TABLE III. Numerical properties of the model.

Critical-point parameters T_c, ρ_c, μ_c	$T_c = 3.3095\,°\mathrm{K}$ $\rho_c = 0.041\,341\ \mathrm{gm/cm^3}$ $\mu_c = 3.3764 \times 10^{-15}$ erg
Polynomial coefficients $P(n) = \sum_{l=1}^{5} a_l \rho^l$	$a_1 = -3.912 \times 10^{-14}\ \mathrm{erg(cm^3/gm)}$ $a_2 = 5.995 \times 10^{-13}\ \mathrm{erg(cm^3/gm)}$ $a_3 = -4.7885 \times 10^{-12}\ \mathrm{erg(cm^3/gm)^3}$ $a_4 = 5.5925 \times 10^{-12}\ \mathrm{erg(cm^3/gm)^4}$ $a_5 = 1.0 \times 10^{-10}\ \mathrm{erg(cm^3/gm)^5}$
Singular term parameters	$\lambda = 1.0 \times 10^{-9}\ \mathrm{erg(cm^3/gm)^x}$ $x = 4.01$ $\rho_c = 0.041341\ \mathrm{gm/cm^3}$
Critical exponents	$\alpha = 0$ (no log) $\alpha' = 0.333$ $\gamma = 1$ $\gamma' = 1$ $\beta = 0.332$ $\delta = 4.01$

range many-body interactions must be considered if the model is to be solved. However, the only analogous short-range model of a many-body system which can be solved exactly at the present time is the two-dimensional Ising model.[13]

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Behavior of Eigenphases near a Resonance*

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Eigenphases near a resonance are shown to follow an equation which is a simple generalization of the usual equation for the behavior of the phase shift near a resonance in single-channel scattering. The parameters in the equation are shown to have a simple graphical interpretation.

Burke, Cooper, and Ormonde[1] have presented extensive calculations of excitation cross sections for e^--He collisions. The cross sections exhibited several resonances, whose effect on eigenphase shifts was studied using a formula [Eq. (9)] derived by the author. The influence of resonances on eigenphase shifts was studied earlier by Goebel and McVoy[2] for a specific model. However, they concluded that the resulting eigenphase curves were so complicated that a study of resonances using eigenphases was not fruitful. The complications arise because all eigenphases increase rapidly near a resonance. As one eigenphase increases it will cross another eigenphase curve. The crossing is actually avoided, however, and both eigenphases exhibit sharp bends near a resonance, as in Fig. 1. In this paper, we will show that eigenphase curves near a resonance, including the effects of crossing, are described by a simple generalization of the expression

$$E - E_r = \tfrac{1}{2}\Gamma \cot[\delta_0 - \delta(E)], \qquad (1)$$

valid for single-channel scattering. To a certain extent, the simplicity of the single-channel theory is preserved in the generalization of Eq. (1). Since the analysis of Burke *et al.*[1] indicates that a study of resonant eigenphases is useful, it seems desirable to present the derivation of Eq. (18) of their paper. In this regard, Lu and Fano[3] have shown that the quantum defects of three Rydberg series of Xe converging to the $5p^5\,{}^2P_{3/2}$ state of Xe$^+$ are perturbed by states from Rydberg series converging to the $5p^5\,{}^2P_{1/2}$ state, and that the perturbation is described by exactly the same equation as the perturbation of eigenphases near a resonance.

