break in d^2 versus E in going 400–500 eV (see Table II) is suggestive that the data from the two different laboratories are not totally consistent. If so, the problem of obtaining accurate absolute normalizations for differential cross sections may still be present.

With the above test case rather satisfactorily accounted for by the simple model described in Sec. II, we expect now to consider the more general problem of looking for an "optical model" for electron scattering by atoms. The problem is more complex for larger-Z atoms both because the static potential is generally less well known and because existent data are perhaps less satisfactory than for He. Preliminary results indicate that the considerations given in Sec. II should be adequate to carry us quite far into the general problem.

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[1]A. E. S. Green, T. Sawada, and D. S. Saxon, *The Nuclear Independent-Particle Model* (Academic, New York, 1968).

[2]A. E. S. Green, D. L. Sellin, and A. S. Zachor, Phys. Rev. 184, 1 (1969).

[3]R. B. Brode, Rev. Mod. Phys. 5, 257 (1933).

[4]M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964).

[5]A. C. Yates and T. G. Strand, Phys. Rev. 170, 184 (1968); and other references cited therein.

[6]M. M. Klein and K. A. Brueckner, Phys. Rev. 111, 1115 (1958); E. J. Robinson, *ibid.* 182, 196 (1969); L. B. Robinson, *ibid.* 127, 2076 (1962); P. M. Stone, *ibid.* 141, 137 (1966); P. M. Stone and J. R. Reitz, *ibid.* 131, 2101 (1963).

[7]R. A. Buckingham, Proc. Roy. Soc. (London) A160, 94 (1937); D. R. Bates and H. S. W. Massey, Trans. Roy. Soc. (London) A239, 269 (1943).

[8]C. E. Kuyatt, J. A. Simpson, and S. R. Mielczarek, Phys. Rev. 138, A385 (1965).

[9]A. T. Jusick, C. E. Watson, L. R. Peterson, and A. E. S. Green, J. Geophys. Res. 72, 3943 (1967); L. R. Peterson (private communication).

[10]P. S. Ganas, S. K. Dutta, and A. E. S. Green, following paper; Phys. Rev. A 2, 112 (1970).

[11]L. Vriens, C. E. Kuyatt, and S. R. Mielczarek, Phys. Rev. 170, 163 (1968); J. P. Bromberg, J. Chem. Phys. 50, 3906 (1969).

[12]R. W. LaBahn and J. Callaway, Phys. Rev. 180, 91 (1969); *ibid.* Addendum (to be published).

[13]S. P. Khare and B. L. Moiseiwitsch, Proc. Phys. Soc. (London) 85, 821 (1965); S. P. Khare, *ibid.* 86, 25 (1965); S. P. Khare, Phys. Letters 29A, 355 (1969).

[14]B. L. Moiseiwitsch and S. J. Smith, Rev. Mod. Phys. 40, 238 (1968).

[15]M. H. Mittleman and K. M. Watson, Phys. Rev. 113, 198 (1959).

[16]M. A. Melkanoff, T. Sawada, and J. Raynal in *Methods in Computational Physics* (Academic, New York, 1966), Vol. 6.

[17]R. W. Stagat (private comunication).

[18]S. K. Dutta, W. A. Wilson, and A. E. S. Green, J. Math. Phys. 9, 578 (1968).

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Modified Born Approximation and the Elastic Scattering of Electrons from Helium†

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We discuss the elastic scattering of electrons from helium atoms in terms of an energy-dependent central-potential model having the form

$$V(r) = 2Z \lim_{\Lambda \to \mu} \left(\frac{-\Lambda^2}{\Lambda^2 - \mu^2} \frac{e^{-\mu r}}{r} + \frac{\mu^2}{\Lambda^2 - \mu^2} \frac{e^{-\Lambda r}}{r} \right) - \frac{\alpha}{(r^2 + d^2)^2},$$

where μ, α are fixed constants (for He, $\mu = 3.375$ and $\alpha = 1.39$), and d is an energy-dependent phenomenological parameter. The method of partial waves is adapted for a generalized Yukawa potential and a polarization potential. Phase shifts and scattering cross sections are calculated from our potential model using the first Born approximation and a modified form of it. We develop an effective-range theory for a generalized Yukawa and a polarization potential, and apply it to generate a set of energy-dependent electron-helium phase shifts in the region 0–500 eV. Recent experimental angular distribution data in the region 100–500 eV are rather satisfactorily accounted for by our potential model. Our results compare favorably with those of LaBahn and Callaway.

I. INTRODUCTION

The general aspects of the elastic scattering of electrons from helium atoms have been discussed in Paper I.[1] Here we examine the same problem within the framework of the first Born approxima-

tion and a modified form of it.

