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Statistical Dynamics of Quantum Oscillators and Parametric Amplification in a Single Mode*

M. T. Raiford†

Department of Physics, Duke University, Durham, North Carolina 27706

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The description of the statistical dynamics of quantum oscillators is formulated in terms of the Wigner distribution, analogous to the more commonly used P distribution, with explicit formulas being obtained for its time evolution and for average values. This formulation is desirable because, e.g., the Wigner distribution always exists whereas the P distribution does not. The formalism is applied to the process of parametric amplification in a single mode, which may be considered as the degenerate form of the well-known two-mode case. This degeneracy gives rise to significantly different properties; for example, the P distribution for the single mode of interest evolves from a circularly symmetric two-dimensional Gaussian into an elliptically symmetric form and ceases to exist after a finite time, even for amplification in the presence of losses. This is contrary to the two-mode case. The corresponding Wigner distribution is found to exist as a well-behaved function for all time as expected, regardless of the amount of losses, and is used to calculate average values of various quantities of interest. It is found, e.g., that in the lossless case the average number of photons in the signal mode always becomes infinite as $t \to \infty$. This is in contrast to the corresponding classical result for the lossless case which allows the signal to decay rather than to grow with time, depending on the relative phase between the signal and the pump. Field fluctuations are discussed and found to have some unusual properties. The combination of frequency up-conversion with single-mode amplification is also described briefly. The effect of the quantization of the pump oscillator is considered in an Appendix.

I. INTRODUCTION

The theoretical formulation that has been developed during the past few years for the description of various basic processes and field properties which are of interest in quantum electronics and quantum optics has included to a considerable extent formulations in terms of quantum-oscillator statistical distribution functions. Glauber,[1-4] especially, has made extensive use of the eigenstates of the oscillator annihilation operator, called coherent states, in his development of the theory. These states have been used to form the basis for an expansion of the density operator in terms of a distribution function.[1-5] The use of a distribution function, rather than the density operator, has the advantage that it is an ordinary function, allowing a graphical representation of the system under study, and allowing the calculation of average values by ordinary integrals very similar to the way in which it is done classically. If the time evolution of the distribution function can be determined, then one has the complete description of the corresponding process as a function of time. Of the possible distribution functions in terms of which a density operator may be expanded, the particular one usually referred to as the P distribution has received the most attention; it is a particular ("diagonal") case of the general distribution function which is obtained when the density operator is expanded in terms of coherent states.[3] The P distribution has a simple form and many convenient properties, and has therefore been used a good deal, but being a special case it leaves open the possibility that it is not adequate to describe all fields. Thus, the applicability or validity of the P distribution has caused considerable debate and study. Whereas fields could be thought of for which the P distribution does not exist (e.g., a field in the pure occupation-number state[6,7]), it seemed at first as if it were adequate for fields which are met in practice. Mollow and

Glauber [8] found, however, in their study of parametric amplification in two modes, that the description of the coupled modes by a single P distribution is not always possible. They then went to another distribution function, the Wigner distribution, [9] and found that it does exist for this case at all times. In fact, as they point out, the Wigner distribution always exists for all quantum states. The relationship between the Wigner and P distributions has been found. [4,8,10]

In this paper, the process of parametric amplification in a single mode, rather than in two modes, is presented and analyzed. It is found that this provides a simple case for which the P distribution breaks down, and one may see just how an initially well-defined P distribution for the mode of interest evolves into a strongly singular function in the region of operation that is of interest. The Wigner distribution is therefore needed for a complete description of this process, and it is obtained and shown to exist as a well-behaved function for all time. Since the Wigner distribution always exists and it is now seen that the P distribution does not exist for all physically realizable electromagnetic fields, it seems desirable to develop a general formulation in terms of the Wigner distribution analogous to what has been done for the P distribution. [11] This is done in Sec. II in two alternative representations: first in terms of the eigenstates of the displacement and momentum operators, and second in terms of the coherent states. The first might seem more natural for the Wigner distribution, but the second turns out to be simpler. In fact, in terms of coherent states, the resulting formulas obtained in this work involving the Wigner distribution are as simple and convenient to use as those involving the P distribution. The formulas obtained are for the determination of the time evolution of the Wigner distribution, and for the calculation of average values by means of the Wigner distribution at any time.

In Sec. III, the results for the Wigner distribution found in Sec. II are applied to the process of parametric amplification in a single mode, after finding that the P distribution for this case breaks down in the region of operation that is of interest. Results for the amplification and for the field fluctuations are obtained, using the Wigner distribution. Some final remarks are made in Sec. IV along with a brief description of the extension of the theory of parametric amplification in a single mode to include the process of frequency up-conversion. In Appendix A, the definitions, properties, and basic relations of the various pertinent states and operators which are used in this work are reviewed. Appendix B is used to present an example (a pair of coupled oscillators) which shows explicitly how an originally well-behaved P distribution becomes highly singular as a pure occupation-number state is approached, while the corresponding Wigner distribution remains well-behaved. These results for the pure occupation-number state are well known [4,7]; these calculations are presented to show by means of a simple example how these results came about. Appendix C is used to evaluate several complex integrals that occur in Sec. III. In Appendix D is considered the quantum-mechanical, rather than classical, treatment of the pump. In the remainder of this section some of the theory and results concerning the P distribution will be reviewed.

The P distribution is defined[3,5] by the following (diagonal) expansion [12] of the density operator ρ over the coherent states $|z\rangle$:

$$\rho = \int \int dx\, dy\, P(x, y)\, |z\rangle \langle z| \;. \tag{1.1}$$

Since $\mathrm{Tr}\rho = 1$, the P distribution is normalized:

$$\int \int P(x, y) dx\, dy = 1. \tag{1.2}$$

A δ-function singularity is allowed for the P distribution, $P(x, y) = \delta(x - x_0)\delta(y - y_0)$, so that the density operator for the pure coherent state may be obtained from Eq. (1.1) as $\rho = |z_0\rangle\langle z_0|$.

Following Robl, [11] some useful results involving the P distribution will now be reviewed. The average value of an observable which is represented by the operator function F in a system described by a density operator ρ is given by

$$\langle F(t)\rangle = \mathrm{Tr}[\rho(0)F(t)] = \mathrm{Tr}[\rho(t)F(0)] \;. \tag{1.3}$$

To study a system of oscillators, let $\rho(t)$ represent the density operator for the system and let $F_j(0)$ be an operator which operates only on the jth oscillator of the system at time zero. Taking the trace over all states except those of the jth oscillator yields $\rho_j(t)F_j(0)$, where $\rho_j(t)$ denotes the trace of the density operator over all states except those of the jth oscillator. Therefore, one may write from the two equivalent expressions for the average value of $F_j(t)$

$$\langle F_j(t) = \mathrm{Tr}[\rho(0)F_j(t)] = \mathrm{Tr}_j[\rho_j(t)F_j(0)] \;, \tag{1.4}$$

where the symbol Tr_j denotes the trace over all states of the jth oscillator. The equality on the right-hand side above allows one to calculate the time evolution of the P distribution corresponding to any particular oscillator in the system. If the system is composed of initially independent oscillators, then the initial density operator $\rho(0)$ can be written as a direct product of the $\rho_i(0)$ for each oscillator. Using Eq. (1.1) to express ρ in terms of P gives, from Eq. (1.4),

$$\int\int dx_j' dy_j' P_j(x_j', y_j', t)\langle z_j' | F_j(0) | z_j'\rangle = \int \dots \int \Pi_i dx_i dy_i P_i(x_i, y_i, 0)\langle\{z_i\}| F_j(t) |\{z_i\}\rangle. \quad (1.5)$$

If the operator $\exp[-\frac{1}{2}i\zeta a_j^\dagger(t)]\exp[-\frac{1}{2}i\zeta^* a_j(t)]$ is substituted for F_j, then this equation may be formally solved for $P_j(x_j', y_j', t)$ by means of a Fourier transform, yielding

$$P_j(x_j', y_j', t) = (2\pi)^{-2}\int\int d\xi\, d\eta e^{i(\xi x_j' + \eta y_j')}\int \dots \int \Pi_i \times dx_i\, dy_i\, P_i(x_i, y_i 0)\langle\{z_i\}| e^{-i(\zeta/2)a_j^\dagger(t)} e^{-i(\zeta^*/2)a_j(t)} |\{z_i\}\rangle, \quad (1.6)$$

where $\zeta = \xi + i\eta$, $z = x + iy$. This result gives the time evolution of the P distribution for the jth oscillator in terms of the initial P distributions of all the oscillators, provided the Fourier transform exists. Whether it exists or not depends on the solutions to Heisenberg's equations of motion for the annihilation operator $a_j(t)$, which pertains to the jth mode of the radiation field as represented by a quantum oscillator, and for the corresponding creation operator $a_j^\dagger(t)$, for the system under study, and on the initial distributions.

If the P distribution exists and is determined at time t, then the average value of any operator $F_j(t)$, pertaining to the jth oscillator at time t, is, from Eqs. (1.1) and (1.4),

$$\langle F_j(t)\rangle = \int\int dx_j' dy_j' P_j(x_j', y_j', t)\langle z_j' | F_j(0) | z_j'\rangle . \quad (1.7)$$

Of particular interest and usefulness are P distributions which are Gaussian in form

$$P_j(x_j, y_j, t) = [\pi\sigma_j(t)]^{-1} e^{-[\sigma_j(t)]^{-1}|z_j - \gamma_j(t)|^2}, \quad (1.8)$$

where σ_j determines the width and γ_j the location of the center of the Gaussian. For example, in the limit as $\sigma_j \to 0$, P_j approaches a δ function, which corresponds to the pure coherent state. In the other extreme, we have thermal equilibrium, e.g., which is represented by a Gaussian centered at the origin of the complex plane ($\gamma_j = 0$), with $\sigma_j = (e^{\hbar\omega_j/\kappa T} - 1)^{-1}$, which also equals the average number of quanta in the jth mode in the Planck thermal equilibrium distribution. The probability that the jth mode of the radiation field, as described by a Gaussian P distribution, contains n photons at time t approaches a Bose-Einstein distribution as $\gamma_j(t) \to 0$ and approaches a Poisson distribution as $\sigma_j(t) \to 0$. A convenient result for the average value of the number operator $N = a^\dagger a$, when $P_j(x_j, y_j, t)$ is a Gaussian as in Eq. (1.8) above, can be obtained for the jth oscillator at any time t from Eq. (1.7) to be

$$\langle N_j(t)\rangle = |\gamma_j(t)|^2 + \sigma_j(t) . \quad (1.9)$$

Robl[11] has examined in particular the time evolution of Gaussian P distributions and has shown that if the solutions to Heisenberg's equations of motion are of the form

$$a_j(t) = \sum_k w_{jk}(t)a_k + \sum_l w_{jl}(t)a_l^\dagger + \psi_j(t), \quad k \neq l \quad (1.10)$$

where the w_j and ψ_j are ordinary functions of time (initial values are written without any time dependence), then the P distribution for the jth oscillator exists and remains Gaussian at any time. Its width and center are given by, for this case,

$$\sigma_j(t) = \sum_k |w_{jk}(t)|^2\sigma_k + \sum_l |w_{jl}(t)|^2(1 + \sigma_l), \quad (1.11)$$

$$\gamma_j(t) = \sum_k w_{jk}(t)\gamma_k + \sum_l w_{jl}(t)\gamma_l^* + \psi_j(t) . \quad (1.12)$$

II. STATISTICAL DYNAMICS OF QUANTUM OSCILLATORS IN TERMS OF WIGNER DISTRIBUTION

We shall use the expression obtained by Moyal[13] as our basic definition of the Wigner distribution. He showed that the Fourier transform of the characteristic function $\chi(\xi, \eta, t)$, where

$$\chi(\xi, \eta, t) = \mathrm{Tr}[\rho(t)e^{-i(\xi Q + \eta P)}] = \mathrm{Tr}[\rho(0)e^{-i[\xi Q(t) + \eta P(t)]}], \quad (2.1)$$

produces the same form of the distribution function as originally given by Wigner.[9] That is, the Wigner distribution $W(q, p, t)$ is given by

$$W(q, p, t) = (2\pi)^{-2}\int\int d\xi\, d\eta \chi(\xi, \eta, t)e^{i(\xi q + \eta p)} . \quad (2.2)$$

The characteristic function $\chi(\xi, \eta, t)$ is defined by an expansion due to Weyl[14] applied to the density operator

$$\rho(t) = (2\pi)^{-1}\int\int d\xi\, d\eta\, \chi(\xi, \eta, t)e^{i(\xi Q + \eta P)} . \quad (2.3)$$

From Eq. (2.2) we have

$$\chi(\xi, \eta, t) = \int\int dq\, dp\, W(q, p, t)e^{-i(\xi q + \eta p)} . \quad (2.4)$$

The Wigner distribution is another of the quasiprobability functions; it is normalized, as may be seen by letting $\xi = \eta = 0$ in Eq. (2.4) and referring to Eq. (2.1),

$$\int\int dqdp\, W(q, p, t) = 1 . \quad (2.5)$$

Substitution of Eq. (2.4) into Eq. (2.3) yields the density operator in terms of the Wigner distribution at any time t,

$$\rho(t) = (2\pi)^{-1}\int\int dq\, dp\, W(q, p, t) \times \int\int d\xi\, d\eta\, e^{-i(\xi q + \eta p)}e^{i(\xi Q + \eta P)} . \quad (2.6)$$

We shall consider a system of oscillators which are assumed to be initially independent, and shall seek expressions for the time evolution of the Wigner distribution, and the average value of any observable at any time t, for the jth oscillator. The method used in the Introduction to obtain the time evolution of the P distribution could also be used here, but we shall proceed directly from the immediately preceding equations. The development of the theory may be carried out either in terms of the operators Q, P and their eigenstates $|q\rangle$, $|p\rangle$, or in terms of the operators a, $a^\dagger$ and the coherent states $|z\rangle$ (see Appendix A for notation and properties). This will be done separately in Secs. II A and II B.

