which means that $\sigma_0^+(t) \geq 0$. Since $\sigma_0 \geq 0$ and $\alpha_1 gt \geq 0$, then from Eq. (D13) we have $\sigma_0^+(t) \geq 0$ for all time, but $\sigma_0^-(t)$ is non-negative only as long as

$$\sigma_0 \geq \alpha_1 gt\,(1 - 2\alpha_1 gt)^{-1} .$$

Keeping terms to only first order in gt, and letting $\alpha_1 g = g_1$, which corresponds to the g in the case of a classical pump, we have the requirement that $\sigma_0 \geq g_1 t$. This is for the quantum treatment of the pump, and $gt \ll 1$. (It may be shown that there is no restriction here on the existence of the corresponding Wigner distribution.)

Comparing to the classical treatment of the pump, we have, from Eq. (3.25) with $\Omega = 0$, $\sigma_0 \geq \tanh gt \times (1 - \tanh gt)^{-1}$. To first order in gt we have then $\sigma_0 \geq gt$ in the classical treatment, with $gt \ll 1$, as the requirement for the P distribution to exist.

Thus, it is seen that the two results are essentially the same, i.e., the curve which determines the breakdown of the P distribution for this case (the $\Omega = 0$ curve in Fig. 1) rises from the origin immediately in the quantum, as well as in the classical, treatment of the pump. As mentioned in the caption of Fig. 1, this means, for example, that the P distribution for an initially coherent state of the signal breaks down immediately, and the results of this Appendix show that treating the pump quantum mechanically does not alter this conclusion.

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Hydrodynamics of Liquid Crystals*

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A consistent set of hydrodynamic equations for liquid crystals is derived from the necessary conservation laws and the requirements of Galilean invariance. In the stationary case, the equations reduce to the Oseen-Frank hydrostatic theory. The equations should be useful in discussing the hydrodynamics of cholesteric and smectic crystals. Linear dissipative effects are also considered.

I. INTRODUCTION

The continuum hydrostatic theory of liquid crystals of Oseen[1] and Frank[2] is well known and firmly established. More recently, Ericksen[3] and Leslie[4] have discussed continuum theories of the dynamics of liquid crystals. The theory of Leslie[4] is deficient in that it does not, in the stationary case, reduce to the Oseen-Frank hydrostatic theory. This situation has been partially rectified by Ericksen.[5] This deficiency is not important in the case of nematic crystals, and some interesting solutions of the linearized equations have been considered by the Orsay group.[6] However, it is desirable to have a consistent set of hydrodynamic equations describing the macroscopic behavior of liquid crystals. It is shown in this paper that a consistent set of hydrodynamic equations can be determined uniquely from the requirements of Galilean invariance, the necessary conservation laws, and certain natural assumptions about how the fluxes transform under Galilean transformations. This method was used by Landau[7] and Khalatnikov[8] to derive the equations of superfluid hydrodynamics, and the procedure followed here is rather similar. These equations should be useful for discussing the hydrodynamics of cholesteric and smectic liquid crystals. It is necessary in order to derive the equations consistently not to neglect nonlinear terms but only to make such approximations in the final equations. The dissipative effects in the hydrodynamic equations in a linear approximation are also considered briefly.

II. CONSERVATION RELATIONS

In simple liquid crystals the molecules have a preferred direction but the centers of mass of the molecules are arranged at random as in an ordinary liquid. We denote the preferred direction of the molecules by the vector $\vec{n}$ and refer to it simply as the director. We will assume that its magnitude as well as its direction may vary and specialize later to the case where $\vec{n}^2$ is a constant. Thus besides the usual types of hydrodynamic motions occurring in ordinary liquids there are also motions connected with the changes in the director. We assume that the dynamical equations in the absence of dissipative effects have the form proposed by Ericksen[5]:

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{J}=0 \quad , \tag{1}$$

$$\frac{\partial J_i}{\partial t}+\frac{\partial}{\partial x_j}T_{ij}=0 \quad , \tag{2}$$

$$\frac{\partial E}{\partial t}+\vec{\nabla}\cdot\vec{Q}=0 \quad , \tag{3}$$

$$\frac{\partial}{\partial t}(\rho s)+\vec{\nabla}\circ(\rho s\vec{v})=0 \quad , \tag{4}$$

$$\frac{\partial}{\partial t}(\rho\dot{n}_i)+\frac{\partial}{\partial x_j}(\rho\dot{n}_i v_j)+\frac{\partial}{\partial x_j}\Pi_{ij}+g_i=0 \quad . \tag{5}$$

Equation (1) describes the conservation of mass; ρ is the mass density, $\vec{J}=\rho\vec{v}$ is the current, and $\vec{v}$ is the fluid velocity. Equations (2)–(4) express the conservation of momentum, energy, and entropy, respectively; T_{ij} is the stress tensor, $\vec{Q}$ is the energy current, and s is the entropy per unit mass. Finally Eq. (5) is the equation of motion of the director; Π_{ij} is a stress tensor, and g_i is the internal body force. We use a Cartesian tensor notation where necessary, and a dot indicates the substantial time derivative. Thus we have

$$\dot{n}_i=\frac{dn_i}{dt}=\frac{\partial n_i}{\partial t}+\vec{v}\cdot\vec{\nabla}n_i \quad , \tag{6}$$

so that $\dot{n}_i$ is the rate of change of the director of a particular molecule or group of molecules. We have omitted external forces in Eqs. (1)–(5), but these can be included in an obvious way by adding terms to the right-hand sides of (2), (3), and (5).

We now transform to a coordinate system moving with the fluid. This system thus has a velocity $\vec{v}$ and an angular velocity $\vec{\omega}=\frac{1}{2}\operatorname{curl}\vec{v}$. The rate of change of the director relative to the moving fluid is given by

$$\vec{N}=\dot{\vec{n}}-(\vec{\omega}\times\vec{n}) \quad . \tag{7}$$

We assume that the quantities in the stationary coordinate system are related to their values in the moving coordinate system (which we denote by a subscript 0) by the following transformation formulas of mechanics:

$$T_{ij}=\rho v_i v_j+T_{0ij} \quad , \tag{8a}$$

$$E=\tfrac{1}{2}\rho\vec{v}^2+\rho\vec{N}\cdot(\vec{\omega}\times\vec{n})+\tfrac{1}{2}\rho(\vec{\omega}\times\vec{n})^2+E_0 \quad , \tag{8b}$$

$$\begin{aligned}Q_j=[\tfrac{1}{2}\rho\vec{v}^2+\rho\vec{N}\cdot(\vec{\omega}\times\vec{n})+\tfrac{1}{2}\rho(\vec{\omega}\times\vec{n})^2+E_0]v_j\\ +v_i T_{0ij}+(\vec{\omega}\times\vec{n})_i\Pi_{ij}+Q_{0j} \quad .\end{aligned} \tag{8c}$$

In the moving coordinate system, the molecules only execute rotations described by $\vec{N}$. Hence the quantities T_{0ij}, E_0, and Q_{0j} will only depend on $\vec{N}$ and other thermodynamic variables. The energy E_0 satisfies the following thermodynamic identity:

$$dE_0=\mu d\rho+Td(\rho s)+g_i dn_i-\Pi_{ij}dn_{i,j}+N_i d(\rho N_i) \quad , \tag{9}$$

where μ is the chemical potential per unit mass and $n_{i,j}=\partial n_i/\partial x_j$ is the director gradient.

In order to illustrate the transformation laws (8a)–(8c) let us take a simple model in which the molecules are like dumbbells. Each molecule consists of two masses M_1 and M_2 rigidly connected. Focus attention on a molecule with M_1 at $\vec{r}_1$ and M_2 at $\vec{r}_2$ and let

$$\vec{R}=\frac{1}{M}(M_1\vec{r}_1+M_2\vec{r}_2) \quad , \quad \vec{n}=\frac{(M_1M_2)^{1/2}}{M}(\vec{r}_1-\vec{r}_2) \quad .$$

