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Analog of the ac Josephson Effect in Superfluid Helium*

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When a chemical potential difference exists between the superfluid helium on two sides of an orifice, superfluid flow takes place at a critical velocity accompanied by the creation of quantized vortices. The frequency with which vortices are created is expected to be equal to the chemical potential difference (the analog of the ac Josephson frequency in superconducting tunneling). An experiment is described in which this frequency condition is verified by a method analogous to that used in superconductors. Experimental conditions were selected such that the gravitational head difference mgZ was the dominant contribution to the chemical potential difference. Z was measured by a capacitance technique. An ultrasonic transducer was used to modulate the flow through the orifice at the frequency ω so that the rate of creation of vortices was synchronized with the modulation frequency. The system was observed to exhibit dynamic stability at values of head difference $Z=n\hbar\omega/n'mg$ corresponding to n quantized vortices created every n' cycles of the modulation. The correspondence between theory and experiment is most convincing under experimental conditions such that the strength of the stability decreases rapidly with increasing values of the integer n'. This was found to occur with small orifices and with moderate modulation amplitude, in agreement with the results of the analogous experiments in superconductors.

I. INTRODUCTION

The superfluid state of liquid He^4 is in many ways remarkably similar to the superconducting state of the electrons in metals at low temperatures. This analogy has been useful to the investigators of both systems. In particular, it suggests the possibility of observing in superfluid liquid helium effects related to the well-known Josephson effects[1] of superconductivity. The major difficulty in attempting such experiments is in the construction of a barrier through which superfluid helium atoms could tunnel. It has been suggested[2] that tunneling could occur through a barrier pierced with holes of atomic dimensions, but this has not yet been experimentally verified. Anderson suggested to the author that this difficulty could be avoided by constructing a superfluid helium analog of the Anderson-Dayem experiment.[3,4] Their experiment, which is schematically illustrated in Fig. 1, is similar to the classic ac Josephson experiment except that a narrow neck in a superconducting thin film takes the place of a tunnel junction. The superfluid He analog of this geometry consists of two vessels of He connected through a small orifice as shown in Fig. 2 (a). Early experiments done on this system in close collaboration with Anderson showed that when the flow through the orifice was modulated with an ultrasonic transducer, the time-average flow ceased for periods of minutes or longer at various values of the head difference Z between

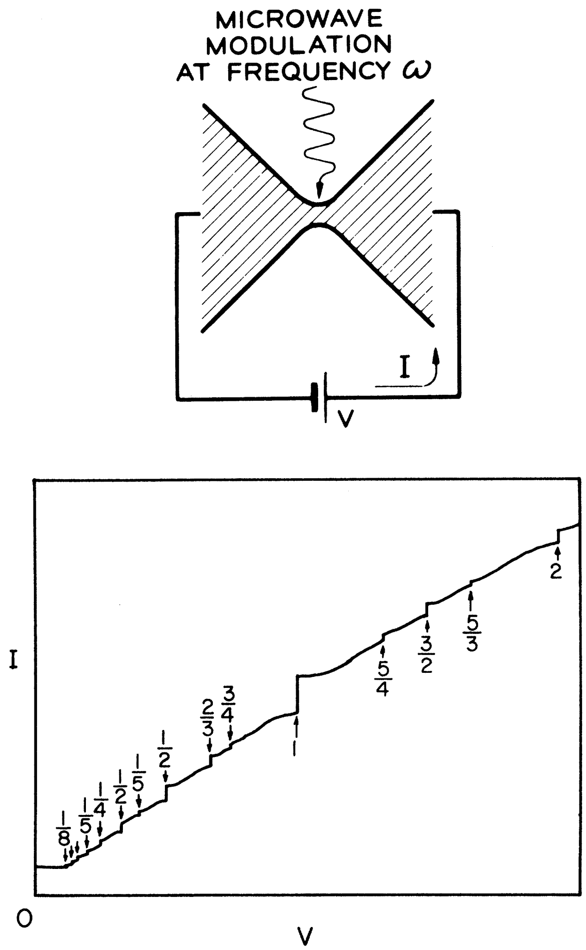

FIG. 1. Anderson and Dayem showed that the dc current-voltage characteristic of a superconducting thin-film bridge exhibits vertical steps in the current when the voltage is modulated at a microwave frequency ω. The steps occur because the flow of quantized fluxoids across the bridge becomes synchronized with the modulation. They represent a stability of the system whenever the natural frequency of fluxoid crossing $2eV/\hbar$ corresponds to n fluxoids every n' cycles of the modulation.

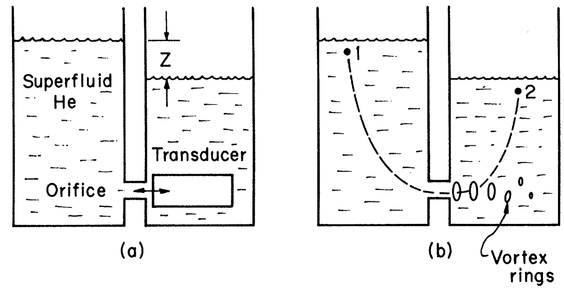

FIG. 2. (a) Schematic diagram of the superfluid helium analog of the Anderson-Dayem experiment. Two vessels of helium connected via a small orifice or short channel are filled to different levels to provide a chemical potential difference mgZ. The resulting flow is modulated by an ultrasonic transducer. (b) Vortex rings crossing the path of integration.

the baths. This phenomonon was interpreted as the analog in superfluid He of the current steps in the Anderson-Dayem experiment. These results were reported and discussed in several places.[5-7] Subsequently, this experiment has been repeated by a number of workers including Khorana and Chandrasekhar and Khorana and Douglass.[8] In this paper we will give a full discussion of the experiment and present improved results.

