Binding of three identical bosons in two dimensions

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Qualitative features are discussed for the binding of three identical bosons interacting through pair potentials in two dimensions. Two special cases, known to yield pathologies in three dimensions, are examined using the Faddeev equation for the bound states. The Thomas effect does not occur: in the model which is treated, the trimer binding energy is finite for a zero-range force with a finite dimer energy. The Efimov effect can only occur under more restrictive conditions than in three dimensions: the number of bound trimer states is finite at the dimer threshold for a range of potential models. The trimer ground-state energy is determined as a function of the coupling constant for a simple model, and variational results for loosely bound Lennard-Jones trimers are shown to reflect a general trend found for the model.

I. INTRODUCTION

The three-body problem in quantum mechanics is already complex enough that surprising possibilities[1-4] are included in the catalog of bound states as a function of coupling constant. For three bosons interacting through pair potentials in three dimensions, the Thomas effect[1,2] arises in the limiting case of zero-range forces and the Efimov effect[3,4] occurs for resonant two-body forces. Here we show that these two effects do not generally appear for the binding of three identical bosons in two dimensions. This emphasizes the role of spatial dimensionality in the formal theory of trimer binding. Further, the results have application to the self-binding of physisorbed submonolayers of quantum gases[5,6] such as helium.

In the Thomas effect, for three identical bosons in three dimensions, the trimer ground-state energy becomes infinitely negative for a zero-range attractive potential of finite scattering length; there is an infinity of trimer bound states.[2] For one dimension, in this limit there is a single trimer bound state with energy proportional to the dimer energy.[7] For two dimensions, we find in this limit that there are 2 trimer bound states, both with energy proportional to the dimer energy. There is no Thomas effect in one or two dimensions. We determine the bound states in two dimensions for two interaction models, as a function of the range parameter, and use the results in a discussion of variational calculations for Lennard-Jones trimers.[8]

In the Efimov effect, for three identical bosons in three dimensions, there is an infinity of trimer bound states when the pair potential is resonant, i.e., at the threshold for a bound dimer.[3,4] For a near-resonant interaction Efimov's construction[9] provides an illustration of how a self-bound trimer can be formed in the absence of a bound dimer.

We find that having the pair potential at the threshold for a bound dimer is not sufficient to produce an infinity of trimer bound states in two dimensions. This limitation on the occurrence of the Efimov type of binding is significant for monolayers of quantum gases because variational calculations with Jastrow trial functions[6] show the dimer and the many-boson thresholds for self-binding are equal in two dimensions. The existence of the Efimov effect would have shown a qualitative failure of the Jastrow trial function for boson trimers near threshold.

In this paper we treat only the case of three identical spinless bosons interacting via pair potentials in two dimensions. The organization of the paper is as follows: In Sec. II we review the formulation of the trimer binding calculation in terms of the Faddeev equation.[10,11] The study of the possibility of a Thomas effect is contained in Sec. III. Model calculations for the trimer energy as a function of the force range, with a comparison to calculations for a Lennard-Jones model, are presented in Sec. IV. The study of the possibility of an Efimov effect is contained in Sec. V. Some concluding remarks are made in Sec. VI.

II. FADDEEV EQUATION FOR THREE-BOSON BOUND STATES

The construction of the Faddeev equation for the bound states of three identical bosons[10,11] has been summarized recently by Lim and Duffy.[12] Here we give a statement of this equation and outline the properties which are used in the following work.

