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One-Dimensional System of Bosons with Repulsive δ-Function Interactions at a Finite Temperature T

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The equilibrium thermodynamics of a one-dimensional system of bosons with repulsive δ-function interaction is found to be intermediate between those of a one-dimensional free-boson and a one-dimensional free-fermion system. Numerical comparisons are given.

I. INTRODUCTION

In a previous paper (which we shall henceforth call I)[1] a method was developed in which the pressure P at any temperature T of a system of bosons with repulsive δ-function interaction in one dimension was shown to be exactly given by

$$P = T/2\pi \int_{-\infty}^{\infty} dk \ln(1 + e^{-\epsilon(k)/T}) , \qquad (1)$$

where $\epsilon(k)$ is the unique solution of the integral equation

$$\epsilon(k) = -A + k^2 - \frac{Tc}{\pi}\int_{-\infty}^{\infty} \frac{dq}{c^2 + (k-q)^2} \times \ln\{1 + \exp[-\epsilon(q)/T]\} , \qquad (2)$$

with c = interaction strength > 0 and A = chemical potential. We shall in this paper discuss the thermodynamics of such a gas.

II. THERMODYNAMICS

A. Behavior of the Pressure

In terms of the fugacity $z = e^{A/T}$ the pressure of a one-dimensional system has the general form

$$\frac{PL}{T} = \ln\left(\sum_{N=0}^{\infty} z^N \sum_{\text{state } s} e^{-E_s/T}\right) ,$$

where L = size of the one-dimensional system and E_s is the energy of state s. In the present case, it is obvious that E_s increases with increasing c. The coefficient of z^N thus decreases with increasing c. For constant z we therefore have $P(c=0) > P(c \text{ finite}) > P(c=\infty)$, from which follows:

Theorem 1

$$P_{\text{BE}} > P(c) > P_{\text{FD}} \text{ at a particular } T \text{ and } z. \qquad (3)$$

Here P_{BE} and P_{FD} stand for the pressure at a fixed

z for a free Bose-Einstein and a free Fermi-Dirac system, respectively. By definition

$$P(c=0)=P_{\rm BE}\ . \qquad (4a)$$

It was shown in I that

$$P(c=\infty)=P_{\rm FD}\ . \qquad (4b)$$

We have performed numerical computations (details described below) at $z=0.1353$, 1.0000, 1.6487 and 7.3890. The $P^{2/3}$-versus-T curves are presented in Fig. 1. They explicitly demonstrate the statement of Theorem 1. Note that for $z>1$, $P_{\rm BE}$ is not defined.

Theorem 2

$$\text{At fixed } z,\quad T\to 0\ ,\quad P(c)/P_{\rm FD}\to 1\ , \qquad (5a)$$

$$T\to\infty\ ,\quad P(c)/P_{\rm BE}\to 1\ . \qquad (5b)$$

Proof: Dimensionally $P\sim 1/L^3$, $z\sim 1$, $T\sim 1/L^2$, $c^2\sim 1/L^2$. Thus, $P^{2/3}/T\sim 1\sim f(T/c^2, z)$

$$\text{or}\quad P^{2/3}=Tf(T/c^2,\ z). \qquad (6)$$

Using Eq. (4b)

$$P/P_{\rm FD}=[f(T/c^2,z)/f(0,z)]^{3/2}\ .$$

Hence Eq. (5a). Similarly, by using Eq. (4a), one proves Eq. (5b).

We now consider the surface $P^{2/3}$ over the z-T plane. Since $z=e^{A/T}$ is always positive, only the positive quadrant where $z>0$, $T>0$ need be examined. From I, Appendix D, P is analytic in z and T for all z, $T>0$. Because of Eq. (3), the $P^{2/3}$ surface is below the $P_{\rm BE}^{2/3}$ and above the $P_{\rm FD}^{2/3}$ surfaces, respectively. Both $P_{\rm BE}^{2/3}$ and $P_{\rm FD}^{2/3}$ are surfaces generated by straight lines parallel to the $z=0$ plane, since

$$P_{\rm BE}=(2\sqrt{\pi})^{-1}T^{3/2}\sum_{n=1}\frac{z^n}{n^{3/2}}$$

and

$$P_{\rm FD}=(2\sqrt{\pi})^{-1}T^{3/2}\sum_{n=1}(-)^{n+1}\frac{z^n}{n^{3/2}}\ .$$

Furthermore, for constant z, Eq. (5a) tells that the slope of $P^{2/3}(c)$ is the same as that of $P_{\rm FD}^{2/3}$ at $T=0$ and Eq. (5b) says that $P^{2/3}(c)$ and $P_{\rm BE}^{2/3}$ are parallel for $T\to\infty$.

If one takes T and the density D^1 as independent variables, one can ask whether z is analytic in T and D. The answer is yes, since D is analytic in T and z, and by a theorem[2] on the existence of thermodynamic limits (generalized to quantum statistical mechanics), D is monotonic in z.