This particular approach to the study of elastic scattering of electrons from helium is motivated by a number of objectives. In the first place, the approach is intended to facilitate the transfer to atomic physics of techniques developed for high-energy nuclear physics, and vice versa. The possibility of doing this rests in part upon certain physical similarities between electron-helium scattering in the 0–500-eV region and nucleon-nucleon scattering in the 0–400-MeV region. For example, in both cases the s waves have large (and positive) phase shifts, while all other phase shifts are small. In addition, the electron cloud in the helium atom and the proton charge cloud are both, to a good approximation, distributed according to an exponential function. Thus the Coulomb interactions between an external electron and these clouds have the same functional form. This similarity will serve as the starting basis of this work.

II. ELECTRON-HELIUM SCATTERING POTENTIAL

We work in atomic units, i.e., all radial distances are in units of the Bohr radius a_0, and all energies are in rydbergs.

By solving Poisson's equation one can show that the interaction of an electron with an exponential cloud of Z electrons is given by

$$V_c(r) = 2Z/r - (2Z/r)(1+\tfrac{1}{2}\mu r)e^{-\mu r} \quad . \tag{1}$$

To establish contact with recent work on meson clouds we note that Eq. (1) may be expressed as the limit of a "well-regulated superposition of three Yukawa functions,"[2]

$$V_c(r) = \lim_{\Lambda \to \mu} \lim_{U \to 0} J(r) \quad ,$$

where
$$J(r) = 2Z\left(\frac{e^{-Ur}}{r} - \frac{\Lambda^2 - U^2}{\Lambda^2 - \mu^2}\,\frac{e^{-\mu r}}{r} + \frac{\mu^2 - U^2}{\Lambda^2 - \mu^2}\,\frac{e^{-\Lambda r}}{r}\right) . \tag{2}$$

The relationships between the electron-helium and nucleon problems will be discussed further in Sec. X.

Our analysis will be based upon the Born approximation for electron scattering by the electron cloud potential given by Eq. (2), the electron nuclear potential $-2Z/r$, and the polarization potential $-\alpha(r^2+d^2)^{-2}$. Explicitly the scattering potential is

$$V(r) = 2Z \lim_{\Lambda \to \mu}\left(-\frac{\Lambda^2}{\Lambda^2-\mu^2}\,\frac{e^{-\mu r}}{r} + \frac{\mu^2}{\Lambda^2-\mu^2}\,\frac{e^{-\Lambda r}}{r}\right) - \frac{\alpha}{(r^2+d^2)^2} \quad . \tag{3}$$

Thus it consists of a generalized Yukawa potential, which provides the major attraction at short ranges, and a polarization potential, which provides the major attraction at long ranges. The parameters μ, α are fixed: $\mu = 3.375\ a_0^{-1}$, $\alpha = 1.39\ a_0^3$ for He; the parameter d is treated phenomenologically.

III. BORN AMPLITUDES FOR YUKAWA AND POLARIZATION POTENTIALS

The scattering amplitude in Born approximation is given by the well-known formula

$$f_B(K) = -K^{-1}\int_0^\infty r \sin Kr\, V(r)\, dr \quad , \tag{4}$$

where
$$K = 2k\sin\tfrac{1}{2}\theta \quad . \tag{5a}$$

Here $V(r)$ is the scattering potential (in rydbergs) and K is the momentum transfer. The wave number k is defined by

$$k = E^{1/2} \quad , \tag{5b}$$

where E is the energy in rydbergs, and the units of k are a_0^{-1}. From Eq. (3) we see that the scattering potential consists of (i) a static term $S(r)$, which is a sum of Yukawa functions,

$$V_S(r) = \textstyle\sum_i (-C_i)e^{-\mu_i r}/r, \tag{6}$$

where C_i and μ_i characterize the strength and range of the Yukawa components, and (ii) a sum of polarization potentials of the form

$$V_P(r) = \textstyle\sum_i -\alpha_i/(r^2+d_i^2)^2 \quad , \tag{7}$$

where α_i and d_i characterize the strength and range of the polarization components. For the Yukawa components, Eq. (4) gives

$$f_B^Y(K) = \textstyle\sum_i C_i/(K^2+\mu_i^2) \quad , \tag{8}$$

whereas for the polarization components

$$f_B^P(K) = \textstyle\sum_i \frac{1}{4}\pi\alpha_i e^{-Kd_i}/d_i \quad . \tag{9}$$

The Born amplitude for the generalized Yukawa of Eq. (3) is

$$f_B^S(K) = 2Z(K^2+2\mu^2)/(K^2+\mu^2)^2 \quad . \tag{10}$$

IV. PARTIAL-WAVE PROJECTIONS OF YUKAWA AND POLARIZATION POTENTIALS

Both the Yukawa and polarization potentials have convenient partial-wave projections. These may be evaluated using

$$f_{B,\,l}(k) = \tfrac{1}{2}\int_{-1}^{1} f_B(K) P_l(\cos\theta)\, d(\cos\theta) \quad , \tag{11}$$

where $P_l(\cos\theta)$ is the Legendre polynomial. For the Yukawa components, Eq. (11) gives

$$f_{B,\,l}^Y(k) = \sum_i \frac{C_i}{2k^2}\, Q_l\left(1+\frac{\mu_i^2}{2k^2}\right) \quad , \tag{12}$$

where Q_l is the Legendre function of the second kind.[3]