The Schrödinger equation for the three-boson bound-state wave function Ψ in the center-of-mass system is transformed to a homogeneous integral equation for a function ϕ related to Ψ by

$$\Psi(\vec{p}_1,\vec{p}_2,\vec{p}_3)=\phi(\tfrac{1}{2}[\vec{p}_1-\vec{p}_2],\ \tfrac{1}{3}[2\vec{p}_3-\vec{p}_1-\vec{p}_2]) + (\text{cyclic permutations}). \quad (1)$$

Defining relative momenta, in Jacobi coordinates,

$$\vec{p}=\tfrac{1}{2}[\vec{p}_1-\vec{p}_2],\quad \vec{q}=\tfrac{1}{3}[2\vec{p}_3-\vec{p}_1-\vec{p}_2], \quad (2)$$

the kinetic energy operator $H_0^{(3)}$ for three particles can be written as

$$H_0^{(3)}=p^2+\tfrac{3}{4}q^2. \quad (3)$$

Except in the discussion of the Lennard-Jones model in Sec. IV, we use units in which the reduced Planck constant $\hbar$ and the single-particle mass M are set equal to unity.

An off-shell two-particle T matrix is defined in the three-particle Hilbert space

$$T_i(z)=V_i+V_i\,G_0^{(3)}(z)\,T_i(z) \quad (4)$$

with the free-particle Green's function

$$G_0^{(3)}(z)=1/(z-H_0^{(3)})$$

and the pair potential

$$V_3=v_{12}(\text{cyclic}).$$

In the plane-wave representation, T_i is simply related to the two-particle t matrix for a shifted energy $z-\frac{3}{4}q^2$, z being the three-particle energy:

$$\langle\vec{p},\vec{q}\,|\,T_3(z)\,|\vec{p}\,',\vec{q}\,'\rangle=\langle\vec{p}\,|\,t(z-\tfrac{3}{4}q^2)\,|\vec{p}\,'\rangle\delta(\vec{q}-\vec{q}\,'). \quad (5)$$

Using the above definitions, the Faddeev equations for three identical bosons become

$$\phi(\vec{p},\vec{q})=2(z-p^2-\tfrac{3}{4}q^2)^{-1}\times\int d\vec{p}\,'\langle\vec{p}\,|\,t(z-\tfrac{3}{4}q^2)\,|\vec{p}+\tfrac{1}{2}\vec{q}\rangle\,\phi(\vec{q}+\tfrac{1}{2}\vec{p}\,';\vec{p}\,'). \quad (6)$$

We use this in a form given by Amado and Noble,[4] after defining a function

$$\psi(\vec{q},\vec{p})=\phi(\vec{p}+\tfrac{1}{2}\vec{q},\vec{q}), \quad (7)$$

$$\psi(\vec{q},\vec{p})=\iint d\vec{q}\,'\,d\vec{p}\,'\,K(\vec{q},\vec{p}\,|\,\vec{q}\,',\vec{p}\,')\,\psi(\vec{q}\,',\vec{p}\,'). \quad (8)$$

The kernel in Eq. (8) is

$$K(\vec{q}\,\vec{p}\,|\,\vec{q}\,'\vec{p}\,')=2(z-p^2-q^2-\vec{q}\cdot\vec{p})^{-1}\times\langle\vec{p}+\tfrac{1}{2}\vec{q}\,|\,t(z-\tfrac{3}{4}q^2)\,|\vec{q}\,'+\tfrac{1}{2}\vec{p}\,'\rangle\delta(\vec{q}-\vec{p}\,'). \quad (9)$$

In this paper we find it convenient to use an energy variable $s=-z$, with bound-state energies corresponding to positive values of s.

As it stands, Eq. (8) is a two-vector integral equation. By specializing to the cases of one- and two-term separable s-wave potentials this simplifies to one-variable integral equations if we use a partial-wave reduction.

The integral equation, Eq. (8), is in the form of an eigenvalue equation for the kernel K. The eigenstate energies $-s_j$ are the values of the parameter s which yield a kernel K with an eigenvalue equal to 1. The pathologies of the three-dimensional trimer binding can be expressed[2,4] in terms of properties of the eigenvalue spectrum of K. Except for the pathological cases K can be reduced to a compact (or completely continuous) kernel.[13] The exceptional cases of the Thomas effect and the Efimov effect can be identified, respectively, as divergences arising from the ultraviolet (high momentum) and infrared (low momentum) behavior of the kernel.

