameters in hydrodynamics, can therefore be independently calculated from the statics and dynamics. We have seen that our results for the speed of sound and the specific-heat ratio are completely consistent in this sense.

This consistency is achieved by using the same approximation on both parts of the mass operator $\Sigma^{(s)}$ and $\Sigma^{(c)}$ and subsequently calculating the initial condition $S^0(\vec{k};\vec{\xi}\vec{\xi}')$ from $\Sigma^{(s)}(\vec{k}) = \lim \Sigma(\vec{k}z)$ as $z \to \infty$. Indeed, we feel that such a procedure should serve as a restriction on better approximation schemes. In quantum statistical mechanics, we know how to obtain dynamical and static properties simultaneously, by using temperature Green's functions. Equation (1.8) has been derived from a similar starting point in two ways.[18]

Since our weak-coupling approximation imposes no restriction on the values of wave number and frequency, Eq. (1.7) is presumably equally valid for all regions of k and ω. For the intermediate region, there is little that can be said, except that the equation fulfills such general requirements as stability and the conservation laws. An analytical solution in this region is, of course, impossible; however, the numerical integration of (1.8) for $S_{nn}(\vec{k}\omega)$ appears feasible if a sufficiently simple force law is provided. To the extent that hard cores are inessential, such calculations should explain sound propagation and attenuation when hydrodynamics is inapplicable.

Note added in manuscript. After this work was completed we learned of work[19] by P. Resibois which employed very similar techniques. Resibois's work is complementary to ours. By not retaining the "destruction fragment," he confines his analysis to a special unphysical initial state, but in this state, he retains all terms in a potential expansion. Since any initial configuration can be used in the hydrodynamic regime, he demonstrates for more general potentials than we have discussed, that the potential expansions of the kinetic theory expression and the Kubo formula agree. However, since his initial configuration omitting the unanalyzed destruction fragment is not the equilibrium configuration, his method is not presently suited to the discussion of the kinetic nonvanishing frequency behavior, and thus to the discussion of inelastic scattering. We have benefited from our correspondence with Resibois.

APPENDIX

In this Appendix, we furnish some of the detail omitted in the calculation of $\Omega_{ij}(\vec{k}z)$ to second order. Specifically, we shall show how the matrix element $\Omega_{33}(\vec{k}z)$ is obtained. (a) First, consider the first line of (6.4), i.e., the matrix element $\langle \epsilon | \Sigma^{(c)}(\vec{k}z) | \epsilon \rangle$. The function $\Sigma^{(c)}(\vec{k}z)$ is invariant under spatial rotation, as is $\epsilon = (1/\sqrt{6})(\xi^2 - 3)$. It is clear, therefore, that to second order, there will be terms $\sim z$, z^2, and k^2, but no term $\sim kz$. We can therefore simplify the bookkeeping by setting $k=0$ first. Substituting, we have

$$\langle\epsilon|\Sigma^{(c)}(0z)|\epsilon\rangle$$

$$=\frac{-2n}{3m^2v_0^2}\,v^2(\vec{\mathrm{k}})\vec{\mathrm{k}}\cdot\vec{\xi}\,\vec{\mathrm{k}}\cdot(\vec{\xi}-\vec{\xi}\,')\,\frac{\phi(\vec{\xi})\,\phi(\vec{\xi}\,')}{v_0\vec{\mathrm{k}}\cdot(\vec{\xi}\,'-\vec{\xi})-z}$$

$$=\frac{2nz}{3m^2v_0^3}\,v^2(\vec{\mathrm{k}})\vec{\mathrm{k}}\cdot\vec{\xi}\,\frac{\phi(\vec{\xi})\,\phi(\vec{\xi}\,')}{v_0\vec{\mathrm{k}}\cdot(\vec{\xi}\,'-\vec{\xi})-z}\,,$$

since $v(\vec{\mathrm{k}})$ is even. Then, substituting $\vec{\mathrm{k}}\to-\vec{\mathrm{k}}$, $\vec{\xi}\to\vec{\xi}\,'$, we obtain