A. Formulation in Terms of Operators Q, P, and Their Eigenstates

Using Eq. (2.6) for the ith oscillator at $t=0$ and substituting into Eq. (2.1) written for the jth oscillator at time t gives

$$\chi_j(\xi,\eta,t)=\mathrm{Tr}\{\textstyle\prod_i(2\pi)^{-1}\int\cdots\int dq_i\,dp_i\,W_i(q_i,p_i,0)$$
$$\times\int\int d\xi' d\eta' e^{-i(\xi' q_i+\eta' p_i)}e^{i(\xi' Q_i+\eta' P_i)}$$
$$\times e^{-i[\xi Q_j(t)+\eta P_j(t)]}\}\,. \quad (2.7)$$

The trace may be evaluated between q eigenstates, in which case the identity operator for p eigenstates is inserted; then the expressions for the bracket $\langle q|p\rangle$ and the Dirac δ function are used, giving a result for $\chi_j(\xi,\eta,t)$ which, when substituted into Eq. (2.2), yields the desired result

$$W_j(q_j',p_j',t)=(2\pi)^{-2}\int\int d\xi\,d\eta\,e^{i(\xi q_j'+\eta p_j')}\textstyle\prod_i[(2/\pi)^{1/2}$$
$$\times\int\cdots\int dq_i\,dp_i\,W_i(q_i,p_i,0)e^{-2iq_ip_i}$$
$$\times\int\cdots\int dq_i''\,dp_i''\,\langle\{p_i''\}|e^{-i[\xi Q_j(t)+\eta P_j(t)]}|\{q_i''\}\rangle$$
$$\times e^{i(2q_ip_i''-q_i''p_i''+2p_iq_i'')}]\,. \quad (2.8)$$

This formula expresses the time evolution of the Wigner distribution for the jth oscillator.

The formula for the average value of any operator function pertaining to the jth oscillator $F_j(t)$ is obtained by substituting Eq. (2.6) for the density operator into Eq. (1.4), yielding

$$\langle F_j(t)\rangle=(2\pi)^{-1}\int\int d\xi\,d\eta\int\int dq_j\,dp_j\,W_j(q_j,p_j,t)$$
$$\times e^{-i(\xi q_j+\eta p_j)}\mathrm{Tr}_j[e^{i(\xi Q_j+\eta P_j)}F_j(0)]\,. \quad (2.9)$$

The trace may be evaluated in terms of the q eigenstates by inserting the identity operator for p eigenstates and using the Baker-Hausdorff rule to effect the decomposition $\exp[i(\xi Q+\eta P)]=\exp(i\frac{1}{2}\xi\eta)\exp(i\xi Q)\exp(i\eta P)$. Then integration over ξ,η gives the general result

$$\langle F_j(t)\rangle=(2/\pi)^{1/2}\int\int dq_j\,dp_j\,W_j(q_j,p_j,t)e^{2iq_jp_j}$$
$$\times\int\int dq_j'\,dp_j'\langle q_j'|F_j(0)|p_j'\rangle e^{-i(2p_jq_j'-p_j'q_j'+2q_jp_j')}\,. \quad (2.10)$$

If $F_j(t)$ can be expressed in the particular form of a sum of a function f of $P_j(t)$ and a function g of $Q_j(t)$ then an especially simple formula for the average value results. Instead of the previous decomposition, we now use the alternative forms

$$e^{i(\xi Q+\eta P)}=e^{i(1/2)\eta P}e^{i\xi Q}e^{i(1/2)\eta P}=e^{i(1/2)\xi Q}e^{i\eta P}e^{i(1/2)\xi Q}$$

The trace in Eq. (2.9) is then taken in a similar manner to that above but now the integrations involved produce δ functions. Then integration over ξ,η leads to the following simple formula for this special case[9]:

$$\langle f[P_j(t)]+g[Q_j(t)]\rangle=\int\int dq_j\,dp_j\,[f(p_j)+g(q_j)]$$
$$\times W_j(q_j,p_j,t)\,. \quad (2.11)$$

In this particular case, the average value is obtained exactly as is done classically by integrating over phase space the ordinary functions weighted by the (Wigner) distribution function. A useful example for which this equation is directly applicable is the number operator $N=a^\dagger a$, which in terms of P and Q for the jth oscillator is

$$N_j(t)=\tfrac{1}{2}[P_j^2(t)+Q_j^2(t)-1]\,.$$

In particular, if $W_j(q_j,p_j,t)$ is a Gaussian whose width is defined by $\sigma_j^w(t)$ and center by $\gamma_j^w(t)$, then Eq. (2.11) gives

$$\langle N_j(t)=\tfrac{1}{2}[|\gamma_j^w(t)|^2+\sigma_j^w(t)-1]\,. \quad (2.12)$$

B. Formulation in Terms of Operators a, $a^\dagger$, and Coherent States

The formulation up to this point has been carried out in terms of the operators Q, P, and their eigenstates, which might seem natural when using the Wigner distribution. However, the general formulas obtained in this way are seen to be rather complex; another inconvenience is that the model Hamiltonian for the system under study is usually expressed, and the equations of motion solved, in terms of a and $a^\dagger$ rather than Q and P. It therefore seems advantageous to develop the formulas in terms of a and $a^\dagger$ (and coherent states and initial P distributions). This will have the added advantage of allowing a more direct comparison between the Wigner- and P-distribution formulations, which is done in Sec. II C.

The basic formulas for the Wigner distribution, Eq. (2.2), and for the characteristic function, Eq. (2.1), are again the starting point, but now we express the $\rho_i(0)$ in terms of the initial P distributions, by means of Eq. (1.1), and Q and P are expressed in terms of a and $a^\dagger$ (see Appendix A). Using now the decomposition

$$e^{-i(\xi Q+\eta P)}=e^{-(1/4)|\zeta|^2}e^{-i(\zeta/\sqrt{2})a^\dagger}e^{-i(\zeta^*/\sqrt{2})a}\,,$$

where $\zeta=\xi+i\eta$, then Eq. (2.1) becomes, for the jth oscillator,

$$\chi_j(\xi,\eta,t)=\mathrm{Tr}[\int\cdots\int\textstyle\prod_i dx_i\,dy_iP_i(x_i,y_i,0)$$

$$\times \left| z_i \right\rangle \langle z_i \left| e^{(-1/4)|\zeta|^2} e^{-i(\zeta/\sqrt{2})a_j^\dagger(t)} e^{-i(\zeta^*/\sqrt{2})a_j(t)} \right] . \tag{2.13}$$

Evaluating the trace over any complete set of state (e.g., the number states), substituting the result into Eq. (2.2) written for the jth oscillator (and replacing ζ by $\zeta/\sqrt{2}$ for convenience) yields the result

$$W_j(q_j', p_j', t) = \tfrac{1}{2}(2\pi)^{-2} \int \int d\xi \, d\eta \, e^{(-1/8)(\xi^2+\eta^2)} \times e^{i[(\xi/\sqrt{2})q_j' + (\eta/\sqrt{2})p_j']} \int \cdots \int \prod_i dx_i \, dy_i P_i(x_i, y_i, 0) \times \langle \{z_i\} | e^{-i(1/2)\zeta a_j^\dagger(t)} e^{-i(1/2)\zeta^* a_j(t)} | \{z_i\} \rangle . \tag{2.14}$$

Equation (2.14) gives the Wigner distribution for the jth oscillator at any time in terms of the initial set of P distributions, the operators $a_j(t)$ and $a_j^\dagger(t)$, and the coherent states. It is an alternative expression to that in Eq. (2.8).

The average value of $F_j(t)$ in this formulation is obtained by again using Eq. (2.6) for $\rho_j(t)$ in Eq. (1.4), but expressing the last factor in Eq. (2.6) as

$$e^{i(\xi Q + \eta P)} = e^{(-1/4)|\zeta|^2} e^{i(\zeta/\sqrt{2})a^\dagger} e^{i(\zeta^*/\sqrt{2})a} .$$

This gives

$$\langle F_j(t) \rangle = (2\pi)^{-1} \int \int d\xi \, d\eta \, e^{-(1/4)|\zeta|^2} \int \int dq_j \, dp_j \, W_j(q_j, p_j, t) \times e^{i(\xi q_j + \eta p_j)} \mathrm{Tr}_j [e^{i(\zeta/\sqrt{2})a_j^\dagger} e^{i(\zeta^*/\sqrt{2})a_j} F_j(0)] . \tag{2.15}$$

Taking the trace over any complete set of states of the jth oscillator and using the identity operator for coherent states gives the general result (after replacing ζ by $\zeta/\sqrt{2}$)

$$\langle F_j(t) \rangle = (2\pi)^{-2} \int \int d\xi \, d\eta \, e^{(-1/8)(\xi^2+\eta^2)} \int \int dq_j \, dp_j \times W_j(q_j, p_j, t) e^{i[(\xi/\sqrt{2})q_j + (\eta/\sqrt{2})p_j]} \int \int dx_j \, dy_j \times \langle z_j | e^{i(1/2)\zeta a_j^\dagger} e^{i(1/2)\zeta^* a_j} F_j(0) | z_j \rangle . \tag{2.16}$$

As an example, consider the operator $a^\lambda a^{\dagger\lambda}$, λ any positive integer, for which the following identity holds:

$$a^\lambda a^{\dagger\lambda} = (a^\dagger a + 1)(a^\dagger a + 2) \cdots (a^\dagger a + \lambda). \tag{2.17}$$

This form, rather than normal form, is useful here in evaluating the trace in Eq. (2.15), since a trace allows cyclic permutation of factors, and this form allows indirectly the calculation of the average value of any power of the number operator $N = a^\dagger a$. Substitution of Eq. (2.17) for $F_j(0)$ into Eq. (2.15), evaluation of the trace between coherent states, and integration over ξ, η (and replacing ζ by $\zeta/\sqrt{2}$ and z by $z/\sqrt{2}$) gives the result

$$\langle a_j^\lambda(t) a_j^{\dagger\lambda}(t) \rangle = (2^\lambda \pi)^{-1} \int \int dq_j \, dp_j \, W_j(q_j, p_j, t) \times \int \int dx_j \, dy_j (x_j^2 + y_j^2)^\lambda e^{-[(x_j + q_j)^2 + (y_j + p_j)^2]} . \tag{2.18}$$

For $\lambda = 1$ and $W_j(q_j, p_j, t)$ a Gaussian, Eq. (2.18) together with Eq. (2.17) yields the same result for $\langle N_j(t) \rangle$ as obtained before from the q, p formulation, Eq. (2.12).

C. Comparison of $W(q, p, t)$ and $P(x, y, t)$

The relationship between the Wigner and P distributions can be found explicitly by a calculation similar to that leading up to Eq. (2.14), this time starting with the density operator written at time t. Then Eqs. (1.1), (2.1), and (2.2) give the result (also obtained by others[4,10])

$$W(q, p, t) = \pi^{-1} \int \int dx \, dy \, P(x, y, t) e^{-[(q - \sqrt{2}x)^2 + (p - \sqrt{2}y)^2]} . \tag{2.19}$$

From this relation one may prove the theorem that if the P distribution is a Gaussian then the Wigner distribution is a Gaussian also, with the relationships between their widths and centers given by

$$\sigma^w(t) = 2\sigma(t) + 1, \qquad \gamma^w(t) = \sqrt{2}\, \gamma(t) . \tag{2.20}$$

The superscript w is used to distinguish quantities referring to the Wigner distribution from those referring to the P distribution. This theorem allows the results concerning the properties and evolution of Gaussian P distributions [see Eqs. (1.8)–(1.12)] to be applied to Gaussian Wigner distributions as well. It is seen, e.g., that the pure coherent state is represented by a Gaussian Wigner distribution of unit width ($\sigma^w = 1$), rather than zero width as for the P distribution, and that thermal equilibrium is represented by a Gaussian Wigner distribution centered at the origin, as with the P distribution. It might be noted that Eq. (2.20) establishes the identity of the results obtained for $\langle N_j(t) \rangle$ by means of either Gaussian Wigner or P distributions, Eqs. (2.12) and (1.9).

Other formulas which may be useful and which give further information on the relation between the quantities $W(q, p, t)$, $P(x, y, t)$, and $\chi(\xi, \eta, t)$ will now be presented. The relation between $P(x, y, t)$ and $\chi(\xi, \eta, t)$ can be obtained by equating Eqs. (2.3) and (1.1) for $\rho(t)$, taking the expectation value of both sides between the coherent states $|z'\rangle$, making use of the Fourier transform, and performing the integration over x', y', yielding the result

$$\chi(\xi, \eta, t) = e^{(-1/4)(\xi^2+\eta^2)} \int \int dx \, dy \, P(x, y, t) e^{-i\sqrt{2}(\xi x + \eta y)} . \tag{2.21}$$

Using the inverse Fourier transform gives

$$P(x, y, t) = 2(2\pi)^{-2} \int \int d\xi \, d\eta \, e^{(1/4)(\xi^2+\eta^2)} \chi(\xi, \eta, t) e^{i\sqrt{2}(\xi x + \eta y)} . \tag{2.22}$$

(These results agree with those of Glassgold and

Holliday,[10] although they use a different definition for the coherent states.) Comparing Eq. (2.21) with Eq. (2.4), and Eq. (2.22) with Eq. (2.2), shows that the difference in the relation between $P(x, y, t)$ or $W(q, p, t)$ with $\chi(\xi, \eta, t)$ is essentially an exponential factor.