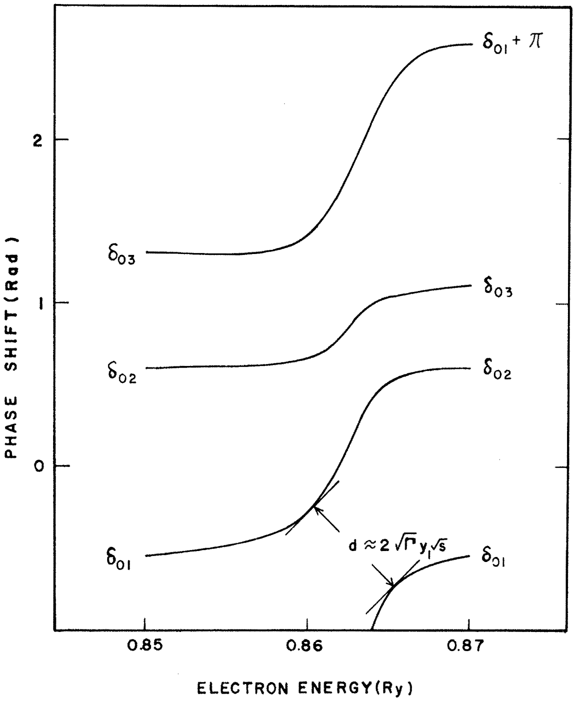

FIG. 1. The ${}^1S$ eigenphases near a resonance below the $n=3$ threshold for e^- - H scattering [taken from Burke *et al.* (Ref. 1)]. This figure illustrates Eqs. (9) and (12) of the text. Here $\sqrt{\Gamma}\,y_1$ is the amplitude for the decay of the resonance into the first eigenchannel and s is the ratio of the radian scale to the energy scale.

The derivation starts with the expression[4] for the S matrix near a resonance

$$S = S_0^{1/2}[1 - i\,\Gamma\vec{\gamma}^{+}\vec{x}\vec{\gamma}/(E - E_r + \tfrac{1}{2} i\,\Gamma)]S_0^{1/2}, \qquad (2)$$

where S_0 is the nonresonant S matrix and is assumed to vary slowly with energy near the resonance pole. The elements of the vector $S_0^{1/2}\vec{\gamma}^{+}$ give the amplitude for the resonance to decay into the physical channels and Γ is the total decay width. The eigenphases are determined by diagonalizing S. The matrix S is diagonalized in two steps. First S is transformed by a unitary transformation U_0, which diagonalizes S_0. Equation (2) becomes

$$S' = U_0^{+} S\, U_0 = e^{i\delta_0}[1 - i\,\Gamma\vec{y}\vec{x}\vec{y}/(E - E_r + \tfrac{1}{2} i\,\Gamma)]e^{i\delta_0}, \qquad (3)$$

where $e^{i\delta_0}\vec{y} = e^{i\delta_0}\,U^{+}\,\vec{\gamma}^{+}$ is a vector whose elements give the amplitude for the resonance to decay into the eigenchannels of S_0, and $e^{i\delta_0}$ is a diagonal matrix whose elements are the eigenvalues of $S_0^{1/2}$. Substituting Eq. (3) into the eigenvalue equation

$$S'\vec{b}_j = e^{2i\delta_j}\vec{b}_j \qquad (4)$$

gives

$$e^{2i\delta_j}\vec{b}_j = e^{2i\delta_0}\vec{b}_j - i\,\Gamma e^{i\delta_0}\vec{y}\,a_j\,[(E - E_r + \tfrac{1}{2} i\Gamma]^{-1}, \qquad (5)$$

$$\text{where}\quad a_j = \vec{y}\,e^{i\delta_0}\cdot\vec{b}_j\,. \qquad (6)$$

Equation (5) gives $\vec{b}_j$ in terms of a_j, and upon substituting the expression for $\vec{b}_j$ from Eq. (5) into Eq. (6), we find

$$a_j = \textstyle\sum_i\; y_i^2\,[\exp 2i\,(\delta_j - \delta_{0i})]^{-1}\,a_j[(E - E_r + \tfrac{1}{2} i\,\Gamma)]^{-1}. \qquad (7)$$

In order for Eq. (7) to hold, the coefficients of a_j on the left- and right-hand sides must be equal. Using the relation

$$[e^{-2i\theta} - 1]^{-1} = \tfrac{1}{2}[1 + i\,\cot\theta] \qquad (8)$$

and the condition $\sum_i y_i{}^2 = 1$ to simplify the right-hand side of Eq. (7), we get the desired relation

$$E - E_r = \tfrac{1}{2}\Gamma\sum_{i=1}^{n}\; y_i^2\cot[\delta_{0i} - \delta_j(E)]. \qquad (9)$$