For the polarization components, Eq. (11) gives

$$f^P_{B,\,l}(k) = \sum_i \frac{\pi\alpha_i}{4d_i}\int_0^{2^{1/2}} \exp(-2^{1/2}kd_i t) \times P_l(1-t^2)t\,dt \quad . \tag{13}$$

The Legendre polynomials may be expressed in terms of the hypergeometric series according to[4]

$$P_l(1-t^2) = F(-l,\ l+1;\ 1;\ \tfrac{1}{2}t^2) = \sum_{n=0}^{l} D_{nl}\ (t^{2n}/2^n) \quad , \tag{14}$$

where the coefficients D_{nl} are given by

$$D_{nl} = [(-1)^n/(n!)^2]\,[(l+n)!/(l-n)!] \quad . \tag{15}$$

Substituting (14) into (13) we obtain

$$f^P_{B,\,l}(k) = \sum_i \frac{\pi\alpha_i}{2d_i}\sum_{n=0}^{l} D_{nl}\,\frac{\gamma(2n+2, 2kd_i)}{(2kd_i)^{2n+2}} \quad , \tag{16}$$

where $\gamma(a,y)$ is the incomplete γ function.[3] Both the incomplete γ function $\gamma(a,y)$ and the Legendre function $Q_l(x)$ may be computed very rapidly from recurrence relations.

The phase shift δ_l is related to the partial-wave projections according to

$$\tan\delta_l = kf^V_{B,\,l} + kf^P_{B,\,l} \quad . \tag{17}$$

V. LOW-ENERGY SCATTERING BY A POLARIZATION POTENTIAL

For a single polarization potential, Eqs. (16) and (17) give

$$\tan\delta_l = \frac{\pi\alpha k}{2d}\sum_{n=0}^{l} D_{nl}\,\frac{\gamma(2n+2,\,2kd)}{(2kd)^{2n+2}} \quad , \tag{18}$$

where the coefficients D_{nl} are given by (15). Now the incomplete γ function $\gamma(a,y)$ may be expanded as a power series in y[4]:

$$\gamma(a,y) = \sum_{p=0}^{\infty} \frac{(-1)^p y^{a+p}}{p!(a+p)} \quad . \tag{19}$$

Using the expansion (19) one may expand $\tan\delta_l$ as a power series in k; Eq. (18) gives

$$\tan\delta_l = \frac{\pi\alpha k}{2d}\sum_{n=0}^{l} D_{nl}\left(\frac{1}{2n+2} - \frac{2kd}{2n+3} + \frac{4k^2d^2}{2n+4} + O(k^3)\right) \quad . \tag{20}$$

Now it may be shown that for any $l>0$,

$$\sum_{n=0}^{l} D_{nl}\,(2n+2)^{-1} = 0 \quad , \tag{21}$$

$$\sum_{n=0}^{l} D_{nl}\,(2n+3)^{-1} = -[(2l+3)(2l+1)(2l-1)]^{-1} \quad . \tag{22}$$

The formulas (21) and (22) can be readily verified for small values of $l>0$. Inserting (21) and (22) into (20) we obtain

$$\tan\delta_l = \frac{\pi\alpha k^2}{(2l+3)(2l+1)(2l-1)} + O(k^3), \quad l>0. \tag{23}$$

The result (23) agrees with the modified effective range theory of O'Malley, Spruch, and Rosenberg[5]; however, they arrived at this result from rather different considerations, by solving Schrödinger's equation with a potential proportional to r^{-4}. It may be noted that our result is independent of d.

Considering the case $l=0$ we have $D_{nl} = D_{00} = 1$, and (20) gives

$$\tan\delta_0 = \frac{\pi\alpha k}{4d}\left(1 - \frac{4d}{3}k + d^2k^2 + O(k^3)\right) \approx \frac{\pi\alpha k/4d}{1+\frac{4}{3}dk+\frac{7}{9}d^2k^2+O(k^3)} \tag{24}$$

for sufficiently small k. Setting $a = -(\pi\alpha/4d)$, Eq. (24) becomes

$$\tan\delta_0 = k/[a_0 + a_1k + a_2k^2 + O(k^3)] \quad , \tag{25}$$

where $a_0 = -1/a$, (26)

$$a_1 = \pi\alpha/3a^2 \quad , \tag{27}$$

$$a_2 = -\tfrac{7}{16}aa_1^2 \quad . \tag{28}$$

The expansion (25) and the coefficients (26) and (27) are formally the same as those obtained by O'Malley, Spruch, and Rosenberg. It may be noted that their analysis also yielded a term proportional to $k^2\log k$. The appearance of a term linear in k is a characteristic of the long-ranged nature of the potential, as has been discussed by Spruch, O'Malley, and Rosenberg.[6]

If one wishes to consider a superposition of polarization potentials, then in the low-energy limit $\tan\delta_l$ will consist of a sum of terms of the form (25) in the case of the s wave, or a sum of terms of the form (23) in the case of $p, d, f, \ldots$, waves.