$$\langle\epsilon|\Sigma^{(c)}(0z)|\epsilon\rangle$$

$$=\frac{-nz}{3m^2v_0^4}\,v^2(\vec{\mathrm{k}})-\frac{nz^2}{3m^2v_0^4}\,v^2(\vec{\mathrm{k}})\,\frac{\phi(\vec{\xi})\,\phi(\vec{\xi}\,')}{v_0\vec{\mathrm{k}}\cdot(\vec{\xi}\,'-\vec{\xi})-z}\,.$$

In the term $\sim z^2$, we set $z=i\epsilon$ in the denominator, and use

$$\int d\vec{\xi}\int d\vec{\xi}\,'\;\phi(\vec{\xi})\,\delta|(v_0\vec{\mathrm{k}}\cdot(\vec{\xi}-\vec{\xi}\,'))\,\phi(\vec{\xi}\,')=(2\sqrt{\pi}v_0\vec{\mathrm{k}})^{-1}.$$

We then find

$$\langle\epsilon|\Sigma^{(c)}(0z)|\epsilon\rangle=-z\alpha-iz^2(\tfrac{1}{2}\gamma_\kappa''-\tfrac{5}{4}\gamma_\eta'')+O(z^3)\,.\qquad\text{(A1)}$$

Now we set $z=i\epsilon$, in the terms of order k^2, obtaining

$$\langle\epsilon|\Sigma^{(c)}(\vec{\mathrm{k}}i\epsilon)|\epsilon\rangle=\tfrac{1}{6}(v_0k)\langle J_3^\epsilon(\vec{\mathrm{k}}i\epsilon)\,|\,\xi^2\rangle\,.$$

At $k=0$, $\langle J_3^\epsilon(0i\epsilon)|\xi^2\rangle=0$; therefore, we must expand $J_3^\epsilon(\vec{\mathrm{k}}i\epsilon)$ to first order in k. The integration proceeds as above:

$$\langle J_3^\epsilon(\vec{\mathrm{k}}i\epsilon)\,|\,\xi^2\rangle=i(v_0k)\,4\gamma_\kappa''\,.\qquad\text{(A2)}$$

Altogether, we obtain, to second order,

$$\langle\epsilon|\Sigma^{(c)}(\vec{\mathrm{k}}z)|\epsilon\rangle=-z\alpha-iz^2(\tfrac{1}{2}\gamma_\kappa''-\tfrac{5}{4}\gamma_\eta'')-\tfrac{2}{3}i(v_0k)^2\gamma_\kappa''\,.\qquad\text{(A3)}$$

We now turn to the second line of (6.4). Using (3.19), we may write it as

$$\sqrt{6}\langle\epsilon|\,[\omega^0(\vec{\mathrm{k}})+\Sigma^{(c)}(\vec{\mathrm{k}}z)]=(v_0k)\,\langle\xi_3(\xi^2-3)+J_3^\epsilon(\vec{\mathrm{k}}z)|\;-z\,\langle E(\vec{\mathrm{k}}z)|\,.$$

Next we note that at $k=0$ and $z=i\epsilon$, E is even under $\xi_3\to-\xi_3$, while $[\xi_3(\xi^2-3)+J_3^\epsilon]$ is odd. Therefore, to second order, we have

$$\langle\epsilon|[\omega^0(\vec{\mathrm{k}})+\Sigma^{(c)}(\vec{\mathrm{k}}z)]\,Q\{z-Q[\omega^0(\vec{\mathrm{k}})+\Sigma^{(c)}(\vec{\mathrm{k}}z)]Q\}^{-1}$$

$$\times Q[\omega^0(\vec{\mathrm{k}})+\Sigma^{(c)}(\vec{\mathrm{k}}z)]|\epsilon\rangle$$

$$=-\tfrac{1}{6}i(v_0k)^2\langle\xi_3(\xi^2-3)+J_3^\epsilon|QK^{-1}Q|\xi_3(\xi^2-3)+J_3^\epsilon\rangle$$