II. THEORY

A. Average Rate of Vortex Crossing

As was first pointed out by Zimmermann,[9] the experiment described here can be understood from two points of view. One is the quantum-mechanical approach used in the first report[5]; the other is to use classical fluid dynamics plus vortex quantization. Both points of view will be reviewed in this section since each contributes to a physical understanding of the effect. Following Feynman[10] we can describe the flow of superfluid helium by a wave function $\Psi = \Psi_0 e^{i\phi(\vec{r})}$, where Ψ_0 is the ground-state wave function of the fluid at rest and the phase $\phi(\vec{r})$ is a potential function for the superfluid velocity distribution $\vec{v}_s(\vec{r})$,

$$\vec{v}_s(\vec{r}) = (\hbar/m)\vec{\nabla}\phi(\vec{r}). \tag{1}$$

The requirement that Ψ be single valued leads to the condition

$$\oint \vec{\nabla}\phi \cdot d\vec{r} = n2\pi, \quad n = 0,\ 1,\ 2, \ldots \tag{2}$$

so that the circulation

$$\kappa \equiv \oint \vec{v}_s \cdot d\vec{r} = nh/m. \tag{3}$$

Equations (2) and (3) are interpreted as describing quantized vortices in superfluid helium. In the case of a vortex line with cylindrical symmetry, Eq. (3)

becomes $v_s = n\hbar/mr$, a velocity field which falls off as r^{-1} and in which each helium atom has angular momentum $n\hbar$. The shape and position of the vortex line is conventionally discussed in terms of the position of the core inside which the above velocity distribution is no longer valid. [11] If the core is closed on itself in a circle of radius R we have a vortex ring whose energy E and momentum p (perpendicular to the plane of the ring) are known from the classical dynamics of a frictionless fluid[12] of density ρ_s:

$$E = \tfrac{1}{2}\rho_s \kappa^2 R\ \tfrac{1}{2}\rho_s \kappa^2 R[\ln(8R/a) - \tfrac{7}{4}] \;, \tag{4}$$

$$p = \rho_s \kappa\, \pi R^2 \;. \tag{5}$$

The existence of quantized vortex rings in superfluid He has been most clearly verified by Rayfield and Reif[13] who found that the core radius $a \simeq 1$ Å and that the quantum of circulation $\kappa = h/m$, that is, that $n = 1$ for their experiment.

Implicit in the above picture is the assumption that the phase factor ϕ is coherent over macroscopic distance, so we can define the phase difference

$$\Delta\phi = \phi(\vec{r}_2) - \phi(\vec{r}_1) \tag{6}$$

between two points and consider its evolution in time. A time evolution of the phase difference is expected if there is a chemical potential difference $\Delta\,\mu$ between the two points. Beliaev[14] showed that for a weakly interacting Bose gas the phase contains the term $\mu t/\hbar$, so we expect that within an additive constant

$$\Delta\,\phi = \Delta\,\mu t\,/\hbar \;. \tag{7}$$

The same argument can be made for superconductors from the work of Gor'kov. [15] Except for a factor of 2 due to pairing the result is the same. Josephson[1] stated this condition in the form $\hbar\partial\Delta\phi/\partial t = 2$ eV.

We can now relate the chemical potential difference to the rate of vortex motion using the experimental geometry of Fig. 2(b). We choose a path of integration between points in the two baths which are far from the orifice so that the fluid is nearly stationary. If the temperature is the same at the two points, then $\Delta\,\mu = mgZ$. From (1) and (7) we obtain

$$\Delta\,\phi = \frac{m}{\hbar}\int_1^2 \vec{v}_s \cdot d\vec{l} = mgZt/\hbar \;, \tag{8}$$

so that the phase difference (and velocity integral) increase linearly in time. These relations are easily understood in the regime of accelerating potential flow through the orifice. Experimentally, however, a steady state is rapidly reached in which the helium flows at a critical velocity accompanied by the production of vorticity. It can be seen from (2) that the phase difference $\Delta\,\phi$ in the presence of a vortex will depend on which side of the vortex core the path of integration passes. In general, the phase difference between any two points will change by $n2\pi$ when a vortex with n quanta of circulation passes between them. [16] We can obtain a description of the experimentally observed steady state if we assume that vortex cores carry quanta across the path of integration at an average frequency

$$\langle\nu\rangle = mgZ/h \;, \tag{9}$$

with the proper sign to cancel, on the average, the phase slippage (8). (Alternatively we can assume vortices which carry n quanta crossing at a frequency $\langle\nu\rangle/n$.) The corresponding frequency in the superconducting case is