VI. LOW-ENERGY SCATTERING BY A YUKAWA POTENTIAL

For a single Yukawa potential, Eqs. (12) and (17) give

$$\tan\delta_l = (C/2k)Q_l(z) \quad , \tag{29}$$

where $z = 1 + \mu^2/2k^2$. (30)

In the low-energy limit we have $z \gg 1$; then using the following property of the Legendre function[4]:

$$Q_l(z) = \frac{l!}{(2l+1)!!}\ \frac{1}{z^{l+1}}\left[1 + O\left(\frac{1}{z^2}\right)\right], \quad z \gg 1, \tag{31}$$

we obtain the effective range expansion

$$\tan\delta_l = k^{2l+1}/[a_0 + a_2k^2 + O(k^4)] \quad , \tag{32}$$

where $a_0 = [(2l+1)!!/(2l)!!]\,[\mu^{2l+2}/C]$, (33)

$$a_2 = [(2l+2)/\mu^2]a_0 \quad . \tag{34}$$

The form (32) is characteristic of nuclear forces, which are short ranged; we note in particular that it does not contain a term linear in k. We see that at sufficiently low energies $\tan\delta_l$ is proportional to k^{2l+1} for all partial waves. This may be contrasted with the corresponding result for a polarization potential, where $\tan\delta_l$ is proportional to k^2 for all partial waves except the s wave.

If one wishes to consider a superposition of Yukawa potentials, then in the low-energy limit $\tan\delta_l$ will consist of a sum of terms of the form (32). More generally, if one considers a superposition of Yukawa and polarization potentials, then at sufficiently low energy, $\tan\delta_l$ will consist of a sum of terms of the form (32) together with a sum of terms of the form (23) or (25). In this case it may be noted that at sufficiently low energy the contributions to the s wave from both polarization and Yukawa potentials are proportional to k, whereas for $p, d, \ldots$, waves the polarization potential contributes a term proportional to k^2 while the Yukawa contributes a term proportional to k^3 for the p wave, k^5 for the d wave, and so on; thus the polarization potential gives the dominant contribution to $p, d, \ldots$, waves at sufficiently low energy.

For future use we note that Eqs. (32) and (33) give for the s wave

$$\tan\delta_0 \approx Ck/\mu^2 \tag{35}$$

to lowest order in k.

VII. PARAMETRIZATION OF LOWER-PARTIAL-WAVE PHASE SHIFTS

We may, of course, accept the numerical values of the lower-partial-wave phase shifts such as have been obtained in the detailed analyses of LaBahn and Callaway[7] and Callaway *et al.*[8] Alternatively it would be interesting to see if such phases can be characterized analytically. Our approach consists in applying the results of our effective-range theory to the low-energy phase shifts, the results of the Born approximation to the high-energy phase shifts, and using a suitable function to interpolate between the low- and high-energy regions.

For the s wave we take

$$V(r) = 4\lim_{\Lambda\to\mu}\left(-\frac{\Lambda^2}{\Lambda^2-\mu^2}\ \frac{e^{-\mu r}}{r} + \frac{\mu^2}{\Lambda^2-\mu^2}\ \frac{e^{-\Lambda r}}{r}\right) - \frac{\alpha}{(r^2+d^2)^2} \quad . \tag{36}$$

For the higher partial waves we take

$$V(r) = -\alpha/(r^2+d^2)^2 \quad . \tag{37}$$

Here $\alpha = 1.39$ and $\mu = 3.375$. This is equivalent to the assumption that the short-ranged part of the potential only affects the s wave. In the numerical computations we set $\Lambda = 1.001\mu$ and use double precision arithmetic.

The Born phase shift from the potential (36) may be expressed as

$$\tan\delta_0 = kf^Y_{B,0} + kf^P_{B,0} \quad , \tag{38}$$

whereas for the potential (37),

$$\tan\delta_l = kf^P_{B,l} \tag{39}$$

The quantities on the right-hand sides of (38) and (39) are given by Eqs. (12) and (16).

The low-energy analysis performed in Secs. V and VI suggests that we consider the following form for the s wave:

$$\tan\delta_0 = \left(\frac{k}{-1/a + (\pi\alpha/3a^2)k} + \frac{8k}{\mu^2}\right) I(k^2) + (kf^Y_{B,0} + kf^P_{B,0})[1 - I(k^2)] \quad . \tag{40}$$

For higher partial waves we use

$$\tan\delta_l = \frac{\pi\alpha k^2 I(k^2)}{(2l+3)(2l+1)(2l-1)} + kf^P_{B,l}\,[1 - I(k^2)]. \tag{41}$$

For a convenient interpolation function we use

$$I(k^2) = (\chi+1)/[\chi + \exp(\beta k^2)] \quad , \tag{42}$$

where χ, β are adjustable constants. The effective range term in (40) is composed of a contribution from the polarization potential as given by (25)–(27) and a contribution from the Yukawa potentials as obtained by applying Eq. (35) to the double Yukawa in (36). The effective range term in (41) is taken from Eq. (23). The interpolation function has the special property that for small values of k it approaches unity; then the expressions (40) and (41) reduce to effective range expansions. For large values of k it approaches zero; then the expressions (40) and (41) reduce to the Born ex-

pressions. In other words, the interpolation function suppresses the effective range terms at high energy and suppresses the Born terms at low energy.