$$-\tfrac{1}{6}i\,z^2\langle E|QK^{-1}Q|E\rangle\,.\qquad\text{(A4)}$$

Combining (A3) and (A4), we find the result

$$\Omega_{33}(\vec{\mathrm{k}}z)=-z\alpha-\tfrac{2}{3}iz^2\gamma_{33}-i(v_0k)^2\bar{\gamma}_{33}\qquad\text{(A5)}$$

quoted in the text in Eqs. (6.12) and (6.15). The calculation of the other matrix elements proceeds in an analogous fashion.

---

*Supported in part by a grant from the National Science Foundation.

†Present address: Department of Chemistry, Columbia University, New York, N. Y. 10027.

[1] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962).

[2] R. Zwanzig, *Lectures in Theoretical Physics*, edited by W. E. Brittin (Wiley, New York, 1961).

[3] G. E. Uhlenbeck and G. W. Ford, *Lectures in Statistical Mechanics* (American Mathematical Society, Providence, R. I., 1963).

[4] M. Nelkin and S. Ranganathan, Phys. Rev. 164, 222 (1967); J. L. Lebowitz, J. K. Percus, and J. Sykes, *ibid.* 171, 264 (1968); T. Morita, Progr. Theoret. Phys. (Kyoto) 40, 758 (1968).

[5] A. Z. Akcasu and E. Daniels, Bull. Am. Phys. Soc. 14, 75 (1969); A. Z. Akcasu and J. J. Duderstadt, Phys. Rev. 188, 479 (1969). A. Z. Akcasu, N. Corngold, and J. J. Duderstadt (Report of work prior to publication).

[6] H. Mori, Progr. Theoret. Phys. (Kyoto) 33, 423 (1965).

[7] R. Brout and I. Prigogine, Physica 22, 621 (1956); I. Prigogine, *Non-Equilibrium Statistical Mechanics* (Wiley, New York, 1963); P. Resibois, *N – Particle Physics*, edited by E. Meeron (Gordon and Breach, New York, 1967); J. M. J. Van Leeuwen and S. Yip, Phys. Rev. 139, A1138 (1965).

[8] L. P. Kadanoff and P. C. Martin, Ann. Phys. (N. Y.) 24, 419 (1963).

[9] P. C. Martin, *Statistical Mechanics in Equilibrium and Non-Equilibrium*, edited by J. Meixner (North-Holland, Amsterdam, 1969), p. 100.

[10] H. Grad, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. XII, p. 205.

[11] B. J. Berne, J. P. Boon, and S. A. Rice, J. Chem. Phys. 45, 1086 (1966).

[12] Expressions for the transport coefficients of this form have been derived by P. Resibois, J. Chem. Phys. 41, 2979 (1964); and Nicolis-Severne, *ibid.* 44, 1477 (1966).

[13] J. L. Lebowitz, J. K. Percus, and J. Sykes, Phys. Rev. 188, 489 (1969).

[14] P. Ortoleva and M. Nelkin (Report of work prior to publication); also, Phys. Rev. 181, 429 (1969).

[15] R. Zwanzig, Phys. Rev. 156, 190 (1967); R. Nossal and R. Zwanzig, *ibid.* 157, 120 (1967).

[16] K. Kawasaki and I. Oppenheim, Phys. Rev. 139, A1763 (1965).

[17] J. R. Dorfman and E. G. D. Cohen, Phys. Letters 16, 124 (1965). Y. Pomeau, Phys. Letters 27A, 601 (1968).

[18] We have developed a purely classical derivation, based upon representations of the operator algebra $f(\vec{\mathrm{r}}\,\vec{\mathrm{p}}\,t)$ and generators of canonical transformations. The result has also been recently obtained by taking the classical limit of the quantum equation for the two particle Green's function by G. Mazenko.

[19] P. Resibois (Report of work prior to publication).