Equation (2.19) relating $W(q, p, t)$ to $P(x, y, t)$ may now be obtained also by using the result given above in Eq. (2.21) in Eq. (2.2). It can also be easily shown that Eq. (2.19) yields $W_j(q_j', p_j', t)$ in terms of the $P_i(x_i, y_i, 0)$ as stated in Eq. (2.14) if Eq. (1.6) is used for $P_j(x_j', y_j', t)$. An alternative derivation of the time-evolution formula for the P distribution, Eq. (1.6), may be obtained using the methods presented here by means of Eqs. (2.22), (2.1), and (1.1).

A comparison of the formulas for $W_j(q_j', p_j', t)$ and $P_j(x_j', y_j', t)$ in terms of $P_i(x_i, y_i, 0)$, Eqs. (2.14) and (1.6), respectively, shows that the formula for the Wigner distribution has a factor $\exp[-\frac{1}{8}(\xi^2+\eta^2)]$ which the P-distribution formula does not. This gives the integrals over ξ, η stronger convergence in the case of the Wigner distribution and indicates why the Wigner distribution may exist at time t even when the P distribution does not. Another indication of this is given, for Gaussian distributions, by the relation $\sigma(t)=\frac{1}{2}[\sigma^w(t)-1]$ from Eq. (2.20), which shows that the width of the P distribution can be zero or negative while that of the Wigner distribution remains positive. A thorough study of the existence properties of the Wigner and P distributions has been made recently by Cahill and Glauber,[7] and they show that in fact the Wigner distribution always exists. Therefore, the formulation in terms of the Wigner distribution presented herein may be very useful in describing the statistical dynamics of quantum oscillators, especially when the P distribution breaks down, as it does, e.g., for the process described in Sec. III.

III. PARAMETRIC AMPLIFICATION IN A SINGLE MODE

We present here a study of parametric amplification in a single mode, in contrast to the usual two-mode case which has been treated rather extensively[8,11,15,16]; this degenerate form has quite different properties as compared to the two-mode case. Summarizing for reference some of the pertinent results for the two-mode case, the solution for $a_j(t)$ for that case is of the form given in Eq. (1.10), so Eqs. (1.8)–(1.12) apply directly, and we have immediately the result that an initial Gaussian P distribution for either mode remains Gaussian in form (circularly symmetric about its center) for all time. Those equations show that the distribution grows in width and moves towards infinity along a hyperbolic path in the complex z plane as time goes on (at least, for the lossless case), which corresponds to amplification.

Parametric amplification in a single mode, including losses, may be described by the Hamiltonian

$$H=\hbar\omega_0(a^\dagger a+\tfrac{1}{2})-\hbar\tfrac{1}{2}g(a^\dagger a^\dagger e^{-2i\omega_0 t}+aae^{2i\omega_0 t})+\hbar\sum_\lambda\omega_\lambda\sum_{\mu=1}^{m_\lambda}(a_{\lambda\mu}^\dagger a_{\lambda\mu}+\tfrac{1}{2})-\hbar\kappa\sum_\lambda\sum_{\mu=1}^{m_\lambda}(a_{\lambda\mu}^\dagger a+a^\dagger a_{\lambda\mu}). \tag{3.1}$$

The first term describes the self-energy of the oscillator representing the mode of interest. The second term describes the coupling of the classical pump to that mode, giving rise to the parametric amplification process. The third term represents the self-energy of a system of damping (loss) oscillators, m_λ of which have the frequency ω_λ, and the last term represents the coupling of the radiation oscillator in the single mode of interest to the reservoir of damping oscillators. The phenomenological coupling constants $\frac{1}{2}g$ and κ are taken to be positive real; they have the dimension of a frequency. The pump is assumed to be an intense coherent beam from a source such as a laser, with frequency twice that of the single mode of interest so that one pump quantum can produce two signal quanta. The phase of the pump is taken to be zero at $t=0$ for convenience. Our model Hamiltonian above is a degenerate form of the usual two-mode Hamiltonian, and a device based on this model might be called a degenerate parametric amplifier.

The operator equations of motion are obtained by using Eq. (3.1) in Heisenberg's equation of motion, along with the usual boson commutation relations (and the fact that operators pertaining to different oscillators commute). They are

$$\dot{a}(t)=-i\omega_0 a(t)+iga^\dagger(t)e^{-2i\omega_0 t}+i\kappa\textstyle\sum_\lambda\sum_\mu a_{\lambda\mu}(t), \tag{3.2}$$

$$\dot{a}_{\lambda\mu}(t)=-i\omega_\lambda a_{\lambda\mu}(t)+i\kappa a(t), \tag{3.3}$$

along with the corresponding adjoint equations. Making the substitutions

$$a(t)=A(t)e^{-i\omega_0 t}, \tag{3.4}$$

$$\sum_{\mu=1}^{m_\lambda}a_{\lambda\mu}(t)=A_\lambda(t)e^{-i\omega_0 t}, \tag{3.5}$$

results in the following simplified set of equations in the rotating frame:

$$\dot{A}(t)=igA^\dagger(t)+i\kappa\textstyle\sum_\lambda A_\lambda(t), \tag{3.6}$$

$$\dot{A}_\lambda(t)+i(\omega_\lambda-\omega_0)A_\lambda(t)=i\kappa m_\lambda A(t), \tag{3.7}$$

and the adjoint equations. The last equation was obtained after summing over μ; it may be formally integrated directly, and Robl[11] has obtained an approximate solution to this very same type of equation. He assumes a uniform density of damping oscillators, which we shall denote by $\rho(\omega_0)$, in the range $\omega_0 - \omega' < \omega_\lambda < \omega_0 + \omega'$, and in this limit obtains the result, written for our Eq. (3.7),

$$\sum_\lambda A_\lambda(t) = \sum_\lambda e^{-i(\omega_\lambda - \omega_0)t} \sum_\mu a_{\lambda\mu} + i\kappa\pi\rho(\omega_0)A(t). \tag{3.8}$$

It should be noted that this solution does not satisfy the initial conditions exactly [see Eq. (3.5)], so care must be taken in matching our general solution to the initial conditions.

Substitution of Eq. (3.8) into Eq. (3.6) gives

$$\mathring{A}(t) = igA^\dagger(t) + i\kappa\sum_\lambda e^{-i(\omega_\lambda-\omega_0)t}\sum_\mu a_{\lambda\mu} - \pi\rho(\omega_0)\kappa^2 A(t), \tag{3.9}$$

and likewise for the corresponding adjoint equations. Combining Eq. (3.9) and its adjoint equation yields the single equation in $A(t)$,

$$\ddot{A}(t) + 2\Omega\dot{A}(t) + (\Omega^2 - g^2)A(t) = \kappa\sum_\lambda[(\omega_\lambda - \omega_0) + i\Omega] \times e^{-i(\omega_\lambda-\omega_0)t}\sum_\mu a_{\lambda\mu} + \kappa g \sum_\lambda e^{i(\omega_\lambda-\omega_0)t}\sum_\mu a^\dagger_{\lambda\mu}. \tag{3.10}$$

The abbreviation $\Omega = \pi\rho(\omega_0)\kappa^2$, which has the dimension of a frequency, has been used. The general solution of this equation is found to be

$$A(t) = (k_1 e^{gt} + k_2 e^{-gt})e^{-\Omega t} - \kappa\sum_\lambda D_\lambda^{-1}[(\omega_\lambda - \omega_0) + i\Omega] \times e^{i(\omega_\lambda-\omega_0)t}\sum_\mu a_{\lambda\mu} - \kappa g\sum_\lambda (D_\lambda^*)^{-1} e^{i(\omega_\lambda-\omega_0)t}\sum_\mu a^\dagger_{\lambda\mu}, \tag{3.11}$$

where

$$D_\lambda = (\omega_\lambda - \omega_0)^2 + 2i\Omega(\omega_\lambda - \omega_0) - (\Omega^2 - g^2). \tag{3.12}$$

The constants k_1 and k_2 are to be evaluated from the initial conditions. The initial value of $A(t)$ is simply obtained from Eq. (3.4) as $A(0) = a$. The initial value of $\dot{A}(t)$ must, however, be obtained from Eq. (3.9) rather than Eq. (3.6), since it is Eq. (3.9) which makes use of the approximate solution (3.8), that was used in obtaining the general solution (3.11). Thus, from Eq. (3.9),

$$\mathring{A}(0) = iga^\dagger + i\kappa\sum_\lambda\sum_\mu a_{\lambda\mu} - \Omega a. \tag{3.13}$$

Matching these initial values to the general solution (3.11), solving for the constants k_1 and k_2, and substituting these expressions into Eq. (3.11), gives the complete solution, which can be written in the form

$$A(t) = R(t)a + S(t)a^\dagger + \sum_\lambda U_\lambda(t)\sum_{\mu=1}^{m_\lambda} a_{\lambda\mu} + \sum_\lambda V_\lambda(t)\sum_{\mu=1}^{m_\lambda} a^\dagger_{\lambda\mu}. \tag{3.14}$$

The following abbreviations are used:

$$R(t) = e^{-\Omega t}\cosh gt, \tag{3.15}$$

$$S(t) = ie^{-\Omega t}\sinh gt = iS'(t), \tag{3.16}$$

$$U_\lambda(t) = i\kappa D_\lambda^{-1}\{e^{-\Omega t}[\Omega\cosh gt + g\sinh gt - i(\omega_\lambda - \omega_0) \times \cosh gt] + i[(\omega_\lambda - \omega_0) + i\Omega]e^{-i(\omega_\lambda-\omega_0)t}\}, \tag{3.17}$$

$$V_\lambda(t) = \kappa(D_\lambda^*)^{-1}\{e^{-\Omega t}[\Omega\sinh gt + g\cosh gt + i(\omega_\lambda - \omega_0) \times \sinh gt] - ge^{i(\omega_\lambda-\omega_0)t}\}. \tag{3.18}$$

The solution for $a(t)$ is then given by Eq. (3.4). It can be shown by a rather long calculation that, in the approximation used in obtaining the solution [see discussion preceding Eq. (3.8) and at the end of Appendix C] this solution satisfies the requirement that $[a(t), a^\dagger(t)] = 1$, and it agrees with its initial value a when $t \to 0$.

The P distribution for the radiation oscillator representing the single mode of interest (which shall be called the signal oscillator or signal mode) in this model will now be studied. We assume that all modes of oscillation can initially be represented by Gaussian P distributions of quantum oscillators (except for the classical pump which has been represented by an ordinary function). We take σ for the width and $\gamma = \alpha + i\beta$ for the center of the Gaussian representing the signal oscillator, and $\sigma_{\lambda\mu}$ for the widths and $\gamma_{\lambda\mu} = 0$ for the centers of the Gaussians representing the damping oscillators, assuming them to be in thermal equilibrium initially. The solution $a(t)$ above is not of the form given in Eq. (1.10), so the formulas related to that equation are not directly applicable here. Therefore, the expression for the P distribution for the signal oscillator at time t is sought by substituting these initial Gaussian P distributions and the solution $a(t)$ above into the general formula Eq. (1.6).