We have parametrized the s, p, and d wave shifts in the region 0–500 eV using expressions (40) and (41). We fix $\chi = 1$ for all partial waves. For the s wave we search on a, β, d; and for the p and d waves we search on β, d. The final values are

s wave: $a = 1.983$, $\beta = 0.235$, $d = 1.4825$;

p wave: $\beta = 0.789$, $d = 0.6107$;

d wave: $\beta = 0.462$, $d = 0.6886$.

The fits are shown in Fig. 1. Thus expressions (40) and (41) with the above parameter values give an analytic representation of the s, p, and d wave phase shifts over the region 0–500 eV.

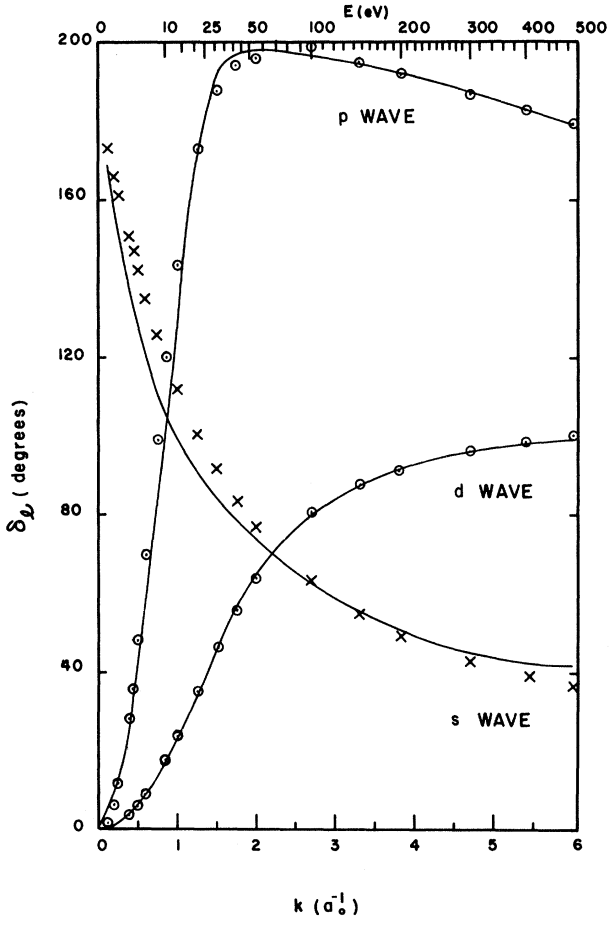

FIG. 1. Electron-helium s-, p-, and d-wave phase shifts. The data points represented by the symbols ⊙ and X are taken from Refs. 7 and 8. The solid curves are the best fits as given by the expressions (40) and (41). The p- and d-wave phase shifts have been enlarged by a factor of 10.

VIII. MODIFIED BORN APPROXIMATION

In the case of elastic scattering the Born approximation should break down at low energies due to the effects of exchange, polarization, the distortion of the incoming and outgoing plane waves in the field of the atom or molecule, and other physical effects. To account for all of these effects would be a large computational task, which loses the simplicity and utility of the closed-form Born results. In this section we describe a technique for modifying the Born approximation so that it applies in the low-energy region. The technique consists in projecting out the lower partial-wave components, say, for $l = 0$, 1, 2, from the closed-form expression for the Born elastic scattering amplitude, and replacing them by components from the exact scattering amplitude obtained from experiment or detailed solutions of the many-electron system. Modification of the Born scattering amplitude along these lines has been discussed by Dutta, Wilson, and Green[9] with reference to Yukawa potential scattering; the present discussion is similar except that we consider a generalized Yukawa and include a polarization potential as well.

For scattering by a central potential we can write the exact scattering amplitude in terms of partial-wave phase shifts as

$$f_e(k,\theta) = \sum_{l=0}^{\infty} (2l+1) A_l(k) P_l(\cos\theta) \ , \qquad (43)$$

where $A_l(k) = (\cos\delta_l \sin\delta_l + i \sin^2\delta_l)/k$. (44)

In calculating the coefficients A_l one may use phase shifts from experiment or from some theoretical calculation which gives cross sections in good agreement with experiment (e.g., the calculations of LaBahn and Callaway[7]). One may also use the parametrizations given by (40) and (41) if one rewrites (44) in the form

$$A_l(k) = (\tan\delta_l + i \tan^2\delta_l)/[(1+\tan^2\delta_l)k] \ . \qquad (45)$$

The theoretical values of LaBahn and Callaway are used in the present calculations.