$$\langle\nu\rangle = 2\ \text{V/h}. \tag{10}$$

We thus identify the ac Josephson frequency with the average rate of crossing of single quantum vortices. It should be noted that the argument presented here does not rely on a specific model of vortex motion. Such a model is, however, helpful in visualizing the process. A model for fluxoid crossing is easily obtained for the Anderson-Dayem bridge. The self-magnetic field of the critical current breaks into the film at one edge as a quantized fluxoid. Under the influence of the Lorentz force this fluxoid migrates across the bridge thus necessarily cutting the path of integration from one side to the other. If we assume perfect symmetry, then fluxoids of opposite sign will enter from opposite sides. They will meet and annihilate because of their mutual attraction. A similar picture for liquid helium can also be found. Let us assume that vorticity is produced in the form of quantized vortex rings in the orifice as shown in Fig. 2 (b). We must assume that the vortex core is produced at the orifice walls so that all possible paths of integration through the orifice initially thread the vortex ring. The vortex ring with radius approximately equal to that of the orifice will then move down stream, lose energy due to collisions with phonons and rotons (normal fluid), and shrink to a roton. [13] The velocity integral of Eq. (8) will increase linearly with time as long as the vortex rings stay threaded on the path of integration. It is, of course, more likely that in the steady state the rings will cross the path and move away at an average frequency equal to the vortex creation frequency. The conditions for the validity of Eq. (9) are thus satisfied.

As was mentioned earlier, the average frequency of vortex crossing (creation) can also be obtained from classical hydrodynamics plus vortex quantization. [9] In the spirit of Feynman's derivation of the critical velocity at an orifice, [10] we assume that the force $\rho g Z \pi R^2$ accelerating fluid through an orifice of radius R is equal to the momentum of a vortex ring (5) times the frequency of creation ν. Assuming that the circulation $\kappa = nh/m$ we obtain the frequency for vortex creation (9) given previously. This argument does not explicitly display the average nature of the vortex creation frequency. It makes the additional assumption that any contribution to the pressure $\rho g Z$ due to normal fluid acts to accelerate normal fluid. A more general argument was given by Anderson[6] who showed by integrating Euler's equation for a classical frictionless fluid that the chemical potential difference between two points in the fluid is equal to the time-average rate at which vorticity $(\vec{\nabla}\times\vec{v})$ is transported across the line joining them. In superfluid helium $\vec{\nabla}\times\vec{v}_s = 0$ everywhere except at the vortex cores around which the circulation $\kappa = nh/m$. Anderson's result, which is not particularly useful in classical fluids, thus gives the vortex crossing frequency in superfluids.

B. Synchronization of Vortex Crossing

A direct observation of the vortex crossing frequency, such as can be done to superconductors, is expected to be difficult in helium. There is no electromagnetic radiation to facilitate observation of the predicted peak in the frequency distribution of the current flow in a free-running orifice. The success of the Anderson-Dayem experiment[3,4] shows that the heterodyne technique suggested by Josephson,[1] and used by Shapiro *et al.*[17] for the case of tunnel junctions, does work in the superconducting bridge. It can best be understood as a synchronization of the vortex crossing to a modulation of the current. If the supercurrent is modulated as shown in Fig. 3, vortex nucleation will be enhanced at times of high velocity and suppressed at times of low velocity. Thus the vortex crossing frequency determined by $\Delta\mu$ will correspond to n vortices crossing every n' cycles of the modulation, where n and n' are positive integers. The existence of an

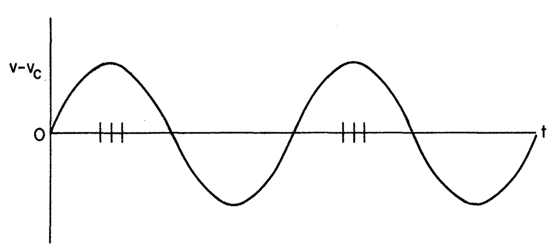

FIG. 3. Schematic representation of vortex synchronization caused by a sinusoidal velocity modulation for the case of a natural frequency of vortex creation equal to three vortices per cycle of the modulation. The tics represent the times at which vortices are created.

observable effect depends on the stability of the system (that is, the constancy of n and n') at values of $\Delta\mu$ which correspond to small values of n and especially of n'. The data in Fig. 1 clearly show such stability of the superconducting bridge at voltage values $V = nh\omega/n'2e$ corresponding to the crossing frequency in Eq. (10). Current steps with $n/n' = 2, 3, \ldots$ are called harmonics of the fundamental $n/n' = 1$ step and those with fractional values of n/n' are called subharmonics. The stability of the more pronounced current steps is indicated by the horizontal lines leading to the top and bottom of such steps. These reveal a tendency for the system to "jump" to the step rather than occupy a range of neighboring voltages. The range of instability is directly related to the current-source impedance used. Although the existence of such steps is plausible from the synchronization argument given above, the detailed nature of the stability, in particular, the selection rule for step height, is not understood in detail. For cases to which the Josephson tunneling theory applies, the nonlinear differential equation governing a junction driven by finite impedance dc and ac sources can be written down. Although solutions are difficult to obtain, this theory seems to account adequately for the observed phenomena.[18] Such a detailed treatment does not exist for the Dayem bridge, but similar behavior is observed. The superfluid He case of interest to us here has been qualitatively discussed by Anderson.[6] Even in the absence of detailed theory, synchronization effects analogous to those in the Dayem bridge can be predicted with confidence.