B. Virial Expansion

One can obtain the fugacity and virial expansions as follows. ϵ is a function of T, z and k. We expand in powers of z

$$\exp\left(\frac{-\epsilon(k)}{T}\right)=\sum_{n=1}^{\infty}A_n(k,T)z^n\ . \qquad (7)$$

The absence of the term a_0 and the following all follow from Eq. (2):

$$a_1=e^{-k^2/T},\qquad a_2=e^{-k^2/T}\underline{O}\,a_1\ , \qquad (8)$$

$$a_3=e^{-k^2/T}[\underline{O}a_2-\tfrac{1}{2}\underline{O}\,(a_1^2)+\tfrac{1}{2}(\underline{O}a_1)^2]\ ,$$

$$\text{where the operator } \underline{O} \text{ is } \frac{c}{\pi}\int_{-\infty}^{\infty}\frac{dq}{c^2+(k-q)^2}\ . \qquad (9)$$

The pressure is then given by

$$P=(T/2\pi)\int_{-\infty}^{\infty}dk\{a_1z+(a_2-a_1^2/2)z^2+(a_3-a_1a_2+a_1^3/3)z^3+\cdots\}$$

$$=P_{\rm FD}+\frac{T^{3/2}}{2\sqrt{\pi}}e^{c^2/2T}\left[\frac{1}{\sqrt{2}}-\left(\frac{2}{\pi}\right)^{1/2}\int^{c/(2T)^{1/2}}\times e^{-y^2}dy\right]z^2+0(z^3)\ . \qquad (10)$$

From this we easily obtain the first nontrivial term in the virial expansion:

$$\frac{P}{T}=\frac{1}{v}\left(1+\left\{\frac{1}{2\sqrt{2}}+e^{c^2/2T}\left[\left(\frac{2}{\pi}\right)^{1/2}\int_0^{c/(2T)^{1/2}}\times e^{-y^2}dy-\frac{1}{\sqrt{2}}\right]\right\}\frac{\lambda}{v}+\cdots\right), \qquad (11)$$

where $v=\frac{L}{N}=(\text{particle density})^{-1}$.

Higher-order terms of both the fugacity and virial expansions can be systematically obtained in this manner by quadratures alone.

C. Hole and Particle Densities in k Space

The hole and particle densities $\rho_h(k)$ and $\rho(k)$ satisfy

$$f(k)=\rho_h(k)+\rho(k)\ , \qquad (12)$$

$$2\pi f(k)=1+2c\int_{-\infty}^{\infty}\frac{\rho(q)\,dq}{c^2+(k-q)^2}\ , \qquad (13)$$

$$\text{and}\quad \epsilon(k)=T\ln[\rho_h(k)/\rho(k)]\ , \qquad (14)$$

where ϵ is the solution of the integral Eq. (2).

Figures 2 and 3 give f, ρ, ρ_h for $A=4$ and $T=0.2$ and 5.0. These graphs are obtained by iteration performed on Eq. (13) (see I, Appendix A and B) using a digital computer with k-mesh sizes chosen to be 0.03 and 0.10, respectively. The convergence is rapid in both cases, requiring around ten iterations per point.

One observes that f is the density of particle "momenta" plus the density of hole "momenta" in k space, as explicitly stated in Eq. (12). For c