Here $\vec{R}$ is the c. m., $\vec{n}$ is the director, and $M=M_1+M_2$. The energy of this molecule is

$$e=\tfrac{1}{2}M\mathring{\vec{R}}^2+\tfrac{1}{2}M\mathring{\vec{n}}^2+U \quad , \tag{10}$$

where U is the potential energy. We now transform to a coordinate system moving with velocity $\vec{v}$ and rotating with angular velocity $\vec{\omega}$. The energy of the molecule measured by an observer in this moving frame is

$$e_0 = \tfrac{1}{2}M(\dot{\vec{R}} - \vec{v})^2 + \tfrac{1}{2}M[\dot{\vec{n}} - (\vec{\omega}\times\vec{n})]^2 + U^\backslash . \qquad (11)$$

The kinetic energy vanishes, as it should, if the coordinate system is rigidly attached to the molecule. In the case where $\vec{v} = \dot{\vec{R}}$ and $\vec{\omega} = \tfrac{1}{2}\mathrm{curl}\vec{v}$, e_0 is the energy of the molecule excluding that part due to the fluid flow. This is exactly the thermodynamic internal energy per molecule. From (10) and (11) for this case we find

$$e = \tfrac{1}{2}M\vec{v}^2 + M(\vec{\omega}\times\vec{n})\cdot[\dot{\vec{n}} - (\vec{\omega}\times\vec{n})] + \tfrac{1}{2}M(\vec{\omega}\times\vec{n})^2 + e_0 \ ,$$

which is in agreement with Eq. (8b). Thus in Eq. (8b), E_0 is the thermodynamic internal energy per unit volume. The remaining transformation formulas (8a) and (8c) may be derived similarly.

The stress tensor T_{0ij} and the energy current Q_{0j} are now determined from the consistency of Eqs. (1)–(5). The details of this calculation are similar to those of Landau in the superfluid case so we will only give the general outline. In the equation of conservation of energy (3) we substitute for E and Q from (8b) and (8c) calculating the time derivative of E_0 from (9). All the time derivatives are eliminated using Eqs. (1)–(5). The resultant equation after some rearrangement has the form

$$\begin{aligned}\frac{\partial E}{\partial t} + \vec{\nabla}\cdot\vec{Q} &= \frac{\partial}{\partial x_j}(Q_{0j} - N_i\Pi_{ij}) \\ &\quad + (E_0 - \mu\rho - T\rho s - \rho\vec{N}^2)\,\vec{\nabla}\cdot\vec{v} \\ &\quad + T_{0ij}\frac{\partial}{\partial x_j}v_i + \Pi_{kj}n_{k,i}\frac{\partial}{\partial x_j}v_i \ . \qquad (12)\end{aligned}$$

This uniquely determines T_{0ij} and Q_{0j} as

$$T_{0ij} = (-E_0 + \mu\rho + T\rho s + \rho\vec{N}^2)\,\delta_{ij} - \Pi_{kj}n_{k,i} \ , \qquad (13)$$

$$Q_{0j} = N_i\Pi_{ij} \ . \qquad (14)$$

With these definitions the hydrodynamic equations (1)–(5) are completely determined.

We now examine the stationary case when all time derivatives vanish. From (2) and (5) we have

$$\frac{\partial}{\partial x_j}T_{0ij} = 0 \ , \qquad (15)$$

$$\frac{\partial}{\partial x_j}\Pi_{ij} + g_i = 0 \ . \qquad (16)$$

From (9) Eq. (16) can be written in the Euler-Lagrange form

$$\frac{\partial}{\partial x_j}\frac{\partial E_0}{\partial n_{i,j}} - \frac{\partial E_0}{\partial n_i} = 0 \ , \qquad (17)$$

where the derivatives are taken at constant density and entropy. This equation is a generalized form of the equilibrium theory of Oseen[1] and Frank.[2]

We now turn to Eq. (15) and from (13) it can be written

$$\begin{aligned}\rho\frac{\partial\mu}{\partial x_i} + \rho s\frac{\partial T}{\partial x_i} - g_k\frac{\partial n_k}{\partial x_i} + \Pi_{kl}\frac{\partial}{\partial x_i}n_{k,l} \\ - \frac{\partial}{\partial x_j}(\Pi_{kj}n_{k,i}) = 0 \ . \qquad (18)\end{aligned}$$

Using (16) to eliminate g_k this reduces to

$$\rho\frac{\partial\mu}{\partial x_i} + \rho s\frac{\partial T}{\partial x_i} = 0 \ , \qquad (19)$$

which is the usual condition for the hydrostatic equilibrium of an ordinary fluid and in that case is equivalent to the pressure being constant. Thus our equations reduce in the stationary case to (17) and (19), which are equivalent to the hydrostatic theory of Oseen and Frank.