III. MEASUREMENT OF CHEMICAL POTENTIAL

A number of contributions to the chemical potential (the Gibbs free energy per atom) may be important in an experiment of the type described here. Bekarevich and Khalatnikov[19] have shown that the gradient of the chemical potential has the form

$$\Delta\mu/m = \Delta P/\rho + (\lambda/\rho)\,\Delta\omega - s\,\Delta T - (\rho_n/2\rho)\,\nabla\,(\vec{v}_n - \vec{v}_s)^2 . \tag{11}$$

Our experimental conditions were selected such that only the first term in (11) is important near the surface of the bath where Z is measured. We thus obtain $\Delta\mu = m\Delta P/\rho = mgZ$; so that stability of the system is expected when

$$Z = nh\omega/n'mg . \tag{12}$$

If one of the baths contained N static vortex lines per unit area of the fluid surface, then the second term in (11) would contribute $\Delta\mu = \lambda hN/\rho$, where $\lambda = -(\rho_s h/2m)\ln a(N)^{1/2}$, the ratio of internal energy to vorticity, depends only weakly on N. Neither this term nor the last term in (11), which depends on the counterflow of super and normal fluid, is expected to be large near the free surface. In the temperature range of interest $1.2 < T < 2.1$ °K the decay of a vortex ring due to collisions with rotons and phonons (mutual friction) is very rapid.[13] In the absence of rotation of the apparatus, only a small number of stable vortex lines should be excited. Effects which can be ascribed to such lines (persistent currents) are described in Sec. V C.

A temperature difference between the two baths will contribute the well-known fountain-effect term $\Delta\mu = -ms\Delta T$. In early experiments[5] such an effect arising from the heat generated by the transducer was clearly seen. A fraction of the measured Z was attributed to preferential flow of normal fluid (excitations) through the orifice away from the warm transducer. It is estimated that the heat conducted between the two baths, which determined ΔT, was shared nearly equally between viscous normal fluid flow through the orifice and a Kapitsa resistance limited heat flow through the capacitor walls. Differential evaporation from the two baths was minimized by the use of connecting tubes of small diameter. In the experiments reported here care was taken to avoid such temperature differences. It proved possible, by carefully minimizing the distance l between the transducer and the orifice, to obtain adequate modulation levels with negligible temperature difference. In principle, experimental situations could be contrived in which any or all terms in (11) contribute significantly to the chemical potential. Such variations of the experiment described here would serve primarily to verify (11), a task which could be accomplished more easily with more conventional equilibrium ($\Delta\mu = 0$) experiments.

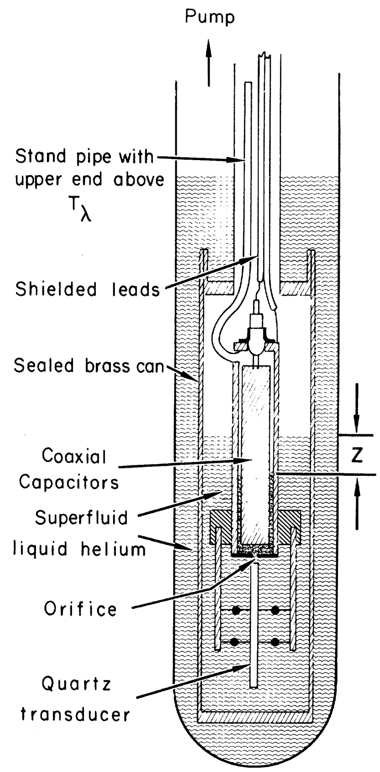

FIG. 4. Apparatus used for the present measurements. The brass can was 4.5 cm o.d. The He inside the capacitor is shaded dark to make it more visible.

IV. APPARATUS

The experimental apparatus shown in Fig. 4 was used to obtain the data presented here. It differs in several details from the apparatus used in our previously reported work.[5] The present experiment was performed in a sealed can immersed in pumped He inside a conventional glass cryostat. The He height inside the can was measured using a coaxial capacitor. Guarded leads connected the capacitor to a commercial capacitance bridge operated at various frequencies near 3 kHz. Changes in level were measured by reading the off-balance signal from the bridge with a lock-in amplifier. The lower end of the capacitor was sealed with a 1-mil Ni foil containing a single orifice.[20] The experimental bath was filled by condensation of room-temperature He gas at an over pressure of ~ 200 Torr. The condensation took place slowly enough to avoid a head difference Z between the inside and outside of the capacitor. The filling was monitored by observing capacitance changes which had been calibrated by raising or lowering the capacitor by a measured amount. The upper end of the capacitor communicated with the He vapor above the experimental bath through a narrow standpipe whose upper end was maintained above T_λ to avoid the conceptual and (perhaps) practical difficulties of a superfluid system multiply connected via film flow. Heat leakage down the standpipe and electrical leads and heat generated by the capacitance measurement may have caused a fountain-effect contribution to Z. Such equilibrium effects, including those caused by surface tension, do not affect the measurement since the "zero" of head difference was obtained from the measured capacitance at equilibrium. Since the surface area of the He in the capacitor is ~ 0.5% of that in the bath, changes in the experimental bath level were neglected except during calibration when a correction was applied for the rather larger change in capacitor displacement as it was raised or lowered. The homogeneity of the capacitor spacing placed a limit of ~ 2% on the absolute accuracy of measurement of Z.