Transformation is made to the rotating frames (denoted by a bar under the symbol) by means of $\zeta = \underline{\zeta}\exp(-i\omega_0 t)$, $\underline{z}' = z'\exp(-i\omega_0 t)$, and we also use $a(t) = A(t)\exp(-i\omega_0 t)$. Since the operators pertaining to different oscillators commute, the bracket expression in Eq. (1.6) factors. We have

$$\underline{P}(\underline{x}', \underline{y}', t) = (2\pi)^{-2}\int\int d\underline{\xi}\, d\underline{\eta}\, e^{i(\underline{\xi}\underline{x}' + \underline{\eta}\underline{y}')}\int\int dx\,dy\,P(x, y, 0)\langle z \,|\, e^{-(1/2)i\underline{\zeta}[R(t)a^\dagger - iS'(t)a]}e^{-(1/2)i\underline{\zeta}^*[R(t)a + iS'(t)a^\dagger]}|\, z\rangle$$
$$\times\int\cdots\int\Pi_\lambda\Pi_\mu dx_{\lambda\mu}\,dy_{\lambda\mu}\,P_{\lambda\mu}(x_{\lambda\mu}, y_{\lambda\mu}, 0)\langle\{z_{\lambda\mu}\}|\,\Pi_\lambda\Pi_\mu e^{-(1/2)i\underline{\zeta}[U^*_\lambda(t)a^\dagger_{\lambda\mu} + V^*_\lambda(t)a_{\lambda\mu}]}$$
$$\times e^{-(1/2)i\underline{\zeta}^*[U_\lambda(t)a_{\lambda\mu} + V_\lambda(t)a^\dagger_{\lambda\mu}]}|\{z_{\lambda\mu}\}\rangle. \tag{3.19}$$

Each bracket is evaluated after first putting the operator function inside in normal form by means of the Baker-Hausdorff rules. [Note that the first two terms of the solution (3. 14), which occur in the first bracket above, do not commute; this, in fact, is the basic mathematical difference between the degenerate and the two-mode cases.] Then putting in the Gaussian expressions for the initial P distributions as described previously and carrying out the integration over x, y and $x_{\lambda\mu}$, $y_{\lambda\mu}$ by means of the formula

$$\int_{-\infty}^{\infty} e^{-A^{-1}(q-q_0)^2+iBq}\, dq=(\pi A)^{1/2}e^{-(A/4)B^2+iq_0B}, \qquad A\geq 0\ , \tag{3.20}$$

yields

$$\underline{P}(\underline{x}',\underline{y}',t)=(2\pi)^{-2}\int\int d\underline{\xi}\,d\underline{\eta}\,\exp\{-\tfrac{1}{4}[(R^2+S'^2)\sigma+S'^2+\textstyle\sum_\lambda|U_\lambda|^2\sum_u\sigma_{\lambda u}+\sum_\lambda|V_\lambda|^2\sum_u(\sigma_{\lambda u}+1)](\underline{\xi}^2+\underline{\eta}^2)\}$$
$$\times\exp\{-\tfrac{1}{4}[\textstyle\sum_\lambda \mathrm{Re}(U_\lambda V_\lambda)\sum_u(2\sigma_{\lambda u}+1)](\underline{\xi}^2-\underline{\eta}^2)\}\exp\{-\tfrac{1}{2}[(2\sigma+1)RS'+\sum_\lambda \mathrm{Im}(U_\lambda V_\lambda)\sum_u(2\sigma_{\lambda u}+1)]\underline{\xi}\underline{\eta}\}$$
$$\times\exp\{i[\underline{x}'-(\alpha R+\beta S')]\underline{\xi}\}\exp\{i[\underline{y}'-(\alpha S'+\beta R)]\underline{\eta}\}\ . \tag{3.21}$$

[In this equation and those that follow, the time dependence of the functions $R(t)$, $S'(t)$, $U_\lambda(t)$, and $V_\lambda(t)$ is not indicated for the sake of simplicity].

The terms with sums involving U_λ and V_λ, which are of the form

$$\sum_\lambda f(\omega_\lambda-\omega_0)\sum_{\mu=1}^{m_\lambda}\sigma_{\lambda\mu},$$

become

$$\sum_\lambda m_\lambda\sigma_\lambda f(\omega_\lambda-\omega_0)$$

after summing over μ. If we assume that only those damping oscillators with frequencies near resonance with the signal frequency are of importance, then $\sigma_\lambda=(e^{\hbar\omega_\lambda/kT}-1)^{-1}$ can be replaced by the constant value $\sigma'=(e^{\hbar\omega_0/kT}-1)^{-1}$, and the expression becomes $\sigma\ \sum_\lambda m_\lambda f(\omega_\lambda-\omega_0)$. If we pass to the limit of a uniform density of damping oscillators $\rho(\omega_0)$ as was done in obtaining the solution $a(t)$, then the sum $\sigma'\sum_\lambda m_\lambda f(\omega_\lambda-\omega_0)$ is replaced by the integral $\sigma'\int_0^\infty d\omega_\lambda\rho(\omega_\lambda)f(\omega_\lambda-\omega_0)$. Using the same assumption as above, letting $\omega=\omega_\lambda-\omega_0$, and extending the limits to infinity, we arrive at the replacement formula

$$\sum_\lambda f(\omega_\lambda-\omega_0)\sum_{\mu=1}^{m_\lambda}\sigma_{\lambda\mu}\rightarrow\sigma'\rho(\omega_0)\int_{-\infty}^{\infty}d\omega f(\omega). \tag{3.22}$$

The terms of this form in Eq. (3.21) without the $\sigma_{\lambda u}$ are of course treated in the same way. This calculation is then carried out for those terms in Eq. (3.21) of this form. This is done in Appendix C. In particular, it is found there that $\sum_\lambda \mathrm{Re}(U_\lambda V_\lambda)$, which becomes $\rho(\omega_0)\int \mathrm{Re}[U(\omega)V(\omega)]d\omega$, is zero, so the $(\underline{\xi}^2-\underline{\eta}^2)$ term in Eq. (3.21) disappears. This allows the elimination of the cross term $(\underline{\xi}\,\underline{\eta})$ by means of a simple 45° rotation (taken counterclockwise) from the $\underline{\xi},\underline{\eta}$ frame to the $\underline{\xi}',\underline{\eta}'$ frame. If a similar rotation is made from the $\underline{x}',\underline{y}'$ frame to the $\underline{x}'',\underline{y}''$ frame, then the expression for the P distribution becomes a simple product of two standard integrals of the form given in Eq. (3.20),

$$\underline{P}'(\underline{x}'',\underline{y}'',t)=(2\pi)^{-1}\int\exp\{-\tfrac{1}{4}[(R+S')^2\sigma+S'(R+S')+\sigma'\rho(\omega_0)\int|U|^2d\omega+(\sigma'+1)\rho(\omega_0)\int|V|^2d\omega$$
$$+(2\sigma'+1)\rho(\omega_0)\int \mathrm{Im}(UV)d\omega]\underline{\xi}'^2\}\exp(i\{\underline{x}''-(1/\sqrt{2})[(\alpha+\beta)(R+S')]\}\underline{\xi}')d\underline{\xi}'(2\pi)^{-1}$$
$$\times\int\exp\{-\tfrac{1}{4}[(R-S')^2\sigma-S'(R-S')+\sigma'\rho(\omega_0)\int|U|^2d\omega+(\sigma'+1)\rho(\omega_0)\int|V|^2d\omega$$
$$-(2\sigma'+1)\rho(\omega_0)\int \mathrm{Im}(UV)d\omega]\underline{\eta}'^2\}\exp(i\{\underline{y}''-(1/\sqrt{2})[(-\alpha+\beta)(R-S')]\}\underline{\eta}')d\underline{\eta}'\ . \tag{3.23}$$

The expressions for the integrals over ω, involving the functions U and V, which occur above in the exponentials, are given in Appendix C, Eqs. (C7), (C8), and (C11), and it is seen there that they are all non-negative. The other two terms which precede these in each exponential involved are also themselves non-negative. Therefore, it is seen that the integral over $\underline{\xi}'$ exists at all times (see Eq. 3.20) whereas the integral over $\underline{\eta}'$ exists only when

$$[(R-S')^2\sigma-S'(R-S')+\sigma'\rho(\omega_0)\int|U|^2d\omega+(\sigma'+1)\rho(\omega_0)$$
$$\times\int|V|^2d\omega-(2\sigma'+1)\rho(\omega_0)\int \mathrm{Im}(UV)d\omega]\geq 0. \tag{3.24}$$

Substituting the expressions for the various quantities, and simplifying, results in the requirement that the following condition be satisfied:

$$\sigma\geq\left(\frac{g-2\sigma'\Omega}{\Omega+g}\right)\frac{\tanh(\Omega+g)t}{1-\tanh(\Omega+g)t}\quad . \tag{3.25}$$

In other words, the P distribution exists only as long as the value of the expression on the right-hand side above does not exceed the value of σ for the initial P distribution. So the greater its initial width is, the longer the P distribution exists. As pointed out in Appendix C, the results for the integrals involved here are valid for all values of Ω and g, so this result is also. A plot of Eq. (3.25) is shown in Fig. 1, with the amount of damping, characterized by $\Omega = \pi\rho(\omega_0)\kappa^2$, as a parameter. By means of

$$\tan_0(\Omega+g)t/[1-\tan_0(\Omega+g)t] = \tfrac{1}{2}\{\exp[2(1+\Omega/g)gt]-1\}$$

the inequality (3.25) can also be expressed in dimensionless form as a function of (gt) with the ratio (Ω/g) as a parameter.

Therefore, it is seen that the P distribution for the signal mode in this model ceases to exist at some finite time when $\Omega < g/2\sigma'$. (That time is determined by the intersection of a given horizontal line $\sigma=$ const with the appropriate curve in Fig. 1.)

In the region for which the P distribution does exist [that is when the inequality (3.25) is satisfied], the integrals over $\underline{\xi}'$, $\underline{\eta}'$ in Eq. (3.23) can be evaluated, using Eq. (3.20). This gives the result

$$\underline{P}'(\underline{x}'', \underline{y}'', t) = [\pi\sigma_+(t)]^{-1/2} e^{-[\sigma_+(t)]^{-1}[\underline{x}''-\alpha(t)]^2} \times [\pi\sigma_-(t)]^{-1/2} e^{-[\sigma_-(t)]^{-1}[\underline{y}''-\beta(t)]^2}, \quad (3.26)$$

where

$$\sigma_\pm(t) = (R\pm S')^2\sigma \pm S'(R\pm S') + \sigma'\rho(\omega_0)\int |U|^2 d\omega + (\sigma'+1)\rho(\omega_0)\int |V|^2 d\omega \pm (2\sigma'+1)\rho(\omega_0)\int \mathrm{Im}(UV)d\omega, \quad (3.27)$$

$$\alpha(t) = (1/\sqrt{2})(\alpha+\beta)(R+S'),\ \beta(t) = (1/\sqrt{2})(-\alpha+\beta)(R-S'), \quad (3.28)$$

and t is such that the inequality (3.25) holds. The expressions for the integrals in Eq. (3.27) are given in Appendix C, Eqs. (C7), (C8), and (C11), and those for R and S' by Eqs. (3.15) and (3.16).

We have denoted rotating frames by a bar under the symbols and 45° rotations by adding a prime to the symbols involved. The result in Eq. (3.26) is expressed in a frame rotated 45° counterclockwise and rotating clockwise with angular frequency ω_0, so that $\alpha(t)$ and $\beta(t)$ are in this frame. The relation of this, the $\underline{x}''$, $\underline{y}''$ frame, to the original x′, y′ frame, is given by

$$\underline{x}'' = x'\cos(\omega_0 t - \tfrac{1}{4}\pi) - y'\sin(\omega_0 t - \tfrac{1}{4}\pi), \quad (3.29)$$

$$\underline{y}'' = x'\sin(\omega_0 t - \tfrac{1}{4}\pi) + y'\cos(\omega_0 t - \tfrac{1}{4}\pi). \quad (3.30)$$

The P distribution is a simple product of Gaussians in the $\underline{x}''$, $\underline{y}''$ frame. As time progresses, the original circularly symmetric Gaussian, in general, becomes elliptically symmetric about its center: It narrows in the $\underline{y}''$ direction and elongates (unless the damping is large) in the $\underline{x}''$ direction. For $\Omega < g/2\sigma'$, the P distribution eventually degenerates into a δ function in the $\underline{y}''$ direction and then ceases to be defined at the finite time dictated by Eq. (3.25), i.e., when $\sigma_-(t) \to 0$. We therefore have an example for which the P distribution ceases to exist as a well-behaved function under certain conditions, and we may see how this comes about as it evolves in time.

The center of the P distribution may go either to zero or to infinity, depending on whether $\Omega > g$ or $\Omega < g$, respectively. The axes of the elliptical cross section of the distribution are at all times parallel to the rotating axes $\underline{x}''$, $\underline{y}''$. When there is no damping, $\Omega = 0$, giving $\sigma_\pm(t) = (c\pm s)^2\sigma \pm s(c\pm s)$ and $\alpha(t) = (1/\sqrt{2})(\alpha+\beta)(c+s)$, $\beta(t) = (1/\sqrt{2})(-\alpha+\beta)\times(c-s)$, where $c = \cosh gt$ and $s = \sinh gt$. This shows that, for $\Omega = 0$, the center always moves towards infinity, approaching the positive or negative $\underline{x}''$ axis asymptotically as $t\to\infty$ (unless it is initially

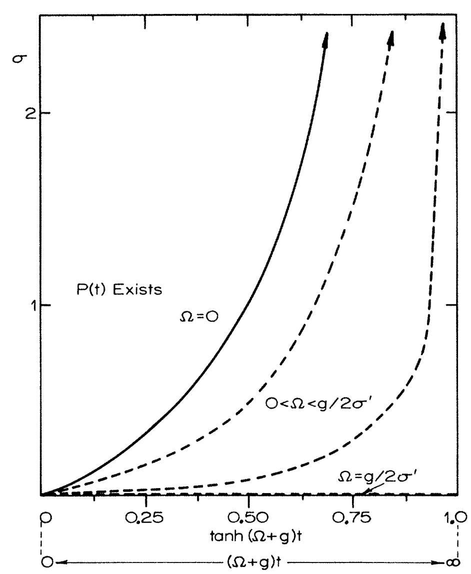

FIG. 1. P distribution for parametric amplification in a single mode exists only in the region above the appropriate curve, according to the inequality (3.25). The amount of damping present is characterized by the parameter Ω; the solid curve is for no damping. Since the condition for amplification is such that $\Omega \ll g/2\sigma'$ for optical frequencies and ordinary temperatures of the medium, it is seen that $P(t)$ ceases to exist at some finite time under these conditions, and this may occur quite soon. In fact, if the initial state is a coherent one ($\sigma=0$) and $\Omega < g/2\sigma'$, then $P(t)$ breaks down immediately.

located at the origin, in which case it remains there). In this case the path is one of the branches of the rectangular hyperbola, with respect to the $\underline{x}''$, $\underline{y}''$ frame, $\alpha(t)\ \beta(t)=\frac{1}{2}(-\alpha^2+\beta^2)$. This compares to the corresponding two-mode case for which the path of the center is a general hyperbola (nonrectangular).