III. SEARCH FOR A THOMAS EFFECT

For three bosons in three dimensions, interacting via attractive pair potentials and with a coupling constant proportional to the depth of the potential and to the square of its range, the Thomas effect of infinite trimer binding occurs for a limit in which the range is taken to zero at (nearly) constant coupling.[1] In this limit, the dimer binding energy is held constant; in the absence of a bound dimer the scattering length is held constant. The essential features of the effect appear in an analysis with a one-term separable potential[2] or with a simple local potential well.[1]

In this section, we present the analogous separable potential calculation for three bosons in two dimensions. In two dimensions a purely attractive well or an attractive one-term separable potential has a bound dimer so that our analysis is for a limit taken at constant dimer binding energy. We have not constructed a satisfactory zero-range limit in two dimensions for a potential without a bound dimer.

The first step is the reduction of the Faddeev equation, Eq. (8), to a one-variable integral equation. We take the zero-range limit as the limit $\beta\to\infty$ in the form factor for the following one-term separable potential[14]

$$\langle\vec{k}\,|\,V\,|\,\vec{k}\,'\rangle=-f(k)f(k') \quad (10)$$

with

$$f(k)=g/(k^2+\beta^2). \quad (11)$$

For Eq. (10), the t matrix of Eq. (5) is

$$\langle\vec{p}\,|\,t(-s)\,|\vec{p}\,'\rangle=-f(p)f(p')/\Delta(-s) \quad (12)$$

with

$$\Delta(-s)=1-\int d\vec{k}\,[f(k)]^2/(k^2+s). \quad (13)$$

The Faddeev equation becomes

$$\phi(\vec{k}) = 2\int d\vec{p}\,(s+p^2+k^2+\vec{p}\cdot\vec{k})^{-1} \times \frac{f(|\vec{p}+\frac{1}{2}\vec{k}|)f(|\vec{k}+\frac{1}{2}\vec{p}|)}{\Delta(-s-\frac{3}{4}p^2)}\phi(\vec{p})$$

$$\equiv \int d\vec{p}\,K(\vec{k},\vec{p})\,\phi(\vec{p}) \qquad (14)$$

for a function $\phi(\vec{p})$ related to ψ by

$$\phi(\vec{p}) = \int d\vec{q}\,\psi(\vec{q},\vec{p})f(|\vec{q}+\tfrac{1}{2}\vec{p}|)\,. \qquad (15)$$

Projecting out the partial wave of zero total angular momentum gives

$$\phi_0(k) = \int_0^\infty p\,dp\,K_0(k,p)\,\phi_0(p) \qquad (16)$$

with

$$K_0(k,p) = \int_0^{2\pi} d\theta K(\vec{k},\vec{p})\,, \qquad (17)$$

where θ is the angle between the vectors $\vec{k}$ and $\vec{p}$.

The reduction in Eqs. (12)–(17) depends on the separable form [Eq. (10)], but not on the specific choice of form factor shown in Eq. (11).

The dimer binding energy $E_2 = -s_2$ is the solution of

$$\Delta(-s_2) = 0\,; \qquad (18)$$

for Eq. (11) this equation is

$$0 = 1 - \frac{\pi g^2}{\beta^4}\left\{\left(1-\frac{s_2}{\beta^2}\right)^{-2}\ln\left(\frac{\beta^2}{s_2}\right) - \left[1-\left(\frac{s_2}{\beta^2}\right)\right]^{-1}\right\}. \qquad (19)$$

To maintain a constant dimer energy E_2 as the range goes to zero ($\beta\to\infty$), the strength g must be adjusted to go as

$$\pi g^2/\beta^4 \simeq 1/(\ln\beta^2 - \ln s_2)\,, \quad \beta\to\infty\,. \qquad (20)$$

The way in which the energy s_2 appears in Eq. (20) is similar to the case for three dimensions where a reduced coupling constant is chosen near the dimer threshold; the deviation from the threshold value is then proportional to $1/\beta a$, where a is the scattering length.