The Born scattering amplitude may be written as

$$f_B(k,\theta) = \sum_{l=0}^{\infty} (2l+1) B_l(k) P_l(\cos\theta) \ , \qquad (46)$$

where $B_l(k) = \frac{1}{2} \int_{-1}^{1} f_B(k,\theta) P_l(\cos\theta) d(\cos\theta)$. (47)

If f_B is given by an expression such as (8) or (9) then the coefficients B_l are simply the partial-wave projections $f_{B,l}$ as given by (12) or (16). We consider a modification of the Born amplitude of the form

$$f_{MB}(k,\theta) = f_B(k,\theta) + f_C(k,\theta) \ , \qquad (48)$$

where $f_C(k,\theta)$ represents a low-energy correction term having the form

$$f_C(k,\theta)=\sum_{l=0}^{N}(2l+1)[A_l(k)-B_l(k)]P_l(\cos\theta)\ , \quad (49)$$

where N denotes the order of the highest partial wave to be corrected. In the calculations which follow we correct for s, p, and d waves. The function f_C is expected to be important at low energy and to approach zero at high energy.

The differential cross section may be calculated using

$$\frac{d\sigma}{d\Omega}=|f_{MB}(k,\theta)|^2=f_B^2+2f_B\Re f_C+|f_C|^2\ , \quad (50)$$

where $\Re f_C$ denotes the real part of f_C.

The total cross section is given by

$$\sigma=2\pi\int_0^{\pi}\frac{d\sigma}{d\Omega}\sin\theta\,d\theta\ . \quad (51)$$

The modified Born approximation technique is useful as an alternative to solving Schrödinger's equation. It has the advantage that it provides an analytic formulation of the problem. However, it clearly becomes unpractical in situations where many partial waves undergo large phase shifts.

IX. NUMERICAL RESULTS

Using the modified Born approximation technique to calculate differential cross sections from our potential model, we have made a direct comparison with the experimental angular distributions at energies 100, 150, 200, 300, 400, and 500 eV. Treating the polarization potential phenomenologically, we find that in order to reproduce the detailed features of the experimental angular distribution data, it is necessary to readjust the value of the range d at each energy; in other words we require the polarization potential to be energy dependent. We find furthermore that the variation of d with energy follows a square-root law:

$$d^2=E/200\ , \quad (52)$$

where E is the energy in eV. Our results are displayed in Fig. 2. The corresponding total cross sections are given in Table I. It is seen that they are very close to the values obtained by LaBahn and Callaway in their "extended polarization potential approximation," the differences being less than 3% in every case. Included in Table I are the values obtained from our potential model using just the ordinary Born approximation. At 100 eV the modified Born approximation introduces a substantial modification to the total cross section; the modification diminishes with increasing energy, becoming almost zero at 500 eV.

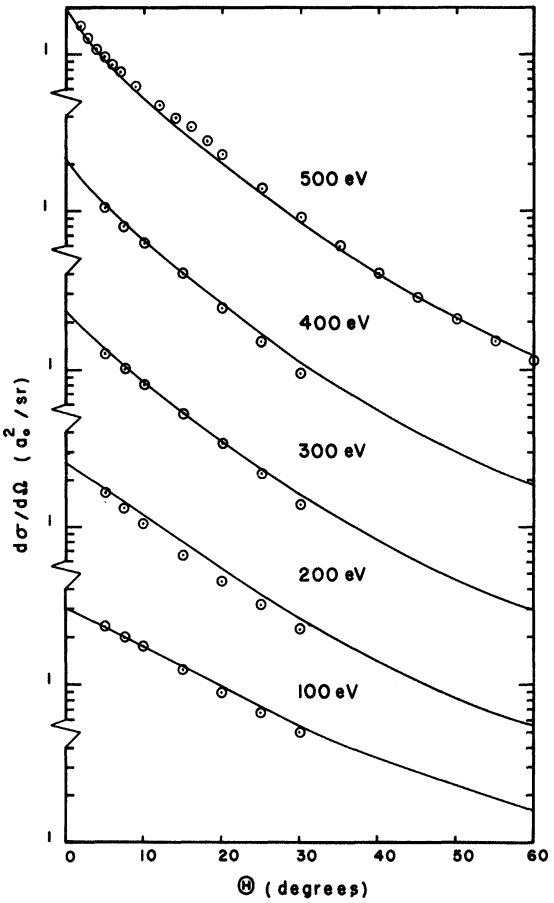

FIG. 2. Differential cross sections for the elastic scattering of electrons from helium. The circled points are the experimental data. The solid curves are the theoretical values from the potential model of this paper.