The corresponding Wigner distribution for this example will now be studied. It is most convenient to use Eq. (2.14), with the initial P distributions assumed to be Gaussian as above. The same procedure as used above is followed, giving the result

$$\underline{W}'(\underline{q}'',\underline{p}'',t)=[\pi\sigma_+^w(t)]^{-1/2}e^{-[\sigma_+^w(t)]^{-1}[\underline{q}''-\alpha^w(t)]^2}\times[\pi\sigma_-^w(t)]^{-1/2}e^{-[\sigma_-^w(t)]^{-1}[\underline{p}''-\beta^w(t)]^2}, \quad (3.31)$$

where

$$\sigma_\pm^w(t)=2\sigma_\pm(t)+1, \quad (3.32)$$

$$\alpha^w(t)=\sqrt{2}\,\alpha(t), \qquad \beta^w(t)=\sqrt{2}\beta(t). \quad (3.33)$$

The expressions for $\sigma_\pm(t)$, $\alpha(t)$, and $\beta(t)$ are given in Eqs. (3.27) and (3.28), which refer to the P distribution. It is noted that these relationships agree with the predictions of Eq. (2.20). We see that the Wigner distribution is quite similar to the P distribution, being a product of Gaussians of unequal width in the frame, analogous to Eqs. (3.29) and (3.30), defined by,

$$\underline{q}''=q'\cos(\omega_0t-\tfrac{1}{4}\pi)-p'\sin(\omega_0t-\tfrac{1}{4}\pi), \quad (3.34)$$

$$\underline{p}''=q'\sin(\omega_0t-\tfrac{1}{4}\pi)+p'\cos(\omega_0t-\tfrac{1}{4}\pi). \quad (3.35)$$

There is an important difference, however, between the two distributions. This difference is expressed in Eq. (3.32). The requirement that must be met in order for the Wigner distribution to exist, and be given by Eq. (3.31), is, similar to the case for the P distribution, that the integral over $\underline{\eta}'$ exists. This is equivalent to the requirement that $\sigma_-^w(t)$ be non-negative. From Eqs. (3.32) and (3.27), it may be shown that this reduces to the requirement that the following condition be satisfied:

$$\sigma\geq[-1/2(\Omega+g)]\{2\sigma'\Omega\,[e^{2(\Omega+g)t}-1]+\Omega e^{2(\Omega+g)}+g^t\}. \quad (3.36)$$

That is, the Wigner distribution for the signal mode exists as long as the value of the expression on the right above does not exceed the value of σ for the initial P distribution. However, it is seen that the expression on the right-hand side is always negative, so the condition is satisfied for all time. Thus, there is no restriction on the existence of the Wigner distribution for the signal mode in this model, in agreement with the general statements made in Secs. I and II.

The motion of the center of the Wigner distribution is similar to that for the P distribution, the distance of its center from the origin being $\sqrt{2}$ times that of the P distribution. The width of the Wigner distribution is also similar to that for the P distribution, being larger according to Eq. (3.32), but it never goes to zero in any direction (except in the limit of no damping and infinite time).

We have found that the Wigner distribution describes the system for all time, whereas the P distribution, in general, does not, and it may be used to calculate average values by means of the general formulas developed in Sec. II. For example, the average number of photons in the signal mode at any time for this model is, from Eq. (3.31) for the Wigner distribution and Eq. (2.11), with $N(t)=\frac{1}{2}[P^2(t)+Q^2(t)-1]$,

$$\langle N(t)\rangle=\tfrac{1}{2}\{|\gamma^w(t)|^2+\tfrac{1}{2}[\sigma_+^w(t)+\sigma_-^w(t)]-1\}. \quad (3.37)$$

Substituting Eqs. (3.32) and (3.33) and using also Eqs. (3.27) and (3.28), the following result is obtained:

$$\langle N(t)\rangle=\tfrac{1}{2}\left[\langle N(0)\rangle+|\gamma|^2\sin2\phi-\frac{2\sigma'\Omega+g}{2(\Omega-g)}\right]e^{-2(\Omega-g)t}+\tfrac{1}{2}\left[\langle N(0)\rangle-|\gamma|^2\sin2\phi-\frac{2\sigma'\Omega-g}{2(\Omega+g)}\right]e^{-2(\Omega+g)t}+\frac{2\sigma'\Omega^2+g^2}{2(\Omega^2-g^2)}. \quad (3.38)$$

We have used $\langle N(0)\rangle=|\gamma|^2+\sigma$ and $\gamma=\alpha+i\beta=|\gamma|e^{i\phi}$. The angle ϕ is the phase of the signal at $t=0$; since the pump phase was taken to be zero at $t=0$, ϕ also represents the relative phase between signal and pump at $t=0$. Although the pump is rotating in the complex plane at twice the angular frequency as that of the signal, so that the relative phase changes with time, it is seen in the above equation for the expectation value of the occupation number that it is twice the signal phase relative to the pump that is of importance, and this quantity remains constant.

From Eq. (3.38) we find that when $\Omega>g$, $\langle N(t)\rangle$ changes from $\langle N(0)\rangle$ to $\frac{1}{2}(2\sigma'\Omega^2+g^2)/(\Omega^2-g^2)$ as t goes from zero to infinity. (It may be noted that, for $\Omega\gg g$, $\langle N(t)\rangle\to\sigma'$ as $t\to\infty$, i.e., the signal oscillator settles into thermal equilibrium with the damping oscillators for very large damping.) This represents a decrease in the signal (except for $\Omega\approx g$, in which case the signal would grow by a limited amount). For $\Omega<g$, $\langle N(t)\rangle$ increases from its initial value without bound. The expression corresponding to Eq. (3.38) for the special case $\Omega=g$ shows that $\langle N(t)\rangle$ is finite at finite times, approaching infinity as t goes to finity. Therefore, it is concluded that the condition for unlimited am-

plification is $\Omega \leq g$.

From Eq. (3.25) it has already been concluded that the P distribution breaks down after a certain time when $\Omega < g/2\sigma'$. But $\sigma' = [\exp(\hbar\omega_0/kT) - 1]^{-1}$, and for ordinary temperatures and optical frequencies, e.g., $\sigma' \ll 1$. Therefore, if the condition for amplification ($\Omega \leq g$) is met for these conditions, then $\Omega \ll g/2\sigma'$ and one is well within the region of operation for which the P distribution for the signal mode ceases to exist after a certain time (see Fig. 1). It is concluded therefore that the ceasing of the P distribution to exist here is not a trivial or uninteresting case, but occurs for the very region of operation that is of interest. It is thus quite desirable to have a description in terms of the Wigner distribution formulation available as present herein.

It is perhaps of interest to compare our quantum theory of the degenerate parametric amplifier to the classical theory; Louisell,[17] for example, has presented the classical treatment for the ideal case of no damping. Our results for no damping are obtained by letting Ω go to zero. From Eq. (3.38), we find the ideal value in this limit to be

$$\langle\langle N(t)\rangle\rangle = \langle\langle N(0)\rangle\rangle (c^2 + s^2) + s^2 + 2cs\,|\gamma|^2 \sin 2\phi, \quad (3.39)$$

where $c = \cosh gt$, $s = \sinh gt$, $\langle\langle N(0)\rangle\rangle = |\gamma|^2 + \sigma$, and $\gamma = |\gamma|\, e^{i\phi}$. We find that $\langle\langle N(t)\rangle\rangle \to \infty$ as $t \to \infty$, regardless of the relative phase angle ϕ at $t = 0$. Therefore, amplification always occurs for this lossless model in the quantum theory. This is in contrast to the classical result which allows the signal to decay rather than to grow, depending on the phase. The difference between the results is the independent term $\sinh^2 gt$ occurring in Eq. (3.39), which is due to quantum noise; it is present even if there is no signal initially [$\langle\langle N(0)\rangle\rangle = 0$, which implies $|\gamma|^2 = 0$ also]. To compare to the classical theory, in which the signal is a classical coherent wave, we may describe this in our notation by letting $\sigma = 0$. This means $\langle\langle N(0)\rangle\rangle = |\gamma|^2$, and even for the most negative contribution of the phase-dependent term in Eq. (3.39), we have $\langle\langle N(t)\rangle\rangle = \langle\langle N(0)\rangle\rangle\, e^{-2gt} + s^2$, which still eventually grows in time, although it may initially decrease if $\langle\langle N(0)\rangle\rangle$ is large enough. The corresponding classical expression is the same except that the spontaneous emission term $\sinh^2 gt$ does not appear, so the signal would decay in the classical theory for this case.

The field fluctuations will now be studied, for which the average values of the observables whose operators correspond to the displacement and to the momentum of the oscillator representing the signal mode are needed. Since the Wigner distribution has already been obtained, Eq. (2.11) may now be used for this purpose. Any power of the displacement therefore has the average value given by the formula, written for the present case,

$$\langle\langle Q^\lambda(t)\rangle\rangle = \int\int d\underline{q}''\, d\underline{p}''\; q^\lambda(\underline{q}'', \underline{p}'', t)\, \underline{W}'(\underline{q}'', \underline{p}'', t). \quad (3.40)$$

Using Eq. (3.31) and the inverse of Eqs. (3.34) and (3.35) gives, for $\lambda = 1$,

$$\langle\langle Q(t)\rangle\rangle = \alpha^w(t)\cos\theta + \beta^w(t)\sin\theta, \quad (3.41)$$

and for $\lambda = 2$,

$$\langle\langle Q^2(t)\rangle\rangle = \{[\alpha^w(t)]^2 + \tfrac{1}{2}\sigma_+^w(t)\}\cos^2\theta + 2\alpha^w(t)\beta^w(t)\sin\theta\cos\theta + \{[\beta^w(t)]^2 + \tfrac{1}{2}\sigma_-^w(t)\}\sin^2\theta, \quad (3.42)$$

where we have used the abbreviation $\theta = \omega_0 t - \frac{1}{4}\pi$. A similar calculation for the momentum operator $P(t)$ produces the results

$$\langle\langle P(t)\rangle\rangle = -\alpha^w(t)\sin\theta + \beta^w(t)\cos\theta, \quad (3.43)$$

$$\langle\langle P^2(t)\rangle\rangle = \{[\alpha^w(t)]^2 + \tfrac{1}{2}\sigma_+^w(t)\}\sin^2\theta - 2\alpha^w(t)\beta^w(t) \times \sin\theta\cos\theta + \{[\beta^w(t)]^2 + \tfrac{1}{2}\sigma_-^w(t)\}\cos^2\theta. \quad (3.44)$$

These results yield the mean-square fluctuations in the variables corresponding to Q and to P, and hence determine the fluctuations in the magnetic and electric fields, respectively, that comprise the signal,

$$(\Delta Q)^2 = \tfrac{1}{2}\sigma_+^w(t)\cos^2\theta + \tfrac{1}{2}\sigma_-^w(t)\sin^2\theta, \quad (3.45)$$

$$(\Delta P)^2 = \tfrac{1}{2}\sigma_+^w(t)\sin^2\theta + \tfrac{1}{2}\sigma_-^w(t)\cos^2\theta. \quad (3.46)$$

At the times when $\cos^2\theta$ is unity and $\sin^2\theta$ is zero, the fluctuations are greater in Q than in P, since $\sigma_+^w(t) > \sigma_-^w(t)$ for $t > 0$. From Eqs. (3.34) and (3.35) it is seen that at these times $\underline{W}''(\underline{q}'', \underline{p}'', t)$ of Eq. (3.31) becomes $W(q', p', t)$ and the product of unequal-width Gaussians is then elongated in a direction parallel to the q' axis. Alternatively, when $\cos^2\theta$ is zero and $\sin^2\theta$ is unity, the fluctuations are greater in P than in Q, and $W(t)$ is elongated parallel to the p' axis. Figure 2 shows the cross section of the Wigner distribution which has evolved after several revolutions in the q', p' frame from an initial distribution centered on the positive $\underline{q}''$ axis in its position at $t = 0$ $(\alpha = \beta > 0)$ as a special case. The angle $\theta = \omega_0 t - \frac{1}{4}\pi$ had been defined so as to be measured positively clockwise, in the same sense as the rotation, from the positive q' axis. The two sets of axes coincide when $\theta = \frac{1}{2}n\pi$, $n = 0, 1, 2, \cdots$, which corresponds to the situations described above regarding fluctuations. For the case shown in Fig. 2, $\beta^w(t) = 0$, so the average values corresponding to P and Q become zero when $W(t)$ is aligned along the q' and p' axes, respectively.