For the limit $\beta\to\infty$ with Eq. (20), the kernel of Eq. (14) becomes

$$K(\vec{k},\vec{p}) = 2/\{\pi[\ln(s+\tfrac{3}{4}p^2) - \ln s_2] \times (s+p^2+k^2+\vec{k}\cdot\vec{p})\} \qquad (21)$$

and the projected kernel is

$$K_0(k,p) = 4[\ln(s+\tfrac{3}{4}p^2) - \ln s_2]^{-1} \times [(k^2+p^2+s)^2 - k^2p^2]^{-1/2}\,. \qquad (22)$$

The kernel K_0 is compact since

$$\mathrm{Tr}K_0K_0^+ = 16\int_0^\infty \frac{p\,dp}{[\ln(s+\frac{3}{4}p^2) - \ln s_2]^2} \times \int_0^\infty \frac{k\,dk}{[(s+k^2+p^2)^2 - k^2p^2]}$$
$$= 4\int_0^\infty \frac{dy}{y[\ln(s+\frac{3}{4}y) - \ln s_2]^2} \times \int_0^\infty \frac{dx}{\{[1+x+(s/y)]^2 - x\}}$$
$$< \infty\,, \quad \text{for } s > s_2\,. \qquad (23)$$

Therefore the Thomas effect of infinite trimer binding does not occur for this two-dimensional model: The largest eigenvalue of K_0 becomes less than 1 for sufficiently large s. In the limit $s\to s_2$, the largest eigenvalue is

$$\lambda_{\max} \simeq -\tfrac{8}{3}\ln[(s/s_2) - 1]\,, \quad s\to s_2\,.$$

With the kernel in Eq. (22), the trimer eigenstate energies $-s_j$ can be scaled with $-s_2$, the dimer energy. We determine the eigenvalues s_j from the dependence of the eigenvalues of the kernel K_0, Eq. (16), on the parameter s.

By numerical solution for the eigenvalues, we find the three largest eigenvalues of the kernel have the dependence shown in Fig. 1. As s ap-

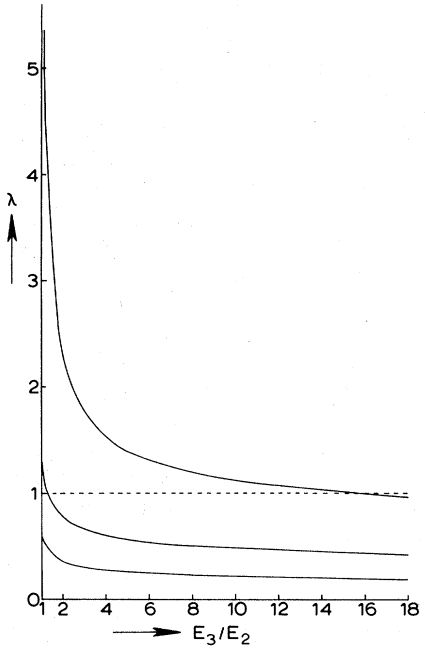

FIG. 1. Three largest eigenvalues of the Faddeev kernel for three bosons in two dimensions interacting with zero-range attracitve forces. The three largest eigenvalues of the zero angular momentum projection of the kernel, Eq. (22), are shown as a function of the ratio of the three-particle energy to the two-particle energy E_3/E_2. The trimer energies are determined from the condition that an eigenvalue λ equals 1. The values obtained in this way for the two trimer bound states are given in Eq. (24) of the text.

proaches s_2, i.e., as the ratio E_3/E_2 goes to 1, there are only two eigenvalues larger than 1. The largest eigenvalue becomes infinitely large at $s=s_2$, which is consistent with Eq. (23), but there is no sign of any other anomaly in the eigenvalue spectrum. The eigenvalues are monotonically decreasing functions of increasing s.