III. CONSERVATION OF ANGULAR MOMENTUM

The conservation of angular momentum puts a restriction on the symmetry properties of the quantities T_{0ij}, Π_{ij}, and g_i. We define the angular momentum tensor by

$$\rho M_{ij} = \tfrac{1}{2}\rho(x_i v_j - x_j v_i) + \tfrac{1}{2}\rho(n_i\dot{n}_j - n_j\dot{n}_i) \ , \qquad (20)$$

and it must satisfy the conservation relation

$$\begin{aligned}\frac{\partial}{\partial t}(\rho M_{ij}) + \frac{\partial}{\partial x_k}(\rho M_{ij}v_k) + \frac{1}{2}\frac{\partial}{\partial x_k}[(x_i T_{0jk} - x_j T_{0ik}) \\ + (n_i\Pi_{jk} - n_j\Pi_{ik})] = 0 \ . \qquad (21)\end{aligned}$$

Substituting (20) into (21) and eliminating the time derivatives by means of (2) and (5) gives

$$T_{0ij} - T_{0ji} + \Pi_{ik}n_{j,k} - \Pi_{jk}n_{i,k} + n_i g_j - n_j g_i = 0 \ . \qquad (22)$$

This is a further symmetry requirement that the stresses must satisfy in order that angular momentum be conserved.[3,4]

In the case of the Oseen-Frank theory, the part of the energy depending on the director is

$$\begin{aligned}E_0^1 = \tfrac{1}{2}k_{11}(\vec{\nabla}\cdot\vec{n} - s_0)^2 + \tfrac{1}{2}k_{22}(\vec{n}\cdot\mathrm{curl}\vec{n} + t_0)^2 \\ + \tfrac{1}{2}k_{33}(\vec{n}\cdot\vec{\nabla}\vec{n})^2 - k_{12}(\vec{\nabla}\cdot\vec{n})(\vec{n}\cdot\mathrm{curl}\vec{n}) \ , \qquad (23)\end{aligned}$$

where s_0 and t_0 are constants and the k_{ii} are elastic constants. We have omitted a term which only contributes to surface energies. From their definitions

$$g_i = \frac{\partial E_0^1}{\partial n_i} = A\,e_{ijk}n_{k,j} + k_{33}(\vec{n}\cdot\vec{\nabla}n_j)n_{j,i} \ , \qquad (24)$$

$$\Pi_{ij} = -\frac{\partial E_0^1}{\partial n_{i,j}} = -A\,e_{kji}n_k - k_{33}(\vec{n}\cdot\vec{\nabla}n_i)n_j - B\delta_{ij} \ , \qquad (25)$$

where e_{ijk} is the completely antisymmetric tensor and

$$A = k_{22}(\vec{n}\cdot\mathrm{curl}\vec{n} + t_0) - k_{12}\vec{\nabla}\cdot\vec{n} \ ,$$

$$B = k_{11}(\vec{\nabla}\cdot\vec{n} - s_0) - k_{12}\vec{n}\cdot\mathrm{curl}\,\vec{n} \quad .$$

Using T_{0ij} given in Eqs. (13), (24), and (25) it is easily verified that Eq. (22) is identically satisfied.