The fluctuations in Q and in P are thus seen to

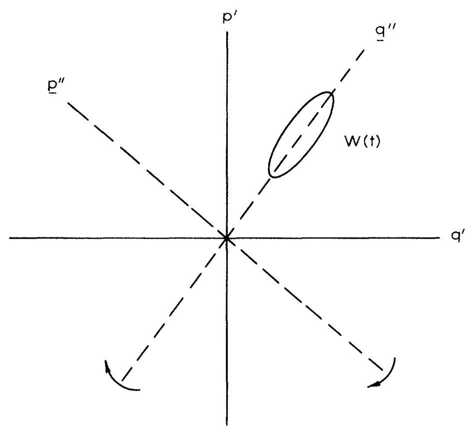

FIG. 2. Wigner distribution for parametric amplification in a single mode, in the special case for which the initial circularly symmetric Gaussian distribution was centered on the positive $\underline{q}''$ axis, (i.e., $\alpha=\beta>0$). The center of the distribution is moving along the $\underline{q}''$ axis away from the origin for small damping or towards the origin for large damping. The eccentricity of the elliptical cross section increases as time goes on, unless the damping is quite large. The ellipse represents the intersection of a plane, parallel to and above the q', p' plane, with the distribution function, and is qualitative only. The distribution, along with the $\underline{q}'', \underline{p}''$ frame, rotates clockwise with respect to the q', p' frame, and for the case shown $W(t)$ remains centered on the positive $\underline{q}''$ axis, with the major axis of its elliptical cross section remaining along the $\underline{q}''$ axis.

alternately and periodically grow and diminish as $W(t)$ rotates clockwise in the q', p' frame. Stating this in another way, the fluctuations in the magnetic and electric fields themselves fluctuate in magnitude as time goes on, this magnitude being proportional to the width of $W(t)$ in a direction parallel to the q', p' axes, respectively, at any time. As t gets larger, the maximum value of these fluctuations increases and the minimum value decreases (unless the damping is large). In particular, these extreme values are proportional to the widths, and as $t\to\infty$, $\sigma_+^w(t)\to\Omega(2\sigma'+1)/(\Omega-g)$ for $\Omega>g$ or $\sigma_+^w(t)\to\infty$ for $\Omega\le g$, and $\sigma_-^w(t)\to\Omega(2\sigma'+1)/(\Omega+g)$ for all Ω and g.

It might be of interest to consider the ideal limit of no damping in connection with the fluctuations. Letting $\Omega\to 0$, we find that as $t\to\infty$, $\sigma_+^w(t)\to\infty$ and $\sigma_-^w(t)\to 0$, so that the fluctuations in Q and in P alternately and periodically become infinite and zero. The product, however, never becomes zero, but obeys the uncertainty principle: From Eqs. (3.45) and (3.46), the uncertainty product squared, considered as a function of θ, has its minimum value when $\theta=\frac{1}{2}n\pi$, $n=0, 1, 2, \cdots$:

$$[(\Delta P)^2(\Delta Q)^2]_{\theta=n\pi/2}=\tfrac{1}{4}\sigma_+^w(t)\sigma_-^w(t). \tag{3.47}$$

If the expressions for the widths, Eqs. (3.27) and (3.32), are used with $\Omega\neq 0$, it is found that $\Delta P\Delta Q>\frac{1}{2}$ as would be expected, but for $\Omega=0$ and $\sigma=0$ substitution of these expressions into Eq. (3.47) gives

$$(\Delta P\Delta Q)_{\min}=\tfrac{1}{2}\,. \tag{3.48}$$

So, in general, we have $\Delta P\Delta Q\geq\frac{1}{2}$, in agreement with the uncertainty principle.

Thus, we see that the uncertainty principle is satisfied for the no-damping case also, i.e., even when the fluctuations in one of the variables become zero while those in the other variable become infinite. We also see from Eq. (3.48) that the absolute minimum uncertainty is attainable (periodically) in this lossless case. This was found to occur if the initial state of the signal mode is a coherent one and at such times that $\theta=\frac{1}{2}n\pi$, $n=0, 1, 2, \cdots$. This was seen earlier to be the same times that the fluctuations are their extreme. For the special case shown in Fig. 2, the Wigner distribution is elongated along one of the axes in the q', p' frame at these times. The minimum values of the fluctuations in the variables corresponding to Q and P occur in this case when their expectation values are zero, and the maximum fluctuations coincide with the peak values of the corresponding variables, and at these times minimum uncertainty obtains. In the lossless case, the minimum values of the fluctuations approach zero as $t\to\infty$. This implies that the fluctuations of the electric and magnetic fields tend to zero whenever their average values are zero. When damping is included this ideal limit of course is not reached.

IV. FINAL REMARKS

The theory given in Sec. III on parametric amplification in a single mode has been extended to include the process of frequency up-conversion, which might be of interest as the basis for an infrared detector. A brief summary of these calculations follows: Neglecting damping, the model Hamiltonian may be written as

$$H=\hbar\omega_1(a_1^\dagger a_1+\tfrac{1}{2})+\hbar\omega_2(a_2^\dagger a_2+\tfrac{1}{2})-\hbar g(a_1^\dagger a_2 e^{-i\omega_I t}+a_2^\dagger a_1 e^{i\omega_I t})-\hbar\tfrac{1}{2}\kappa(a_2^\dagger a_2^\dagger e^{-i\omega_{II}t}+a_2 a_2 e^{i\omega_{II}t}), \tag{4.1}$$

where $\omega_1=\omega_I+\omega_2$ and $\omega_2=\frac{1}{2}\omega_{II}$. A single pump of frequency $\omega_0=\omega_I=\omega_{II}$ might be used for both processes, such that frequency up-conversion would occur at $\omega_1=\frac{3}{2}\omega_0$ and amplification at $\omega_2=\frac{1}{2}\omega_0$. In either case, the solution for $a_1(t)=A_1(t)\exp(-i\omega_1 t)$ is found to be such that

$$A_1(t)=\tfrac{1}{2}(m_1^2-m_3^2)^{-1}\{[-(m_3^2+g^2)a_1-g\kappa a_2^\dagger+i(\kappa/m_1)g^2a_1^\dagger$$
$$+i(g/m_1)(\kappa^2-g^2-m_3^2)a_2]e^{m_1t}+[-(m_3^2+g^2)a_1$$
$$-g\kappa a_2^\dagger-i(\kappa/m_1)g^2a_1^\dagger-i(g/m_1)(\kappa^2-g^2-m_3^2)a_2]e^{-m_1t}$$
$$+[(m_1^2+g^2)a_1+g\kappa a_2^\dagger-i(\kappa/m_3)g^2a_1^\dagger-i(g/m_3)$$
$$\times(\kappa^2-g^2-m_1^2)a_2]e^{m_3t}+[(m_1^2+g^2)a_1+g\kappa a_2^\dagger$$
$$+i(\kappa/m_3)g^2a_1^\dagger+i(g/m_3)(\kappa^2-g^2-m_1^2)a_2]e^{-m_3t}\}\,, \tag{4.2}$$

where

$$m_1=[(\tfrac{1}{2}\kappa^2-g^2)+\kappa(\tfrac{1}{4}\kappa^2-g^2)^{1/2}]^{1/2},$$
$$m_3=[(\tfrac{1}{2}\kappa^2-g^2)-\kappa(\tfrac{1}{4}\kappa^2-g^2)^{1/2}]^{1/2}. \tag{4.3}$$

Whether the expressions for m_1 and m_3 are real, imaginary, or complex determines whether $A_1(t)$ is pure exponential, oscillating, or a combination of the two. For $\frac{1}{2}\kappa>g$, m_1 and m_3 are both positive real, while for $\frac{1}{2}\kappa<g$ they are both complex. For $\kappa=0$, we have the pure imaginary case corresponding to just frequency conversion. Thus, the solution $A_1(t)$ goes from pure oscillatory to a combination of oscillatory and exponential, to pure exponential as $\frac{1}{2}\kappa$ increases from zero.

For the case $\frac{1}{2}\kappa=g$, which corresponds to equal coupling for the amplification ($\frac{1}{2}\kappa$) and conversion (g) processes, the solution is

$$A_1(t)=(\cosh gt-gt\sinh gt)a_1-i(\sinh gt-gt\cosh gt)a_1^\dagger$$
$$+i(gt\cosh gt)a_2-(gt\sinh gt)a_2^\dagger\,. \tag{4.4}$$

This solution is a good deal simpler than that in Eq. (4.2) and has been used to obtain the distribution functions for mode 1 in this case. The P distribution is again found to cease to exist after a certain time, whereas the Wigner distribution does not and is given by

$$\underline{W}_1'(\underline{q}_1'',\underline{p}_1'',t)=[\pi\sigma^w_{\underline{q}_1''}(t)]^{-1/2}e^{-[\sigma^w_{\underline{q}_1''}(t)]^{-1}[\underline{q}_1''-\alpha_1^w(t)]^2}$$
$$\times[\pi\sigma^w_{\underline{p}_1''}(t)]^{-1/2}e^{-[\sigma^w_{\underline{p}_1''}(t)]^{-1}[\underline{p}_1''-\beta_1^w(t)]^2}\,, \tag{4.5}$$

where

$$\sigma^w_{\underline{q}_1''}(t)=2[(1+gt)^2(c-s)^2\sigma_1+(gt)^2(c-s)^2\sigma_2$$
$$-s(c-s)+gt(c-s)^2+(gt)^2(c-s)^2]+1, \tag{4.6}$$

$$\sigma^w_{\underline{p}_1''}(t)=2[(1-gt)^2(c+s)^2\sigma_1+(gt)^2(c+s)^2\sigma_2$$
$$+s(c+s)-gt(c+s)^2+(gt)^2(c+s)^2]+1, \tag{4.7}$$

$$\alpha_1^w(t)=(\alpha_1+\beta_1)(1+gt)(c-s)+(\alpha_2-\beta_2)(gt)(c-s), \tag{4.8}$$

$$\beta_1^w(t)=(-\alpha_1+\beta_1)(1-gt)(c+s)+(\alpha_2+\beta_2)(gt)(c+s). \tag{4.9}$$

This result is in a form similar to that found in Sec. III but the shorter width of the distribution is now parallel to the $\underline{q}''$ axis rather than the greater width as before. In reference to Fig. 2 the distribution $W(t)$ would now be oriented perpendicular to that shown so that the minor axis of the elliptical cross section would be along the $\underline{q}''$ axis. The average number of photons in mode 1 is obtained as

$$\langle N_1(t)\rangle=\langle N_1(0)\rangle\{[1+(gt)^2](c^2+s^2)-4(gt)cs\}$$
$$+\langle N_2(0)\rangle[(gt)^2(c^2+s^2)]+2|\gamma_1|^2\sin2\phi_1\{(gt)(c^2+s^2)$$
$$-[1+(gt)^2]cs\}+|\gamma_2|^2\sin2\phi_2[(gt)^2(c^2+s^2)]$$
$$+2|\gamma_1|\,|\gamma_2|\sin(\phi_1-\phi_2)[(gt)(c^2+s^2)-2(gt)^2cs]$$
$$+2|\gamma_1|\,|\gamma_2|\cos(\phi_1+\phi_2)[(gt)^2(c^2+s^2)-2(gt)cs]$$
$$+(gt)^2(c^2+s^2)+2(gt)cs+s^2\,. \tag{4.10}$$

The notation is similar to that in Sec. III.

Average values, fluctuations, and the uncertainty product for the variables corresponding to $\boldsymbol{Q}$ and P are given by equations of the same form as those obtained previously, Eqs. (3.41)–(3.47). However, due to the reversal of the relative magnitude of the widths of the distribution, the maximum values of the fluctuations for the case shown in Fig. 2 occur when the average values are zero, and the minimum fluctuations coincide with the peak average values. The absolute minimum uncertainty is not attained in this model; the equation corresponding to Eq. (3.48), i.e., for $\sigma=0$ and $\theta=\frac{1}{2}n\pi$, is

$$[(\Delta P)^2(\Delta Q)^2]_{\min}=\tfrac{1}{4}+(gt)^4\,. \tag{4.11}$$

In summary, we have used in this paper distribution functions of quantum oscillators to describe parametric amplification in a single mode, with emphasis on the usefulness of the Wigner distribution. This degenerate form of the two-mode process was seen to be significantly different from that nondegenerate case. The P distribution for the signal mode was found to cease to exist after a finite time for amplification at optical frequencies and ordinary temperatures of the amplifying medium. The Wigner distribution was obtained by means of the general theory developed herein, and was found to be well behaved at all times. Amplification was found to always occur over an extended period of time in this quantum treatment, in the limit of no damping, whereas the corresponding classical treatment allows the signal to decay for certain phase relationships. The field fluctuations in the signal mode were studied and in the limit of no damping were found to alternately and periodically approach zero and infinite values as time goes on, and to be periodically in a state of minimum uncertainty. The combined processes of parametric amplification in a single mode and frequency up-conversion was also treated, with explicit results given for the case where the coupling for the two processes is the same.

The approach used in this paper may also be used to some extent in the description of two-level[10] and three-level[18] systems, e.g., and their interaction with radiation, and appears to be quite applicable to the description of a charged particle in a magnetic field. Boson operators corresponding to quantum oscillators may be used in the latter case, [19] making them especially adaptable to the technique used in this paper. However, fermion operators are used in describing the atoms in the former case and these give rise to difficulties in obtaining exact operator solutions and distribution functions; use of the Wigner distribution seems to be advantageous to the use of the P distribution for that case, due to the stronger convergence of the integrals involved as noted at the end of Sec. II.