We find the two trimer bound-state energies of the zero-range model to be

$$E_{3,0}=(16.1\pm0.2)E_2, \quad E_{3,1}=(1.25\pm0.05)E_2. \tag{24}$$

For three identical bosons in one dimension, a limiting procedure similar to our Eqs. (19)–(21) leads to an integral equation treated by Dodd. He finds[7] a single trimer bound state, with energy 4 times the dimer energy.

The absence of the Thomas effect is related to the question of the relative ordering of the threshold coupling constants for self-binding of two and three bosons, Γ_2 and Γ_3. For a pair potential model $V(r)=V_0 f(r/l)$, the coupling constant is $\Gamma=MV_0 l^2/\hbar^2$. If Γ_2 is finite and nonzero, a threshold Γ_3 less than Γ_2 would permit the construction of a zero-range limit ($l\to0$) with finite dimer binding and infinite trimer binding.

IV. MODEL CALCULATION

With the assumption of a one-term separable potential and the angular momentum projection of Eq. (16), the Faddeev equation reduces to a one-dimensional integral equation which can be solved by standard methods.[15] The angular momentum projection of Eq. (17) was done analytically, Eq. (22), for the zero-range limit; it can also be done analytically for form factors similar to Eq. (11),

$$f_m(k)=g/(k^2+\beta^2)^m, \quad m=1,2,\ldots, \tag{25}$$

without the zero-range limit being taken.

We show the results of solving the integral equation for the cases $m=1$ and $m=2$ in Fig. 2. The ratio of the trimer ground-state energy $E_{3,0}$ to the dimer energy E_2 is shown as a function of the coupling constant $\Gamma\equiv\pi g^2/\beta^{4m}$. The coupling constant Γ is given in terms of the ratio E_2/β^2 by Eqs. (13) and (18). Through these equations, Fig. 2 can also be read as a presentation of the ratio $E_{3,0}/E_2$ as a function of the range $1/\beta$.

The ratio $E_{3,0}/E_2$ goes to 16.1 as the coupling constant Γ goes to zero, in accordance with the zero-range result Eq. (24). The shape dependence of the ratio can be judged from a comparison of the results for the two form factors. Extrapolation of the ratios to small coupling constants without prior knowledge of these results could easily give misleading conclusions. The ratios for both form factors show a sharp rise with decreasing Γ followed by a leveling off at small Γ values. The precision of most variational calculations of trimer energies, such as for the Lennard-Jones model,[16] is limited enough that it would be difficult to observe the leveling off.

We also show ratios $E_{3,0}/E_2$ for a Lennard-Jones (12, 6) potential[8,16] in Fig. 2. The correspondence of the Lennard-Jones coupling constant to the Γ in Fig. 2 is established in the following way. For the Lennard-Jones (12, 6) pair potential

$$v(r)=4\epsilon[(\sigma/r)^{12}-(\sigma/r)^6], \tag{26a}$$

a coupling constant K is defined by

$$K=4\epsilon\sigma^2M/\hbar^2. \tag{26b}$$

The threshold coupling constant for a bound dimer in two dimensions is found by numerical integration of the Schrödinger equation to be $K_{2,c}=14.74\cdots$. Similarly, near threshold the dimer binding energy goes as

$$1/\ln|E_2|\simeq\gamma(K-K_{2,c})+\cdots \tag{26c}$$

with $\gamma\simeq\frac{1}{28}$. From Eq. (19), at small Γ the dimer binding goes as

$$\ln|E_2|\simeq1/\Gamma+\cdots.$$

Therefore we take the correspondence between Γ and K to be

$$\Gamma=\gamma(K-K_{2,c}) \tag{27}$$

with the γ and $K_{2,c}$ values fitted to the numerical results[8] for the Lennard-Jones dimer energy.