The experimental angular distributions in the region 100–400 eV have been taken from Vriens, Kuyatt, and Mielczarek[10] subject to correction. The data in Table II are the renormalized values for the differential cross sections at 5°.[11] The remaining angular data are obtained by renormalizing the data of Vriens *et al.* to match the 5° values listed above. The experimental angular data at 500 eV are taken from Bromberg.[12]

TABLE I. Total cross sections in units of a_0^2 for the elastic scattering of electrons from helium.

Energy (eV)	Born	Modified Born	LaBahn and Callaway
100	4.205	2.264	2.230
150	1.975	1.408	1.377
200	1.213	0.995	0.977
300	0.655	0.608	0.609
400	0.442	0.430	0.441
500	0.333	0.331	•••

TABLE II. Renormalized values for differential cross sections at 5°.

E(eV)	100	150	200	300	400
$d\sigma/d\Omega$ (a_0^2/sr)	2.34	1.98	1.64	1.27	1.04

In Table III we list the phase shifts from our energy-dependent potential model. They have been calculated in ordinary Born approximation on the one hand, and by solving Schrödinger's equation on the other. They are quite consistent with the values of LaBahn and Callaway.

X. DISCUSSION AND CONCLUSION

As we indicated in the Introduction there are several purposes to this study. In part we wish to advance our understanding of electron-atom elastic scattering and electron-helium in particular, and in part we wish to facilitate the transfer of mathematical techniques between atomic physics and high-energy nuclear physics. In the former context we must note that the potential [Eq. (1)] corresponding to an exponential charge density is very restrictive and only works well for very light atoms. The potential used by Green *et al.*[13] is more versatile and can be used to treat heavy atoms. We have established, however, that a sum of two potential terms of the type given by Eq. (1) can approximate rather well the analytic potential of Ref. 13.[14] Thus it is possible to deal with more complex atoms by simply doubling the number of Yukawa components.

One must then face the fact that for heavy atoms the first Born approximation breaks down as the electron-atom potential becomes stronger, and hence the modification procedure used here would become more cumbersome. Here the Schrödinger equation approach, with polarization corrections for the higher partial-wave phase shifts based upon the Born approximation, still should preserve its usefulness.

In recent years several people have discussed the analogs between the structure of elementary particles and the structure of atoms. We have already noted here the similarity of the interaction involved in elastic electron scattering from helium and elastic scattering of GeV electrons by protons. Recent phenomenological analyses of inelastic scattering of electrons by protons involving both discrete and continuous energy loss[15,16] also bear a resemblance to analyses of inelastic scattering of electrons by helium with discrete and continuous energy loss.[17] Undoubtedly it would be fruitful to both fields to pursue further the analogies in detail.

While similar in magnitude, the array of e-He phase shifts in the 0–500-eV range is much simpler than N-N phase shifts in the 0–400-MeV range, which have a marked dependence on the quantum numbers of the N-N system. In all likelihood when polarization experiments are carried out, the e-He system will also show a spin-orbit dependence, so that the differences in degree of complication will be reduced. Here we might note that Eqs. (40)–(42) in effect are a semiempirical energy-dependent set of phase shifts for the e-He system which join the effective range theory to the Born region. They are analogous to the corresponding energy-dependent phase-shift formulations for the N-N problem which have provided a useful way of analyzing and parametrizing experimental data.

Finally we should comment here on the polarization potentials which are recognized and are becoming moderately well understood in atomic scattering. The rather soft structure of protons suggests that corresponding polarization phenomena – perhaps more analogous to what arises in He-He scattering – should occur in the N-N problem. However, little effort has been given to the introduction of an explicit polarization potential into the N-N interaction problem.

TABLE III. Partial-wave phase shifts in radians for electron-helium scattering. In each block the first row gives the Born phase shifts and the second row the Schrödinger phase shifts from the energy-dependent potential model of this paper. The third row gives the values from the calculations of LaBahn and Callaway (Ref. 7).