IV. DISSIPATIVE EFFECTS

Dissipative effects have been extensively considered by Ericksen[3] and Leslie,[4] and we will briefly discuss them. In the presence of dissipative effects the equations of motion (2)–(5) are generalized to

$$\frac{\partial J_i}{\partial t} + \frac{\partial}{\partial x_j}(T_{ij} + t_{ij}) = 0 \quad , \tag{26}$$

$$\frac{\partial E}{\partial t} + \vec{\nabla}\cdot(\vec{Q} + \vec{Q}^1) = 0 \quad , \tag{27}$$

$$\frac{\partial}{\partial t}(\rho s) + \vec{\nabla}\cdot(\rho s\vec{v} + \vec{q}/T) = R \quad , \tag{28}$$

$$\frac{\partial}{\partial t}(\rho\dot{n}_i) + \frac{\partial}{\partial x_j}(\rho\dot{n}_i v_j) + \frac{\partial}{\partial x_j}(\Pi_{ij} + \pi_{ij}) + g_i + g_i^1 = 0 \quad , \tag{29}$$

where the extra fluxes and forces t_{ij}, $\vec{Q}^1$, $\vec{q}$, π_{ij}, and g_i^1 arise from dissipative effects. In particular $\vec{q}$ is the heat current and R is the entropy production. These fluxes and forces are subject to the restrictions (i) that they also satisfy the symmetry property (22), (ii) that the entropy production R be positive, and (iii) that they vanish in the steady-state condition.

The quantities $\vec{Q}^1$ and R are determined as before by substituting Eqs. (9) and (11) in (27), the time derivatives being eliminated by means of (1) and (26)–(29). It is found that

$$Q_j^1 = q_j + v_i t_{ij} + \dot{n}_i \pi_{ij} \quad , \tag{30}$$

$$R = -t_{ij}\frac{\partial v_i}{\partial x_j} - \pi_{ij}\dot{n}_{i,j} + g_i^1\dot{n}_i - \frac{q_i}{T}T_{,i} \quad . \tag{31}$$

Using the symmetry property (22) we can write R in a more convenient form. Thus we have

$$t_{ij}\frac{\partial v_i}{\partial x_j} = t_{ij}d_{ij} + t_{ij}\,\omega_{ij}$$
$$= t_{ij}d_{ij} + \tfrac{1}{2}(t_{ij} - t_{ji})\,\omega_{ij} \quad ,$$

where

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \text{ and } \omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \quad .$$

Using (22) to eliminate $t_{ij} - t_{ji}$ we get

$$t_{ij}\frac{\partial v_i}{\partial x_j} = t_{ij}d_{ij} - \pi_{ij}\,\omega_{ik}n_{k,j} + g_i^1\,\omega_{ij}n_j \quad ,$$

and on substituting this result in (31) we find

$$R = t_{ij}d_{ij} - \pi_{ij}N_{ij} + g_i^1 N_i - (q_i/T)\,T_{,i} \quad , \tag{32}$$

where $N_{ij} = \dot{n}_{i,j} - \omega_{ik}n_{k,j}$ and N_{ij} is the change of the director gradient relative to a coordinate system moving with the fluid.

We now confine ourselves to constitutive relations which are linear in the quantities d_{ij}, N_i, and $T_{,i}$. We also make the assumptions that the fluid has a center of symmetry and that $\vec{n}$ and $-\vec{n}$ are indistinguishable. Further we will neglect as small, products of the director gradient and the above quantities and terms in N_{ij}. This strictly confines us to the case of nematic liquid crystals. More general constitutive relations are easily written down but the extra terms would be expected to be small in most cases and the relations become extremely lengthy.

The constitutive relations consistent with the requirements are exactly those of Leslie[4] and are

$$\begin{aligned} t_{ij} = &-(\mu_1 n_k N_k + \mu_2 d_{kk} + \mu_3 n_k n_p d_{kp})\,\delta_{ij} \\ &-(\mu_4 n_k N_k + \mu_5 d_{kk} + \mu_6 n_k n_p d_{kp})\,n_i n_j \\ &-\mu_7 n_i N_j - \mu_8 n_j N_i - \mu_9 d_{ij} - \mu_{10} n_i n_k d_{kj} \\ &-\mu_{11} n_j n_k d_{ki} \quad , \end{aligned} \tag{33}$$

$$\begin{aligned} g_i^1 = &(\lambda_1 n_k N_k + \lambda_2 d_{kk} + \lambda_3 n_k n_p d_{kp})\,n_i + \lambda_4 N_i \\ &+ \lambda_5 n_j d_{ji} \quad ,' \end{aligned} \tag{34}$$