With this identification, the results of variational calculations for the boson trimer ground-state energy were plotted in Fig. 2; the point at $\Gamma\simeq0.26$ corresponds to the use of the De Boer–Michels ^{4}He parameters[16] for the Lennard-Jones potential. These points are lower bounds on the ratio $E_{3,0}/E_2$ for the Lennard-Jones model. Given the amount of shape dependence found for the two form factors in the separable potential model, the magnitude and trend of the Lennard-Jones model results are in good accord with these models. At the Γ value of 0.15, the ratio $E_{3,\text{trial}}/\epsilon$ for the variational calculation is roughly 1×10^{-3}. It decreases rapidly with decreasing Γ and with the present precision of the variational calculations it would be difficult to follow the trend to the asymptotic small-Γ region found for the separable model.

The second trimer bound state, noted for the zero-range model in Sec. III, persists for finite ranges in both separable models at least to Γ values of 0.25. No second bound state has been sought in the Lennard-Jones trimer calculations but this suggests one might be present there.

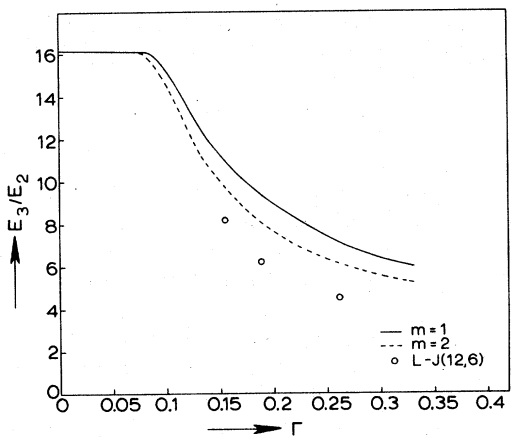

FIG. 2. Ratio of trimer and dimer ground-state energies E_3/E_2 as a function of coupling constant for three models. The lines $m=1$ and $m=2$ are smooth curves drawn through results for the ground-state energy of three bosons in two dimensions interacting with the one-term attractive separable potentials, Eq. (25). The LJ (12,6) points are taken from variational calculations for a Lennard-Jones (12,6) pair potential model. For a discussion of the shape dependence shown here and a statement of how the coupling constant Γ is related to the conventional Lennard-Jones coupling constant, see Sec. IV of the text.

V. SEARCH FOR AN EFIMOV EFFECT

In three dimensions, Amado and Noble[4] treated the Efimov effect[3] by analyzing the Faddeev equation for a one-term attractive separable potential. At the threshold for dimer binding the eigenvalue spectrum of the zero angular momentum projection of the Faddeev kernel had an accumulation point at a value larger than 1. Further analysis showed that this accumulation point implied the existence of an infinity of bound trimer states. For three identical bosons in two dimensions, we show here that the kernel does not generally develop such an accumulation point at the threshold for dimer binding. As a result the infinity of Efimov-type states then does not occur.

In two dimensions, the one-term attractive separable potential has a bound dimer at all values of the coupling constant. To examine the spectrum of the trimer near the dimer threshold we use a two-term separable potential for which the dimer threshold occurs at a finite coupling constant:

$$\langle \vec{k}|V|\vec{k}'\rangle = g(k)g(k') - f(k)f(k'). \tag{28}$$

The form factors $f(k)$ and $g(k)$ are assumed to be smooth real functions[14] which are square integrable, but are otherwise general. The threshold condition for a bound dimer is that the quantity A defined by

$$A = \pi([g(0)]^2 - [f(0)]^2) - 2\pi^2 \int_0^\infty dk\,[f(k)g(0) - f(0)g(k)]^2/k \tag{29}$$

is zero. For $A<0$ the dimer is bound; there is at most one bound two-body state for this model.