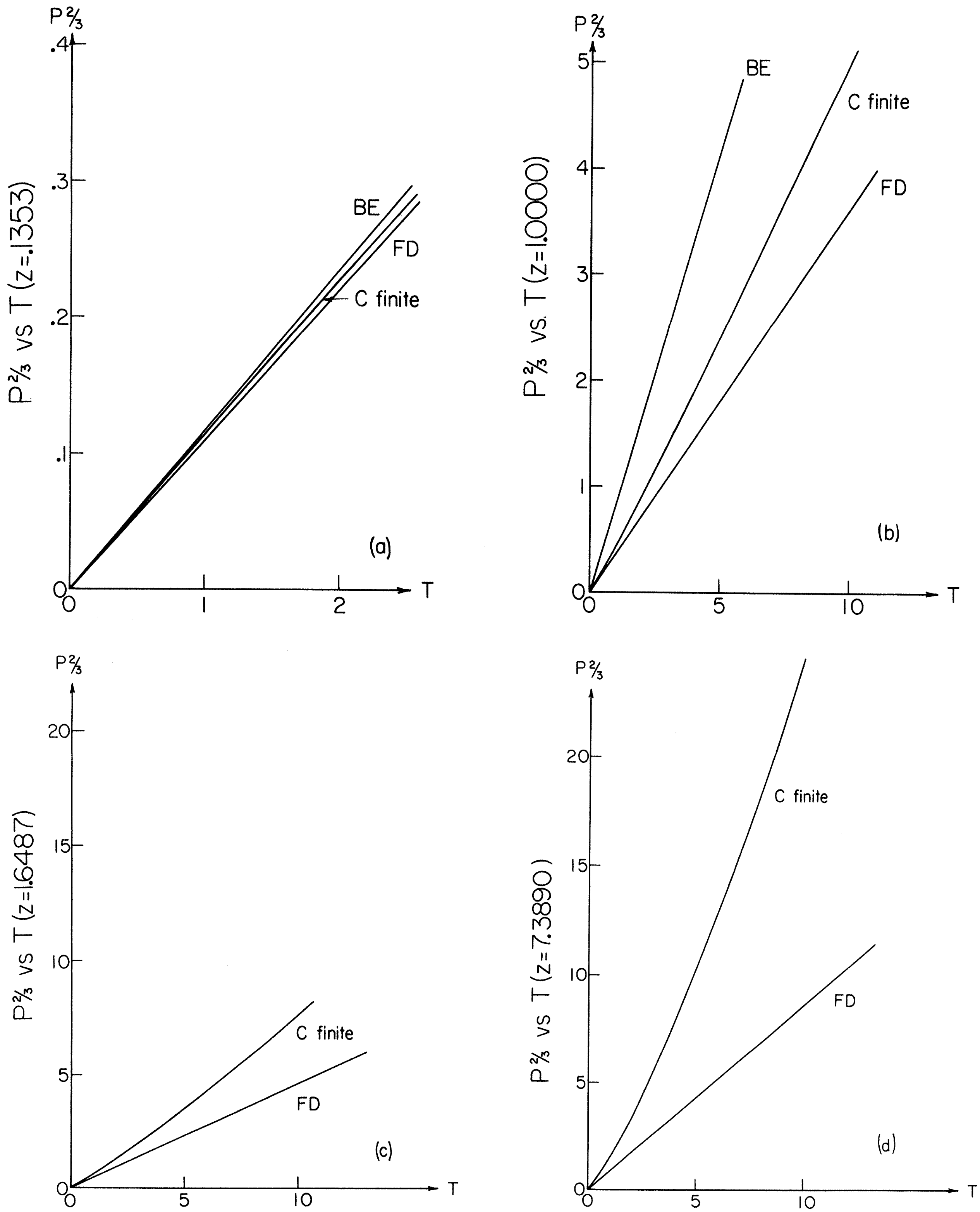

FIG. 1. a, b, c, d Temperature dependence of pressure to the $\frac{2}{3}$ power for fugacity $z = .1353$, 1.0000, 1.6487, and 7.3890. The strength of interaction c is chosen to be unity. The corresponding straight lines for the free bosé gas and the free fermi gas at the same fugacity values are shown for comparison. Note that for $z > 1$, P_{BE} is not defined.

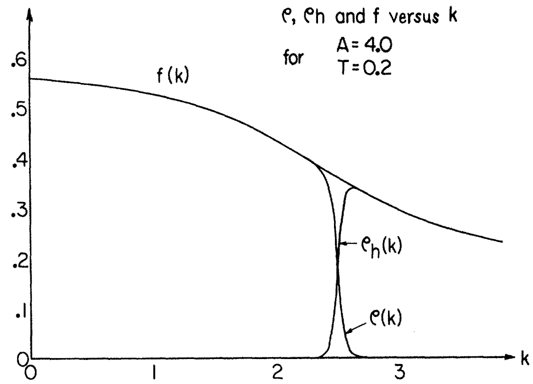

FIG. 2. The momentum dependence of the density of particle "momenta" ρ, the density of the hole "momenta" ρ_h and their sum f for the chemical potential $A=4.0$, and the temperature $T=0.2$, respectively.

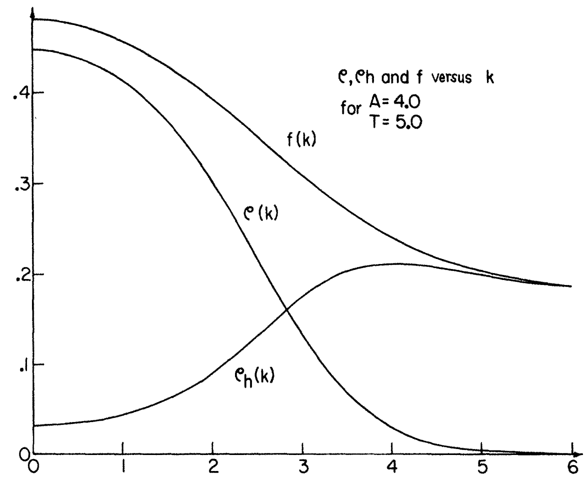

FIG. 3. The momentum dependence of the density of particle "momenta" ρ, the density of the hole "momenta" ρ_h and their sum f for the chemical potential A = 4.0 and the temperature $T=0.5$, respectively.

$=0$, Eq. (13) shows that

$$f=(2\pi)^{-1}.$$

Thus, in general, Eq. (13) shows that

$$f(c>0)>f(c=0).$$

In other words, the repulsive interaction allows more "single-particle momenta" to exist in momentum space.

The curve $\rho(k)$, the density of particle "momenta" per unit k, is characteristically peaked at $k=0$. On the other hand, $\rho_h(k)$, the density of hole "momenta" per unit k, has characteristically a valley at $k=0$.

At $T=0$, all low $|k|$ states up to a $|k|=k_0$ are occupied by particles, while all states with $|k| \geq k_0$ are unoccupied. This is the special case discussed by Lieb and Liniger.[3]

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