l	100 eV	150 eV	200 eV	300 eV	400 eV	500 eV
0	0.8946	0.7670	0.6954	0.6198	0.5772	0.5478
	1.3739	1.0611	0.9142	0.7696	0.6931	0.6429
	1.1118	0.9628	0.8678	0.7540	0.6872	0.6417
1	0.4367	0.3886	0.3544	0.3194	0.3035	0.2943
	0.5742	0.4625	0.4060	0.3551	0.3326	0.3194
	0.3469	0.3397	0.3342	0.3258	0.3187	0.3121
2	0.1869	0.1921	0.1868	0.1771	0.1733	0.1724
	0.2119	0.2107	0.2012	0.1879	0.1828	0.1811
	0.1410	0.1527	0.1592	0.1672	0.1720	0.1751
3	0.0834	0.0976	0.1022	0.1033	0.1041	0.1058
	0.0880	0.1024	0.1065	0.1070	0.1076	0.1094
	0.0686	0.0811	0.0888	0.0982	0.1042	0.1086
4	0.0412	0.0526	0.0585	0.0630	0.0651	0.0674
	0.0422	0.0539	0.0600	0.0644	0.0665	0.0689
	0.0371	0.0473	0.0543	0.0631	0.0688	0.0729
5	0.0227	0.0304	0.0353	0.0401	0.0424	0.0444
	0.0229	0.0308	0.0358	0.0406	0.0430	0.0450
	0.0216	0.0202	0.0350	0.0429	0.0481	0.0518
6	0.0137	0.0189	0.0225	0.0266	0.0287	0.0302
	0.0138	0.0190	0.0227	0.0268	0.0289	0.0305
	0.0135	0.0189	0.0234	0.0301	0.0348	0.0382
7	0.0090	0.0125	0.0152	0.0184	0.0201	0.0213
	0.0090	0.0125	0.0152	0.0185	0.0203	0.0214
	0.0089	0.0128	0.0162	0.0217	0.0257	0.0289
8	0.006 20	0.008 70	0.0107	0.0133	0.0146	0.0155
	0.006 21	0.008 71	0.0107	0.0133	0.0147	0.0156
	0.006 17	0.008 98	0.0115	0.0159	0.0194	0.0222
9	0.004 47	0.006 31	0.007 82	0.0099	0.0110	0.0117
	0.004 47	0.006 32	0.007 84	0.0099	0.0111	0.0117
	0.004 45	0.006 52	0.008 46	0.0119	0.0148	0.0172
10	0.003 33	0.004 74	0.005 91	0.007 57	0.0085	0.0091
	0.003 33	0.004 74	0.005 92	0.007 59	0.0085	0.0091
	0.003 31	0.004 88	0.006 37	0.009 11	0.0115	0.0136

We see that there is much that atomic scattering can borrow from and contribute to high-energy nuclear physics. Clearly, because of costs, atoms and their excited states are more accessible to many experimental physicists than are nucleons and their excited states. It is hoped that this work, which explores only a few of the connections, will serve a useful function towards furthering the theoretical interchange between the two fields.

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[1]J. E. Purcell, R. A. Berg, and A. E. S. Green, preceding paper, Phys. Rev. A 2, 111 (1970).

[2]T. Ueda and A. E. S. Green, Phys. Rev. 174, 1304 (1968); A. E. S. Green and T. Sawada, Rev. Mod. Phys. 39, 594 (1967).

[3]*Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).

[4]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965).

[5]T. F. O'Malley, L. Spruch, and L. Rosenberg, J. Math. Phys. 2, 491 (1961).

[6]L. Spruch, R. F. O'Malley, and L. Rosenberg, Phys. Rev. Letters 5, 375 (1960).

[7]R. W. LaBahn and J. Callaway, Phys. Rev. 180, 91 (1969).

[8]J. Callaway, R. W. LaBahn, R. T. Pu, and W. M. Duxler, Phys. Rev. 168, 12 (1968).

[9]S. K. Dutta, W. A. Wilson, and A. E. S. Green, J. Math. Phys. 9, 578 (1968).

[10]L. Vriens, C. E. Kuyatt, and S. R. Mielczarek, Phys. Rev. 170, 163 (1968).

[11]R. W. LaBahn (private communication).

[12]J. P. Bromberg, J. Chem. Phys. 50, 3906 (1969).

[13]A. E. S. Green, D. L. Sellin, and A. Zachor, Phys. Rev. 184, 1 (1969).

[14]A. E. S. Green (unpublished).

[15]W. K. H. Panofsky, in *Proceedings of the* 14th *International Conference on High-Energy Physics, Vienna*, 1968, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968).

[16]J. D. Bjorken and E. A. Paschos, Stanford Linear Accelerator Report No. SLAL-PUB-572 (unpublished).

[17]A. E. S. Green and S. K. Dutta, J. Geophys. Res. 72, 3933 (1967).

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Photoionization of Argon and Xenon Including Final-State Correlation*

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Effects of final-state correlation on the photoionization cross sections for the $3p$ shell in argon and the $4d$ shell in xenon are calculated by the reaction-matrix method, starting from the Herman-Skillman model Hamiltonian, for photoelectron energies $0 \leq \epsilon \leq 1.2$ a.u. and $0 \leq \epsilon \leq 2.52$ a.u., respectively. The results for argon complement Hartree-Fock and other final-state correlation calculations; the results for xenon are the first *ab initio* calculations of this kind. Although length and velocity forms of the transition matrix element agree in the initial local-field approximation, they are shown to diverge necessarily when only the final state is improved.

I. INTRODUCTION

Independent electron theory, whether based on a local central-field potential or a nonlocal (Hartree-Fock) potential, fails conspicuously to account for the photoionization cross section of the $3p$ shell in argon and the $4d$ shell in xenon near threshold, where electron correlation is important.[1] For argon, Hartree-Fock calculations[2,3] fail to reproduce the experimentally observed knee in the cross section profile, while for both argon[2] and xenon,[4] calculations of the cross section using the Herman-Skillman[5] (HS) local potential give shifted narrow peaks two or more times larger than experiment. Fano and Cooper[6] have classified correlation effects in the continuum as