The two-variable Faddeev equation, Eq. (8), now reduces to a pair of coupled one-variable equations. The steps in the reduction are straightforward. The off-shell t matrix, Eq. (5), for the model Eq. (28) is available in closed form.[17] With the definition of one-variable functions ϕ_i analogous to Eq. (15),

$$\phi_1(\vec{q}) = \int d\vec{p}\, f(|\vec{p}+\tfrac{1}{2}\vec{q}|)\,\psi(\vec{p},\vec{q}),$$
$$\phi_2(\vec{q}) = \int d\vec{p}\, g(|\vec{p}+\tfrac{1}{2}\vec{q}|)\,\psi(\vec{p},\vec{q}), \tag{30}$$

the coupled integral equations are written in a matrix form

$$\phi_i(\vec{k}) = \sum_{j=1}^{2} \int d\vec{p}\, K_{ij}(\vec{k},\vec{p})\,\phi_j(\vec{p}), \quad i=1,2 \tag{31}$$

where the matrix kernel is

$$K_{ij}(\vec{k},\vec{p}) = 2k_{ij}(\vec{k},\vec{p})/\Delta(-s-\tfrac{3}{4}p^2)(s+p^2+k^2+\vec{k}\cdot\vec{p}). \tag{32}$$

The components k_{ij} and Δ are given in Appendix A; here we list the properties of these functions which enter in the discussion of a possible Efimov effect.

The asymptotic behavior of the function Δ at small argument is

$$\Delta(-x) \simeq A\ln(1/x) + B, \quad x \to 0, \tag{33}$$

where A is defined in Eq. (29) and B is a positive constant. The dimer thresholds which we discuss correspond to $A=0$ and $B>0$. The case $A=B=0$ has not been excluded as a possible occurrence for some model; the twofold condition $A=B=0$ would reflect a strong specialization of the model in place of the general models of Efimov's treatment.[3]

The matrix $k_{ij}(\vec{k},\vec{p})$ has zero determinant. Further for zero arguments and zero energy s its trace is proportional to A:

$$k_{11}(0,0) + k_{22}(0,0) = -A/\pi. \tag{34}$$

These features have the consequence, in the treatment outlined in Appendix A, that the small-argument portion of the Faddeev kernel, which becomes singular at the dimer threshold in three dimensions, has zero strength at the dimer threshold in two dimensions.

The zero angular momentum projection of Eq. (31) is

$$\phi_i^{(0)}(k) = \sum_{j=1}^{2} \int_0^\infty p\,dp\,K_{ij}^{(0)}(k,p)\,\phi_j^{(0)}(p), \tag{35}$$

where as in Eq. (17)

$$K_{ij}^{(0)}(k,p) = \int_0^{2\pi} d\theta K_{ij}(\vec{k},\vec{p}). \tag{36}$$

For the discussion of possible accumulation points of the spectrum of the Faddeev kernel we define

$$K_1(s) = \mathrm{Tr}K^{(0)} = \sum_i \int_0^\infty k\,dk K_{ii}^{(0)}(k,k); \tag{37}$$

$$K_2(s) = \mathrm{Tr}K^{(0)}K^{(0)+} = \sum_{i,j} \int_0^\infty p\,dp\,[K_{ij}^{(0)}(k,p)]^2. \tag{38}$$

For a one-term attractive separable potential in three dimensions, Amado and Noble[4] show that the quantity corresponding to K_2 is finite except in the special case that there is a zero-energy dimer, infinite scattering length in the two-body problem. At the dimer threshold, both K_1 and K_2 are logarithmically divergent for small s from contributions of the low-momentum portion of the kernel, an infrared divergence. In Efimov's treatment[3] this corresponds to an effective long-range interaction among the three bosons at the resonance.