Equation (9) gives δ_j as a function of E. For any given energy, it has n solutions $j = 1, n$, one corresponding to each zeroth eigenphase. The qualitative behavior of the eigenphases has been discussed in Ref. 1 and is illustrated in Fig. 1. We may gain further insight into the behavior of the eigenphases by considering their variation with energy. Taking the derivative of Eq. (9) with respect to E, and supposing that $\delta_{0i} \approx$ const, we get

$$1 = \tfrac{1}{2}\Gamma \frac{\partial\delta_j}{\partial E} \sum_{i=1}^{n} y_i^2 \sec^2[\delta_{0i} - \delta_j(E)], \tag{10}$$

which shows that δ_j always increases with energy. Near $E = E_r$ the eigenphases change rapidly with energy. The sharpness of the bend in the curve for δ_i near $E \lesssim E_r$ is a measure of the magnitude of y_i, i.e., a measure of the coupling of the resonance to the ith eigenchannel. This can be stated quantitatively by considering the point at which a tangent to the δ_j curve meets the energy axis at 45°. Then $\partial\delta_j/\partial E$ just equals the scale factor s, which gives the ratio of the radian scale to the energy scale. At such an energy near $E \sim E_r$, $\delta_j - \delta_{0j}$ is small, and we can neglect all terms on the right-hand side of Eq. (10) with $i \neq j$. Approximating the sin by its argument, we find

$$(\tfrac{1}{2}\Gamma)^{1/2} y_j \simeq |\delta_{0j} - \delta_j| / \sqrt{s}\ , \tag{11}$$

for an energy such that $\partial\delta_j/\partial E = s$ and $E \lesssim E_r$. From Eq. (11), it follows that the width for the decay of the resonance into the jth eigenchannel of S_0 is approximately one-half of the perpendicular distance d between the jth and $(j+1)$th curves divided by $\sqrt{s}$:

$$\sqrt{\Gamma}\, y_j \simeq \tfrac{1}{2} d / \sqrt{s}\ . \tag{12}$$

This relation is illustrated in Fig. 1. We emphasize that this construction gives only a rough determination of y_j. It is presented here to give precise meaning to the statement that the degree of curve repulsion measures the partial decay width for the resonance to decay into the jth eigenchannel.

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Interpretation of Transport Measurements in Electronically Conducting Liquids. III. Metals in Liquid Ammonia

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This note presents data on liquid metal-ammonia solutions that were omitted from an earlier tabulation and discussion of the transport characteristics of electronically conducting liquids. Similarities and differences between the metal-ammonia solutions and the liquids included in the earlier publication are noted and briefly discussed.

The transport properties of over 100 electronically conducting liquids were tabulated and discussed in two earlier publications[1] hereafter identified as ECL. The data were arranged according to the magnitude of the electrical conductivity σ. Three ranges of σ, labeled A, B, and C, were identified, covering the values $10^5 - 5\times10^3$, $5\times10^3 - 10^2$, and $10^2 - 10^{-12}\ \Omega^{-1}\,\mathrm{cm}^{-1}$, respectively.

As discussed in ECL, liquids in range A exhibit more or less conventional metallic characteristics. These can be understood in terms of Ziman's nearly free-electron theory of liquids.[2] It seems clear that liquids in range B should also be classified as metallic, despite the semiconductorlike behavior of their conductivity and thermoelectric power.[1]

Solutions of various metals in liquid ammonia show electronic conduction when the metallic concentration is high enough, and they should have been included in ECL. The soluble metals are Li, Na, K, Rb, Cs, Ca, Sr, Ba, Yb, and Eu.[3,4]

In traversing the composition range from pure Cs to pure NH_3, for example, electronic conductivity is observed in σ ranges A and B,[5] while a transition to ionic conductivity occurs in range C when the conductivity drops below $10^{-1}\ \Omega^{-1}\,\mathrm{cm}^{-1}$ or so.[4]

Some transport properties of solutions of Cs