The flow through the orifice was modulated using a quartz frequency-control crystal driven at its longitudinal resonant frequency of $\omega_0/2\pi = 104.6$ kHz. The 1×6-mm end face of the crystal was placed as close as practical ($l \sim 0.2$ mm) to the orifice. The rms velocity v of the end of the crystal was estimated by measuring its resonant Q while immersed in the He bath and also the total heat P generated for a given drive voltage. For a driven harmonic oscillator with a uniform mass distribution the stored energy $Mv^2/4 = PQ/\omega_0$. To obtain the velocity of the fluid in the orifice, the velocity of the end of the transducer must be multiplied by a correction factor. In the approximation of incompressible flow, this factor is $D^2/ld \sim 100$, where D is the effective diameter of the end of the transducer which is slightly larger than its thickness and d is the orifice diameter. The calibration of orifice velocity in terms of drive voltage varied greatly from run to run because of different values of l. Typical estimated velocities ranged from 1–100 cm/sec.

A drive voltage threshold ~ 10 times the largest useful values was observed above which the signal from the capacitance bridge became very noisy. This threshold was assumed to mark the onset of cavitation in the He.[21]

V. EXPERIMENTAL RESULTS

A. Critical Velocity

The critical velocity for flow through the orifice was measured by raising or lowering the capacitor assembly suddenly and monitoring the capacitance as a function of time. Typical results are shown in Fig. 5 for a 12-μ-diam orifice. They correspond to a head-dependent critical velocity $v_c \sim 32$ cm/sec for $Z = 0.5$ mm decreasing to ~ 1 cm/sec for $Z = 0.05$ mm.[22] Flow at smaller values of Z was masked by periodic sloshing of the He in the experimental vessel excited by the change in level of the capacitor. The Feynman[10] expression for the critical velocity in an orifice

$$v_c = (n\hbar/mR)\,[\ln(8R/a) - \tfrac{7}{4}] \qquad (13)$$

can be obtained by equating the rate of loss of potential energy $\rho g Z \pi R^2 v_c$ as fluid flows through the orifice with velocity v_c, to the energy of a vortex ring (4) times their rate of creation (9). Under our experimental conditions, the predicted $v_c = 3$ cm/sec is somewhat larger than our smallest observed value of ~ 1 cm/sec. To fit our data for larger values of Z, we would have to assume either that vortices smaller than the orifice size are created, or that vortices with more than one quantum of circulation are created in this range. Careful measurements of the limiting critical velocity for small Z by Trela and Fairbank[23] revealed reasonable agreement with (13) for $n = 1$ in certain cases

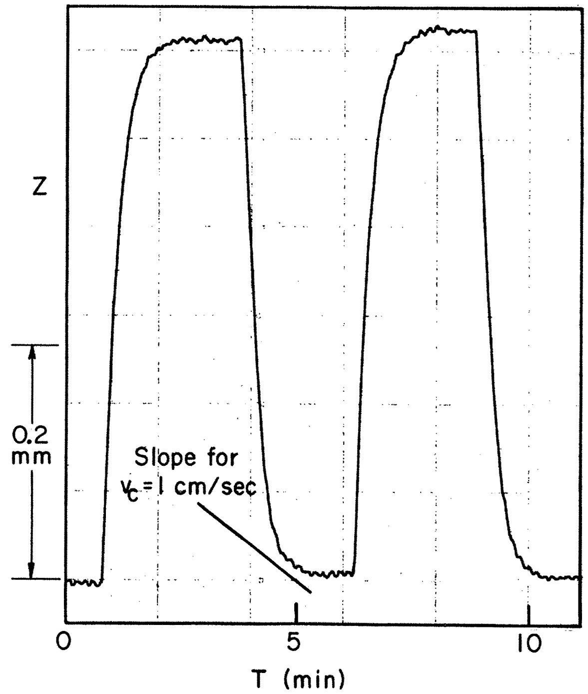

FIG. 5. Decay of a head difference Z between the two He baths which was created by repeatedly raising and lowering the capacitor. The critical velocity drops from 32 to about 1 cm/sec as the head Z decreases.

with larger orifices and lower temperatures. The estimated values of rms velocity in the orifice driven by the modulation are at least comparable to the limiting critical velocity. This appears to be necessary for vortex synchronization to be effective.