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APPENDIX A: DEFINITIONS AND NOTATION

As is well known, any mode of the electromagnetic field is mathematically equivalent to a linear harmonic oscillator. That is, the total energy per mode is $\frac{1}{2}(p^2+\omega^2 q^2)$, where q is the mode amplitude and $p=\dot{q}$. In the quantum theory of the electromagnetic field, $[q_i, p_j]=i\hbar\delta_{ij}$ and p and q determine the electric and magnetic field strengths, respectively. [20] Letting $p=(\hbar\omega)^{1/2}P$ and $q=(\hbar/\omega)^{1/2}\times Q$, then the Hamiltonian for any given mode becomes $H=\frac{1}{2}\hbar\omega(P^2+Q^2)$, where $[Q,P]=i$. The annihilation operator is defined by $a=(1/\sqrt{2})(Q+iP)$ and the creation operator is defined by $a^\dagger=(1/\sqrt{2})\times(Q-iP)$; they are dimensionless. Their inverses are $Q=(1/\sqrt{2})(a^\dagger+a)$ and $P=(i/\sqrt{2})(a^\dagger-a)$. We now have $H=\hbar\omega(a^\dagger a+\frac{1}{2})$, where $[a,a^\dagger]=1$. The operator $N=a^\dagger a$ is identified as the number operator so that its operation on its eigenket $|n\rangle$ yields the eigenvalue n, giving the correct energy eigenvalue for the quantum oscillator, $E_n=\hbar\omega(n+\frac{1}{2})$. With these definitions, we have the following basic relations:

$$N|n\rangle=n|n\rangle,\quad a|n\rangle=n^{1/2}|n-1\rangle,$$

$$a^\dagger|n\rangle=(n+1)^{1/2}|n+1\rangle,\quad a|0\rangle=0,$$

$$|n\rangle=\frac{(a^\dagger)^n}{(n!)^{1/2}}|0\rangle,\ \langle n_i|n_j\rangle=\delta_{ij},\qquad \sum_{n=0}^{\infty}|n\rangle\langle n|=I.$$

The coherent states are defined in terms of the number states as [3]

$$|z\rangle=\exp(-\tfrac{1}{2}|z|^2)\sum_{n=0}^{\infty}(n\,!)^{-1/2}z^n|n\rangle\ . \tag{A1}$$

Their relationship to the number states is expressed in

$$|\langle n|z\rangle|^2=(n!)^{-1}|z|^{2n}e^{-|z|^2}\ . \tag{A2}$$

They are eigenstates of the annihilation operator, such that

$$a|z\rangle=z|z\rangle,\qquad \langle z|a^\dagger=z^*\langle z|. \tag{A3}$$

We denote the coherent state by $|z\rangle$, where $z=x+iy$ is a complex number (our z corresponds to Glauber's α). The expectation value of the number operator in a pure coherent state is seen to be $\langle z|N|z\rangle=|z|^2$. The identity operator for coherent states is

$$\pi^{-1}\int\int dx\,dy\,|z\rangle\langle z|=I. \tag{A4}$$

These states are normalized, $\langle z|z\rangle=1$, and overcomplete. They are not orthogonal, but obey the relation

$$\langle z|z'\rangle=\exp(z^*z'-\tfrac{1}{2}|z|^2-\tfrac{1}{2}|z'|^2).$$

The usual relations are used for the "displacement" and "momentum" operators Q and P, their eigenstates $|q\rangle$ and $|p\rangle$, and their eigenvalues q and p, respectively. That is,

$$Q|q\rangle=q|q\rangle,\qquad \int dq|q\rangle\langle q|=I$$

and

$$\langle q_i|q_j\rangle=\delta_{ij},$$

with similar relations for the quantities referring to momentum, and

$$\langle q_i|p_j\rangle=[1/(2\pi)^{1/2}]e^{iq_ip_j}.$$

APPENDIX B: BEHAVIOR OF P DISTRIBUTION FOR AN OSCILLATOR AS IT APPROACHES A PURE n STATE

Here we present a simple example which allows us to follow the approach of the P distribution towards a strongly singular function as the quantum oscillator it describes approaches a pure n state. The example is that of two coupled quantum oscillators with the same frequency ω, described by the Hamiltonian

$$H=\hbar\omega(a_1^\dagger a_1+\tfrac{1}{2})+\hbar\omega(a_2^\dagger a_2+\tfrac{1}{2})-\hbar g(a_1^\dagger a_2+a_2^\dagger a_1). \tag{B1}$$

The corresponding solutions to Heisenberg's equations of motion are

$$a_1(t)=(a_1\cos gt+ia_2\sin gt)e^{-i\omega t}, \tag{B2}$$

$$a_2(t)=(a_2\cos gt+ia_1\sin gt)e^{-i\omega t}. \tag{B3}$$

(These are also the solutions obtained for frequency conversion. [15])

We assume that oscillator 1 is initially in a P distribution of coherent states, which we will take to be Gaussian, while oscillator 2 is initially in a pure n state. After a time $t_n=\pi/2g$, the roles of the two oscillators will have reversed and oscillator 1 will be in an n state at that time. We wish to follow the time evolution of the P distribution for oscillator 1 as it approaches the n state. We have the following formula, obtained in a manner similar to that presented in the Introduction preceding Eq. (1.6):

$$P_1(x_1', y_1', t)=(2\pi)^{-2}\int\int d\xi\, d\eta\, e^{i(\xi x_1'+\eta y_1')}\int\int dx_1\, dy_1 \times P_1(x_1, y_1, 0)\langle z_1|\langle n_2|\, e^{-i(1/2)\xi a_1^\dagger(t)} \times e^{-i(1/2)\xi^* a_1(t)}|n_2\rangle|z_1\rangle\,. \tag{B4}$$

Substituting Eq. (B2), transforming to the rotating frame, and carrying out the rather lengthy evaluation of the expression in Eq. (B4), we arrive at the result for the P distribution of oscillator 1 at any time t

$$\underline{P}_{1,n}(\underline{x}_1', \underline{y}_1', t)=[(\sigma_1\cos^2 gt-\sin^2 gt)^n/\pi(\sigma_1\cos^2 gt)^{n+1}] \times \exp\{-(\sigma_1\cos^2 gt)^{-1} \times[(\underline{x}_1'-\alpha_1\cos gt)^2+(\underline{y}_1'-\beta_1\cos gt)^2]\} \times L_n\{[\sin^2 gt/\sigma_1\cos^2 gt(\sin^2 gt-\sigma_1\cos^2 gt)] \times[(\underline{x}_1'-\alpha_1\cos gt)^2+(\underline{y}_1'-\beta_1\cos gt)^2]\}. \tag{B5}$$

The function denoted by L_n is the Laguerre function. The index n refers to the n state, which was originally occupied by oscillator 2 and designated n_2. This result satisfies the requirements that it agree with the initial Gaussian when $t\to 0$ and that it is normalized as stipulated in Eq. (1.2). [The special case where $\sigma_1\cos^2 gt-\sin^2 gt=0$ must be treated separately, but calculation shows that a normalized P distribution results for this case as well; however, for present purposes, we shall assume this special case does not arise for the times of interest here, and examine Eq. (B5) only.]

Examining the P distribution as the n state is approached, we have, when t has almost reached t_n,

$$\underline{P}_{1,n}(\underline{x}_1', \underline{y}_1', t\lesssim t_n)\approx(-1)^n[\sigma_1(t)]^{-n} \times L_n\{[\sigma_1(t)]^{-1}|\underline{z}_1'-\gamma_1(t)|^2\} \times[\pi\sigma_1(t)]^{-1}e^{-[\sigma_1(t)]^{-1}|\underline{z}_1'-\gamma_1(t)|^2}, \tag{B6}$$

where

$$\sigma_1(t)=\sigma_1\cos^2 gt, \qquad \gamma_1(t)=\gamma_1\cos gt\,. \tag{B7}$$

The expression (B6) is the product of an oscillating function, whose magnitude grows as the distance from $\gamma_1(t)$ increases in the complex plane, and a normalized Gaussian, which cuts off the preceding function as the distance from $\gamma_1(t)$ increases. In the limit as $t\to t_n$, then $\sigma_1(t)\to 0$ and $\gamma_1(t)\to 0$, which causes the Gaussian to degenerate into a δ function at the origin and the factors preceding it to become infinite. Therefore, it is seen that the P distribution moves toward the origin and becomes highly singular as $t\to t_n$, i.e., as the n state is approached. Our example allows one to see how this highly singular P distribution is approached as a limit of a well-behaved function.

We conclude this Appendix by finding the corresponding Wigner distribution for this example by means of the theory of Sec. II. From Eqs. (2.2) and (2.13), we find the result at any time t to be

$$\underline{W}_{1,n}(\underline{q}_1, \underline{p}_1, t)=\frac{[(2\sigma_1\cos^2 gt+1)-2\sin^2 gt]^n}{\pi(2\sigma_1\cos^2 gt+1)^{n+1}} \times\exp\{-(2\sigma_1\cos^2 gt+1)^{-1}[(\underline{q}_1-\sqrt{2}\,\alpha_1\cos gt)^2 +(\underline{p}_1-\sqrt{2}\,\beta_1\cos gt)^2]\} \times L_n\left(\frac{\sin^2 gt}{(2\sigma_1\cos^2 gt+1)[\sin^2 gt-\frac{1}{2}(2\sigma_1\cos^2 gt+1)]} \times[(\underline{q}_1-\sqrt{2}\,\alpha_1\cos gt)^2+(\underline{p}_1-\sqrt{2}\,\beta_1\cos gt)^2]\right). \tag{B8}$$

It may be shown that the normalization and initial conditions of this expression are satisfied. The form of this expression is similar to that for the P distribution, Eq. (B5), but it remains well defined as $t\to t_n$:

$$\underline{W}_{1,n}(\underline{q}_1, \underline{p}_1, t=t_n)=(-1)^n\pi^{-1}e^{-(\underline{q}_1^2+\underline{p}_1^2)}L_n[2(\underline{q}_1^2+\underline{p}_1^2)]\,. \tag{B9}$$

This is the product of a Gaussian and an oscillating function, which is normalized and well behaved, although it may assume negative values as is typical of quasiprobability functions such as W and P. Therefore, unlike the P distribution, the Wigner distribution is well defined for the pure n state, and is given by Eq. (B9) above. This result for W_n agrees with that obtained by Glauber[4,7] in a different way and using a different notation.

APPENDIX C: EVALUATION OF INTEGRALS

According to Eqs. (3.21) and (3.22), the integrals

$$\int|U(\omega)|^2 d\omega,\ \int|V(\omega)|^2 d\omega, \times\int\mathrm{Re}[U(\omega)V(\omega)]d\omega,\ \int\mathrm{Im}[U(\omega)V(\omega)]d\omega$$

need to be evaluated. The functions U and V are given in Eqs. (3.17) and (3.18), where now $\omega_\lambda-\omega_0=\omega$, and the integrals extend from $-\infty$ to ∞.

(a) Taking the absolute square of Eq. (3.17), simplifying, and formally integrating, we find

$$\int |U(\omega)|^2 d\omega = \kappa^2 \Big\{ [e^{-2\Omega t}(\Omega \cosh gt + g \sinh gt)^2 + \Omega^2] \times \int \frac{d\omega}{|D(\omega)|^2} + (e^{-2\Omega t}\cosh^2 gt + 1) \int \frac{\omega^2}{|D(\omega)|^2}\, d\omega + [-2\Omega e^{-\Omega t}(\Omega \cosh gt + g \sinh gt)] \times \int \frac{\cos\omega t}{|D(\omega)|^2}\, d\omega + (-2\Omega e^{-\Omega t}\cosh gt) \int \frac{\omega^2 \cos\omega t}{|D(\omega)|^2}\, d\omega + (2ge^{-\Omega t}\sinh gt) \int \frac{\omega \sin\omega t}{|D(\omega)|^2}\, d\omega \Big\}, \tag{C1}$$

where

$$|D(\omega)|^2 = \omega^4 + 2(g^2 + \Omega^2)\omega^2 + (g^2 - \Omega^2)^2$$

from Eq. (3.12).