$$q_i = -\beta_1 n_j T_{,j} n_i - \beta_2 T_{,i} \quad , \tag{35}$$

$$\pi_{ij} = 0 \quad . \tag{36}$$

The symmetry relation (22) requires that

$$\lambda_4 = \mu_7 - \mu_8 \;, \quad \lambda_5 = \mu_{10} - \mu_{11} \quad .$$

In the case of incompressible fluids and when $n_i^2 = \mathrm{const}$ the constitutive relations simplify somewhat. In this case we have the extra relations

$$d_{kk} = 0 \;, \quad n_k N_k = 0 \quad , \tag{37}$$

and the constitutive relations may be taken in the form

$$\begin{aligned} t_{ij} = &-\alpha\delta_{ij} - \alpha_1 n_k n_p d_{kp} n_i n_j - \alpha_2 n_i N_j - \alpha_3 n_j N_i \\ &-\alpha_4 d_{ij} - \alpha_5 n_i n_k d_{kj} - \alpha_6 n_j n_k d_{ki} \quad , \end{aligned} \tag{38}$$

$$g_i^1 = \gamma n_i + \gamma_1 N_i + \gamma_2 n_j d_{ji} \quad , \tag{39}$$

$$q_i = \beta_1 n_j T_{,j} n_i - \beta_2 T_{,i} \quad . \tag{40}$$

Here α and γ are arbitrary scalar functions of position and time. These terms arise because for an incompressible fluid the stress is indeterminate to within such a scalar, and in the case where $\vec{n}^2$ has a constant magnitude, g_i^1 is indeterminate within a scalar multiple of the director. These scalar functions are determined by the equations of motion. The symmetry relation (22) requires that

$$\gamma_1 = \alpha_2 - \alpha_3 \;, \quad \gamma_2 = \alpha_5 - \alpha_6 \quad . \tag{41}$$

In addition Parodi[9] has recently shown from the

Onsager relations that

$$\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5 \ . \tag{42}$$

Thus the theory contains seven dissipative coefficients.

In cholesteric crystals, as shown by Leslie,[10] certain additional terms may occur in (38)–(40). In these crystals the molecules are optically active and not invariant under reflections in planes containing the director $\vec{n}$. We assume that $\vec{n}$ and $-\vec{n}$ are still indistinguishable. In this case we may have additional terms,

$$\begin{aligned} t^1_{ij} &= -\alpha_7 e_{ipq} n_j n_p T_{,q} - \alpha_8 e_{jpq} n_i n_p T_{,q} \ , \\ g^{11}_i &= \gamma_3 e_{ijk} n_j T_{,k} \ , \quad q^1_i = -\beta_3 e_{ijk} n_j N_k \ , \end{aligned} \tag{43}$$

with $\gamma_3 = \alpha_8 - \alpha_7$. These terms indicate that a thermal gradient gives rise to a torque on the molecules. The restrictions imposed on the coefficients α_i and β_i by the requirement that the entropy production be positive have been given by Leslie,[4,10] and for completeness we reproduce them below:

$$\begin{aligned} &\alpha_4 \geq 0 \ , \quad 2\alpha_4 + \alpha_5 + \alpha_6 \geq 0 \ , \\ &2\alpha_1 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 \geq 0 \ , \\ &4\gamma_1 (2\alpha_4 + \alpha_5 + \alpha_6) \geq (\alpha_2 + \alpha_3 + \gamma_2)^2 \ , \\ &\beta_1 + \beta_2 \geq 0 \ , \quad \beta_2 \geq 0 \ , \\ &\frac{4\beta_2 \gamma_1}{T} \geq \left(\frac{\beta_3}{T} + \gamma_3 \right)^2 \ . \end{aligned} \tag{44}$$

In conclusion we note that within the present model of a liquid crystal Eqs. (1) and (26)–(29) are a consistent and complete set of hydrodynamic equations. They should be useful in discussing the hydrodynamics of cholesteric and smectic crystals, and some applications will be given elsewhere.

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