B. Pumping Effect

Experiments were usually begun by applying an ac drive voltage to the transducer with the superfluid in a quiescent state at $Z = 0$. Below a minimum drive amplitude no effect was observed. For larger amplitudes He was steadily pumped toward the bath with the transducer (out of the capacitor). This pumping effect was also observed in He above T_λ which had been supercooled to avoid bubble formation. No threshold for pumping was found in the few experiments done for $T > T_\lambda$. This observation is explained by assuming that the transducer is acting as a classical fluid pump. Since the transducer cannot be located symmetrically over the orifice, large fluid velocities transverse to the orifice axis are expected. The asymmetry of the geometry means that fluid is driven across one side of the orifice so that a head difference is produced. The apparatus acts as a Pitot tube for the measurement of this transverse velocity. No

Bernoulli pressure is obtained until vorticity is created. A threshold velocity is required for vortex creation in the superfluid state but not above T_λ. The observed pumping saturated at $Z = v^2/g$ from 1–10 mm, corresponding to transverse velocities of 10–30 cm/sec.

C. Evidence for Vortex Synchronization

Without vortex synchronization, the head difference Z would rise smoothly to its saturation value determined by the pumping action described above. The vortex synchronization phenomena described in Sec. II B produce dynamic stability of the system at values of Z given in Eq. (12). These correspond to n vortices crossing every n' cycles of the modulation. This stability is dramatically illustrated by the data in Figs. 6 and 7. In these figures, the ordinate has been adjusted so that $Z = 10$ divisions corresponds to one vortex crossing per cycle, that is, to $Z = \hbar\omega/mg$. In each run the equilibrium value of Z was established and then the transducer was turned on at the time marked "start." In the lowest curve of Fig. 6 the transducer amplitude was such as to pump a saturation head value close to one full step ($n/n' = 1$). The system showed remarkable stability at this step, returning to it twice after short excursions. These excursions were probably caused by mechanical shocks. The $n/n' = \frac{3}{2}$ subharmonic step is also in evidence. The upper two curves correspond successively to larger pump saturation values as well as higher modulation velocities in the orifice. The center curve shows strong stability for n/n' from 1 through 6 and rather weaker stability on identifiable $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ steps in between. It is expected that the apparent stability of a step increases as the pump approaches its saturation value and ceases to drive Z upward. Perhaps this effect counteracts an intrinsic weakening of the stability for larger values of n/n' such as is observed in the superconducting data shown in Fig. 1. The three curves shown in Fig. 6 were obtained on widely separated occasions with different values of l and so the orifice modulation velocity is not directly related to the pump drive voltage V. The correlation between modulation velocity and saturation height is reasonably close, however, and we can conclude from the inspection of these data and from the many hundreds of runs from which they were selected that for relatively low modulation levels the harmonic steps are rather more stable than the subharmonic steps. At larger modulation levels such as in the top curve, this preference for harmonic steps disappears and a quantitative verification of (12) is more difficult. Our first reported results[5] were obtained with relatively large modulation velocities compared with those shown here.

The modulation voltage and transducer placement for the lower curve in Fig. 7 are identical with those for the center curve in Fig. 6. The differences between the two curves appeared to be due to the previous history of the baths (trapped vorticity) and to uncontrolled shocks and vibrations reaching the apparatus. Despite obvious differences, the main features of the curves are similar. The upper curve in Fig. 7 shows the effect of increasing the orifice size. Again, as with large modulation amplitude, the preference for whole steps is reduced. The 20-μ orifice size is estimated to be large enough that a vortex line may not be carried across the orifice by the transverse velocity in one cycle of the modulation. We thus expect several vortices to be simultaneously present in the neighborhood of the orifice. No steps at all were seen for orifices larger than 50 μ diam. In the presence of large transverse velocities due to misalignment of the transducer the idealized picture of vortex rings presented in Sec. II is almost certainly not correct. All that is necessary to explain the observed effects, however, is to postulate vortex crossing at the proper rate. This experiment does not, therefore, give detailed information on the actual configuration of the vortices.

Although no detailed theory exists for the occurrence of subharmonic steps, the observations reported here are reasonable on fairly general grounds. The Josephson current of a tunnel junction biased at a constant voltage is a pure sinusoidal function of time. The application of a sinusoidal modulation creates harmonic, but not subharmonic steps. The current flow in an orifice or Anderson-Dayem bridge with fixed bias is certainly not a sinusoid. It follows from Eq. (8) that the flow increases linearly until a vortex crossing occurs. The harmonic content of this flow creates the subharmonic steps when a modulation is applied. It is certainly plausible that this harmonic content is richer when the modulation amplitude greatly exceeds the critical velocity or when the orifice is large enough to accommodate several vortices at once.

One frequently observed feature of the data is that Z returns to a value somewhat below its starting value when the transducer is switched off. A similar effect occurs in the published data of Khorana and Chandrasekhar.[7] One explanation is that some residual vorticity remains in the bath containing the transducer for a period of many minutes after the transducer is turned off. The equilibrium, which corresponds to $\Delta\mu = 0$, would then occur at a lower He level in the transducer bath because of the second term in Eq. (11).