Each of the five integrals involved has an even integrand with the same denominator $|D(\omega)|^2$. This denominator has four simple roots at $\omega = \pm i(g \mp \Omega)$. In the complex plane, these roots lie along the imaginary axis, two above the origin and two below it. Whether $i(g-\Omega)$ or $i(\Omega - g)$ lies above or below the origin depends of course on whether $g < \Omega$ or $g > \Omega$; the calculations were done for both cases. All five integrals may be evaluated by integrating around a semicircle in the upper half of the complex plane and using the residue theorem. The results are

$$\int [\omega^2 \cos\omega t / |D(\omega)|^2]\, d\omega = (\pi e^{-gt}/2\Omega g)(\Omega \cosh\Omega t - g \sinh\Omega t), \quad g > \Omega$$
$$= (\pi e^{-\Omega t}/2g\Omega)(g \cosh gt - \Omega \sinh gt), \quad g < \Omega \tag{C2}$$

$$\int [\cos\omega t / |D(\omega)|^2]\, d\omega = [\pi e^{-gt}/2\Omega g(g^2 - \Omega^2)](g \sinh\Omega t + \Omega \cosh\Omega t), \quad g > \Omega$$
$$= [\pi e^{-\Omega t}/2\Omega g(\Omega^2 - g^2)](\Omega \sinh gt + g \cosh gt), \quad g < \Omega \tag{C3}$$

$$\int [\omega \sin\omega t / |D(\omega)|^2]\, d\omega = (\pi e^{-gt}/2\Omega g)\sinh\Omega t, \quad g > \Omega$$
$$= (\pi e^{-\Omega t}/2\Omega g)\sinh gt, \quad g < \Omega \tag{C4}$$

$$\int (\omega^2/|D(\omega)|^2)\, d\omega = \pi/2g, \quad g > \Omega$$
$$= \pi/2\Omega, \quad g < \Omega \tag{C5}$$

$$\int d\omega / |D(\omega)|^2 = \pi[2g(g^2 - \Omega^2)]^{-1}, \quad g > \Omega$$
$$= \pi[2\Omega(\Omega^2 - g^2)]^{-1}, \quad g < \Omega . \tag{C6}$$

Equations (C2)–(C6) are now substituted into Eq. (C1). It is found that the same result is obtained whether the $g > \Omega$ or the $g < \Omega$ solutions in the above equations are used. Therefore, for any value of Ω and g, we find after multiplying by $\rho(\omega_0)$ and using our abbreviation $\Omega = \pi\rho(\omega_0)\kappa^2$, and considerable simplification, the result

$$\rho(\omega_0)\int |U(\omega)|^2 d\omega = \tfrac{1}{2}(\Omega^2 - g^2)^{-1}[(2\Omega^2 - g^2) - \tfrac{1}{2}\Omega(\Omega + g)e^{-2(\Omega - g)t} - \tfrac{1}{2}\Omega(\Omega - g)e^{-2(\Omega + g)t} - (\Omega^2 - g^2)e^{-2\Omega t}]. \tag{C7}$$

This expression starts at the value zero at $t = 0$ and becomes infinite as $t \to \infty$ for $\Omega \lesssim g$, except it is zero for $\Omega = 0$, and it approaches the value

$$\tfrac{1}{2}(2\Omega^2 - g^2)/(\Omega^2 - g^2) \qquad \text{for } \Omega > g.$$

(b) Similarly, using Eq. (3.18), there are four integrals involved, which are the same as those just encountered, Eqs. (C3)–(C6). We find

$$\rho(\omega_0)\int |V(\omega)|^2 d\omega = \tfrac{1}{2}(\Omega^2 - g^2)^{-1}[g^2 - \tfrac{1}{2}\Omega(\Omega + g)e^{-2(\Omega - g)t} - \tfrac{1}{2}\Omega(\Omega - g)e^{-2(\Omega + g)t} + (\Omega^2 - g^2)e^{-2\Omega t}]. \tag{C8}$$

(c) Taking the real part (denoted by Re) of the product of $U(\omega)$ and $V(\omega)$, we obtain

$$\int \mathrm{Re}[U(\omega)V(\omega)]d\omega = \kappa^2 \Big(g(e^{-2\Omega t} + 1) \int \frac{\omega}{|D(\omega)|^2}\, d\omega - 2ge^{-\Omega t}\cosh gt \int \frac{\omega \cos\omega t}{|D(\omega)|^2}\, d\omega + (g^2 - \Omega^2)e^{-\Omega t}\sinh gt \int \frac{\sin\omega t}{|D(\omega)|^2}\, d\omega - e^{-\Omega t}\sinh gt \int \frac{\omega^2 \sin\omega t}{|D(\omega)|^2}\, d\omega \Big). \tag{C9}$$

Since $|D(\omega)|^2$ is an even function of ω, it is seen that all the integrands above are odd, so each of the integrals is zero. Therefore we have

$$\rho(\omega_0)\int \mathrm{Re}[U(\omega)V(\omega)]d\omega = 0. \tag{C10}$$

(d) Taking the imaginary part (denoted by Im) of $U(\omega)V(\omega)$, we find the integrals in Eqs. (C2), (C3), (C5), and (C6) involved again, and the result is

$$\rho(\omega_0)\int \mathrm{Im}[U(\omega)V(\omega)]d\omega = \tfrac{1}{2}(\Omega^2 - g^2)^{-1} \times [\Omega g - \tfrac{1}{2}\Omega(\Omega + g)e^{-2(\Omega - g)t} + \tfrac{1}{2}\Omega(\Omega - g)e^{-2(\Omega + g)t}]. \tag{C11}$$

The general behavior of this expression and that in Eq. (C8) is similar to that given for Eq. (C7), and the results are the same whether the $g < \Omega$ or $g > \Omega$ solutions in Eqs. (C2)–(C6) are used.

The results given in Eqs. (C7), (C8), (C10), and (C11) are used in Sec. III. It might also be mentioned that the proof that $[A(t), A^\dagger(t)] = 1$ for the solution (3.14) involves the same five integrals that are given in this Appendix, viz., Eqs. (C2)–(C6).

APPENDIX D: QUANTUM-MECHANICAL TREATMENT OF THE PUMP

The effect of treating the pump quantum mechanically rather than classically is considered in this Appendix. In particular, we are interested in seeing if the breakdown of the P distribution pertaining to parametric amplification in a single mode, as studied in Sec. III, might be due to the lack of a full quantum-mechanical treatment. For comparison, let us consider the no-damping case for which Eq. (3.1), in which the pump is treated classically, reduces to

$$H=\hbar\omega_0(a^\dagger a+\tfrac{1}{2})-\hbar\tfrac{1}{2}g[\,a^\dagger a^\dagger e^{-2i\omega_0 t}+aae^{2i\omega_0 t}]\,. \quad \text{(D1)}$$

The pump is represented classically in this expression by an ordinary function of the form $\alpha(t)=\alpha\exp(-2i\omega_0 t)$, the amplitude α being real and included in the coupling constant $\frac{1}{2}g$. From this Hamiltonian we get

$$\ddot{A}(t)-g^2A(t)=0\;, \quad \text{(D2)}$$

for which the solution

$$a(t)=(a\cosh gt+ia^\dagger\sinh gt)e^{-i\omega_0 t} \quad \text{(D3)}$$

is easily found [Eq. (3.14) with $\kappa=0$], and the breakdown of the P distribution is governed by the $\Omega=0$ curve in Fig. 1. This shows, e.g., that the P distribution breaks down immediately for the signal mode initially in a coherent state.

We now represent the pump by the quantum-mechanical operator $a_1(t)$, and put a subscript 0 on the operator representing the signal mode to correspond to its frequency ω_0:

$$H=\hbar\omega_0(a_0^\dagger a_0+\tfrac{1}{2})+\hbar\omega_1(a_1^\dagger a_1+\tfrac{1}{2})-\hbar\tfrac{1}{2}g(a_0^\dagger a_0^\dagger\,a_1+a_0a_0a_1^\dagger)\;. \quad \text{(D4)}$$

Here g does not include the pump amplitude. This fully quantum-mechanical Hamiltonian leads to after making the substitutions

$$a_0(t)=A_0(t)\,e^{-i\omega_0 t}\,,$$

$$a_1(t)=A_1(t)\,e^{-i\omega_1 t}\,, \qquad \omega_1=2\omega_0\,,$$

the equations

$$\dot{A}_0(t)=igA_0^\dagger(t)\,A_1(t)\;, \quad \text{(D5)}$$

$$\dot{A}_1(t)=i\tfrac{1}{2}g\,A_0(t)\,A_0(t)\;, \quad \text{(D6)}$$

along with the adjoint equations. From these we get, with $N_i=A_i^\dagger A_i$,

$$\ddot{A}_0(t)-g^2[N_i(t)-\tfrac{1}{2}N_0(t)]A_0(t)=0\;. \quad \text{(D7)}$$

Compared to Eq. (D2) it is seen that in the classical treatment of the pump the bracketed expression above is replaced by unity [or really by the square of the amplitude of the pump, since the g in Eq. (D2) includes the pump amplitude]. However, this quantity is not constant; making use of Eqs. (D5) and (D6) to find $\dot{N}_i(t)$, we find

$$\frac{d}{dt}[\,N_1(t)-\tfrac{1}{2}N_0(t)]=ig\,[\,A_0^2(t)A_1^\dagger(t)-A_0^{\dagger 2}(t)A_1(t)]\;. \quad \text{(D8)}$$

Similarly it is found that

$$\frac{d}{dt}[\,N_1(t)+\tfrac{1}{2}N_0(t)]=0\;, \quad \text{(D9)}$$

which is just a statement of the conservation of total photon energy for this process. From Eq. (D9), we find that

$$N_1(t)-\tfrac{1}{2}N_0(t)=N_1(0)+\tfrac{1}{2}N_0(0)-N_0(t),$$

but Eq. (D7) is still nonlinear when this result is substituted.

Since Eq. (D7) is nonlinear and not readily solvable, we consider a Maclaurin series expansion of the solution $A_0(t)$. Using the values

$$A_0(0)=a_0\,, \quad \dot{A}_0(0)=iga_0{}^\dagger a_1\,,$$

$$\ddot{A}_0(0)=g^2[\,a_1^\dagger a_1-\tfrac{1}{2}a_0^\dagger a_0]a_0\,,$$

we have

$$A_0(t)=a_0+igt\;a_0^\dagger a_1+\tfrac{1}{2}(gt)^2\times[\,a_1^\dagger a_1-\tfrac{1}{2}a_0^\dagger a_0]a_0+\cdots\;. \quad \text{(D10)}$$

To first order in the parameter gt, we have then an approximate solution which is valid for $gt\ll 1$,

$$A_0(t)=a_0+igt\;a_0^\dagger a_1\;. \quad \text{(D11)}$$

We now substitute $a_0(t)=A_0(t)e^{-i\omega_0 t}$ into Eq. (1.6) to determine the time evolution of the P distribution. We are interested in comparing to the previous case in which the pump was treated classically (and had its initial phase taken as zero). The closest quantum-mechanical state to this one is a coherent state with a large number of photons, which we represent as a δ-function P distribution, so that

$$P_1(x_1,\,y_1,\,0)=\delta(x_1-\alpha_1)\,\delta(y_1-0),$$

where $\alpha_1\gg 1$ and $\beta_1=0$ (corresponding to $\phi=0$). Taking $P_0(x_0,\,y_0,\,0)$ to be a general Gaussian as before, then Eq. (1.6) leads to the expression (retaining terms to first order in gt only)

$$\underline{P}_0'(\underline{x}_0'',\underline{y}_0'',t)=[\,\pi\sigma_0^+(t)]^{-1/2}\times\exp\{-[\,\sigma_0^+(t)]^{-1}[\,\underline{x}_0''-\alpha_0(t)]^2\}[\,\pi\sigma_0^-(t)]^{-1/2}\times\exp\{-[\,\sigma_0^-(t)]^{-1}[\,\underline{y}_0''-\beta_0(t)]^2\}\;, \quad \text{(D12)}$$

where

$$\sigma_0^\pm(t)=\sigma_0\pm\alpha_1 gt\,(2\sigma_0+1)\;, \quad \text{(D13)}$$

$$\alpha_0(t)=(1/\sqrt{2}\,)\,(\alpha_0+\beta_0)\,(1+\alpha_1 gt)\;, \quad \text{(D14)}$$

$$\beta_0(t)=(1/\sqrt{2}\,)\,(-\alpha_0+\beta_0)\,(1-\alpha_1 gt)\;. \quad \text{(D15)}$$

The expression in Eq. (D12) is, of course, valid only if the integrals involved in its derivation exist, which means that $\sigma_0^+(t) \geq 0$. Since $\sigma_0 \geq 0$ and $\alpha_1 gt \geq 0$, then from Eq. (D13) we have $\sigma_0^+(t) \geq 0$ for all time, but $\sigma_0^-(t)$ is non-negative only as long as

$$\sigma_0 \geq \alpha_1 gt\,(1 - 2\alpha_1 gt)^{-1} .$$

Keeping terms to only first order in gt, and letting $\alpha_1 g = g_1$, which corresponds to the g in the case of a classical pump, we have the requirement that $\sigma_0 \geq g_1 t$. This is for the quantum treatment of the pump, and $gt \ll 1$. (It may be shown that there is no restriction here on the existence of the corresponding Wigner distribution.)

Comparing to the classical treatment of the pump, we have, from Eq. (3.25) with $\Omega = 0$, $\sigma_0 \geq \tanh gt \times (1 - \tanh gt)^{-1}$. To first order in gt we have then $\sigma_0 \geq gt$ in the classical treatment, with $gt \ll 1$, as the requirement for the P distribution to exist.

Thus, it is seen that the two results are essentially the same, i.e., the curve which determines the breakdown of the P distribution for this case (the $\Omega = 0$ curve in Fig. 1) rises from the origin immediately in the quantum, as well as in the classical, treatment of the pump. As mentioned in the caption of Fig. 1, this means, for example, that the P distribution for an initially coherent state of the signal breaks down immediately, and the results of this Appendix show that treating the pump quantum mechanically does not alter this conclusion.

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†Present address: Physics Department, Guilford College, Greensboro, N. C. 27410.

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Hydrodynamics of Liquid Crystals*

Michael J. Stephen

Physics Department, Rutgers, The State University, New Brunswick, New Jersey 08903

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A consistent set of hydrodynamic equations for liquid crystals is derived from the necessary conservation laws and the requirements of Galilean invariance. In the stationary case, the equations reduce to the Oseen-Frank hydrostatic theory. The equations should be useful in discussing the hydrodynamics of cholesteric and smectic crystals. Linear dissipative effects are also considered.

I. INTRODUCTION

The continuum hydrostatic theory of liquid crystals of Oseen[1] and Frank[2] is well known and firmly established. More recently, Ericksen[3] and Leslie[4] have discussed continuum theories of the dynamics of liquid crystals. The theory of Leslie[4] is deficient in that it does not, in the stationary case, reduce to the Oseen-Frank hydrostatic theory. This situation has been partially rectified by Ericksen.[5] This deficiency is not important in the case of nematic crystals, and some interesting solutions