It should be noted that the experiment described here differs in one important respect from the Anderson-Dayem experiment. There a high-imped-

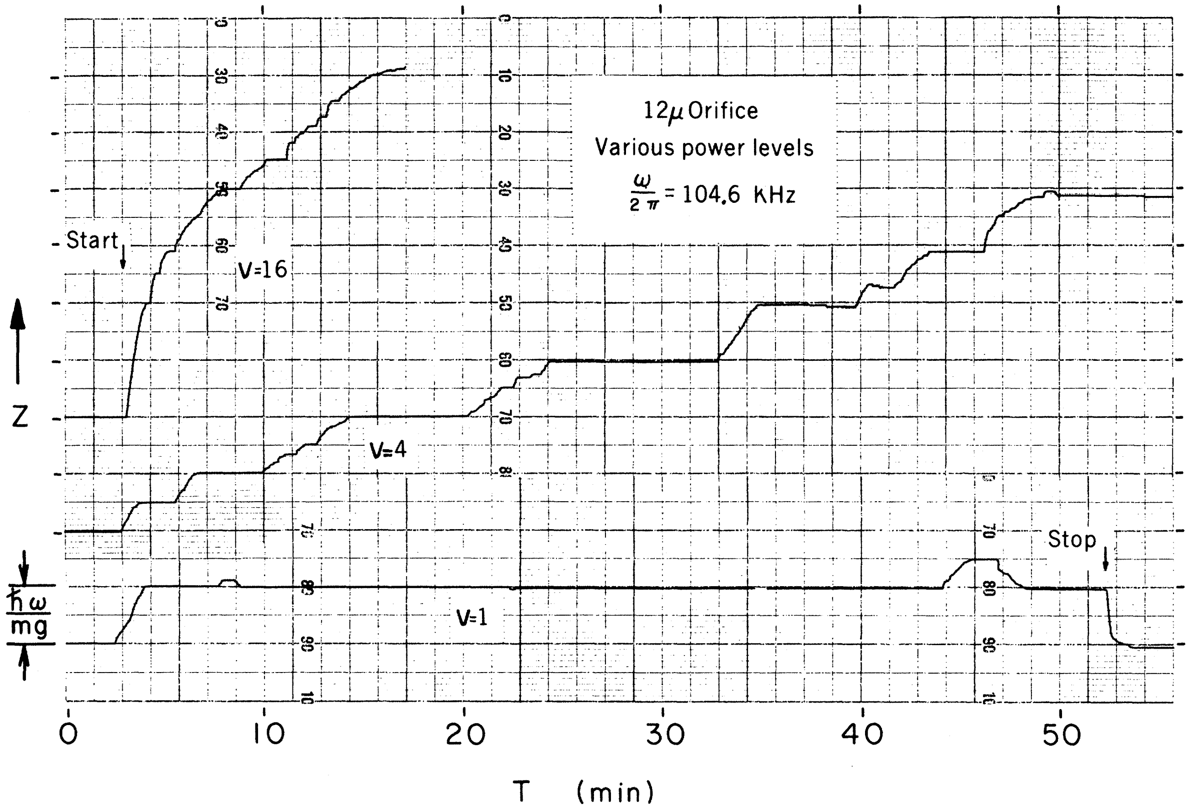

FIG. 6. Measured values of head difference Z versus time for various values of the ac voltage drive on the transducer. The horizontal regions indicate dynamic stability of the He at finite Z due to vortex synchronization.

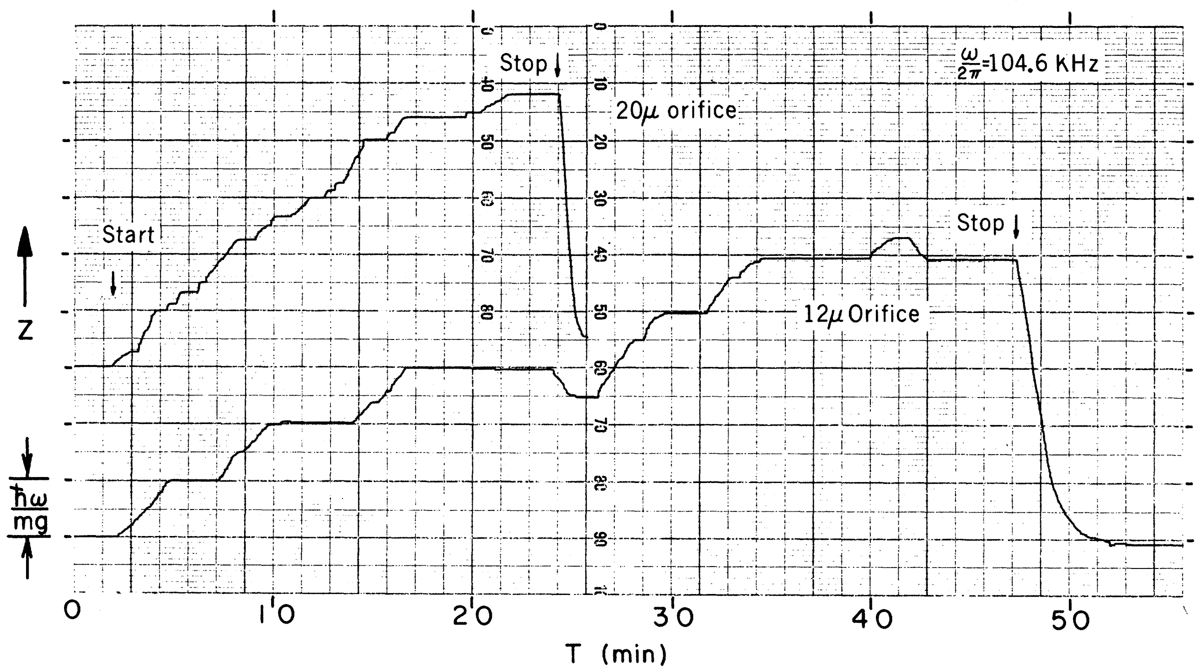

FIG. 7. Measured values of head difference Z versus time for two diameters of orifice. The modulation amplitude is approximately the same as for the center curve of Fig. 6.

ance-constant current source was used and the I–V curve was observed to jump across regions of instability as discussed in Sec. II B. Here the dc current source is the analog of a charged capacitor and its impedance is low compared with that of the orifice. Consequently, a more nearly continuous approach to dynamic equilibrium is observed. The ac source impedance is also low.

One other detailed point is that because of the nature of the current source a position of dynamic equilibrium at constant Z corresponds to no net flow of He. This is indeed fortunate since the normal fluid is at least partially clamped by the orifice and any He flow produces a temperature difference between the baths and thus an $s\Delta T$ contribution to $\Delta\mu$. Such effects can be neglected after the system has reached thermal equilibrium on any given step.

VI. CONCLUSIONS

Nearly five years have passed since the first positive results[5] were announced for the superfluid He analog of the ac Josephson effect. During that time the experiment has been repeated in more than six laboratories and the data have improved greatly. The experimental situation, however, is still unsatisfactory in several respects.

The experiments remain difficult to perform. That is, not all of the variables are under control. Although stability of the bath for various values of Z occurs regularly, data of the quality shown in Figs. 6 and 7 must be selected from many hundreds of runs. Even though plausible reasons for the selection usually exist, it is not clear to what extent the natural tendency to select data which agree with a preconceived theoretical idea has concealed important features of the He experiment. This is especially serious when an attempt is being made to test a theoretical prediction in the form of the ratio of two integers with no reliable information about selection rules. What can be said with confidence is that conditions of dynamic stability with $Z \neq 0$ (and thus $\Delta\mu \neq 0$) can be produced regularly, and that experimental conditions can be found (Figs. 6 and 7) for which the measured values of Z can be fit quantitatively by the expression $Z = n\hbar\omega/n'mg$ with a plausible systematic variation of the integers n and n'. We thus conclude with reasonable confidence that the theory has been verified.

All of the successful experiments known to the author are performed in apparatus identical in most important respects to that originally used. So far only one He analog of the many superconducting Josephson-effect phenomena has been observed. This problem seems important enough to justify a wider range of experimental efforts.

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Statistical Dynamics of Quantum Oscillators and Parametric Amplification in a Single Mode*

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The description of the statistical dynamics of quantum oscillators is formulated in terms of the Wigner distribution, analogous to the more commonly used P distribution, with explicit formulas being obtained for its time evolution and for average values. This formulation is desirable because, e.g., the Wigner distribution always exists whereas the P distribution does not. The formalism is applied to the process of parametric amplification in a single mode, which may be considered as the degenerate form of the well-known two-mode case. This degeneracy gives rise to significantly different properties; for example, the P distribution for the single mode of interest evolves from a circularly symmetric two-dimensional Gaussian into an elliptically symmetric form and ceases to exist after a finite time, even for amplification in the presence of losses. This is contrary to the two-mode case. The corresponding Wigner distribution is found to exist as a well-behaved function for all time as expected, regardless of the amount of losses, and is used to calculate average values of various quantities of interest. It is found, e.g., that in the lossless case the average number of photons in the signal mode always becomes infinite as $t \to \infty$. This is in contrast to the corresponding classical result for the lossless case which allows the signal to decay rather than to grow with time, depending on the relative phase between the signal and the pump. Field fluctuations are discussed and found to have some unusual properties. The combination of frequency up-conversion with single-mode amplification is also described briefly. The effect of the quantization of the pump oscillator is considered in an Appendix.

I. INTRODUCTION

The theoretical formulation that has been developed during the past few years for the description of various basic processes and field properties which are of interest in quantum electronics and quantum optics has included to a considerable extent formulations in terms of quantum-oscillator statistical distribution functions. Glauber,[1-4] especially, has made extensive use of the eigenstates of the oscillator annihilation operator, called coherent states, in his development of the theory. These states have been used to form the basis for an expansion of the density operator in terms of a distribution function.[1-5] The use of a distribution function, rather than the density operator, has the advantage that it is an ordinary function, allowing a graphical representation of the system under study, and allowing the calculation of average values by ordinary integrals very similar to the way in which it is done classically. If the time evolution of the distribution function can be determined, then one has the complete description of the corresponding process as a function of time. Of the possible distribution functions in terms of which a density operator may be expanded, the particular one usually referred to as the P distribution has received the most attention; it is a particular ("diagonal") case of the general distribution function which is obtained when the density operator is expanded in terms of coherent states.[3] The P distribution has a simple form and many convenient properties, and has therefore been used a good deal, but being a special case it leaves open the possibility that it is not adequate to describe all fields. Thus, the applicability or validity of the P distribution has caused considerable debate and study. Whereas fields could be thought of for which the P distribution does not exist (e.g., a field in the pure occupation-number state[6,7]), it seemed at first as if it were adequate for fields which are met in practice. Mollow and