For the two-term separable model, Eq. (28), in two dimensions K_2 is finite for $s \geq 0$ short of the threshold ($A>0$). At the threshold $A=0$, K_2 is logarithmically divergent for small s, with the singular contribution coming from small momenta. For $A>0$, K_1 is logarithmically divergent at small s. Thus, short of the dimer threshold the only possible accumulation point which can occur in the spectrum of the kernel is at zero. The nature of the spectrum for $A=0$ is left open.

We establish, in Appendix A, that the spectrum of the Faddeev kernel has no finite nonzero accumulation point even at the dimer threshold. The first iterate of the kernel, $K^{(0)2}$, is shown to be compact for $s=0$ in the limit that A goes to zero. The compactness rests on the relation Eq. (34), with B of Eq. (33) being positive. The weak low-energy scattering for the "resonant" pair potential in two dimensions also follows from the relation Eq. (34).

The Efimov type of pathology at the dimer threshold does not occur with the same generality in two dimensions as in three dimensions. We have not established that it never occurs for any potential model at threshold; however, Eq. (33) and the qualitative relation Eq. (34) do not appear to be special features of the two-term separable potential. The contrast that the resonant two-body interaction scatters strongly in three dimensions and weakly in two dimensions makes it appear unlikely for a long-range effective interaction to appear at the dimer threshold in two dimensions.

VI. CONCLUSIONS

The binding of boson trimers is a mathematically more regular phenomenon in two dimensions than in three dimensions, judging from the absence of two special cases, the Thomas and the Efimov effects.

In three dimensions the Efimov pathology of an infinity of bound states for a finite number of particles interacting with finite-range pair potentials occurs only[18] for the trimer. Our work places a further restriction on the generality of this pathology: We do not find it for the boson trimer in two dimensions for a range of potential models. We believe that the reasons for the absence of the Efimov phenomenon in our results are quite general and that it would require rather contrived potential models to obtain the phenomenon.

The absence of a Thomas effect for the boson trimer in two dimensions permits construction of a zero-range-force formalism for the trimer binding energy. General trends in the binding of Lennard-Jones trimers of small effective coupling constant in two dimensions are reflected in a model calculation with a one-term separable potential with finite-range form factors. With parameters for ^{4}He atoms, though, the trimer binding does show shape dependence of the potential rather than the simple limiting behavior of a zero-range approximation.

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APPENDIX A: FADDEEV KERNEL FOR THE TWO-TERM SEPARABLE POTENTIAL

The components k_{ij} and Δ of the Faddeev kernel in Eq. (32) of the text are defined with functions α, β, γ and a notation

$$s_p = s + \tfrac{3}{4}p^2; \tag{A1}$$

$$\alpha(-x) = -1 + \int d\vec{k}\,[f(k)]^2/(k^2+x),$$

$$\beta(-x) = 1 + \int d\vec{k}\,[g(k)]^2/(k^2+x), \tag{A2}$$

$$\gamma(-x) = \int d\vec{k}\,g(k)f(k)/(k^2+x),$$

by

$$\Delta(-s_p)=[\gamma(-s_p)]^2-\alpha(-s_p)\beta(-s_p) \tag{A3}$$

$$\begin{aligned}
k_{11}&=\beta(-s_p)f(|\vec{p}+\tfrac{1}{2}\vec{k}|)f(|\vec{k}+\tfrac{1}{2}\vec{p}|)\\
&\quad-\gamma(-s_p)f(|\vec{p}+\tfrac{1}{2}\vec{k}|)g(|\vec{k}+\tfrac{1}{2}\vec{p}|),\\
k_{12}&=\alpha(-s_p)f(|\vec{p}+\tfrac{1}{2}\vec{k}|)g(|\vec{k}+\tfrac{1}{2}\vec{p}|)\\
&\quad-\gamma(-s_p)f(|\vec{p}+\tfrac{1}{2}\vec{k}|)f(|\vec{k}+\tfrac{1}{2}\vec{p}|),\\
k_{21}&=\beta(-s_p)g(|\vec{p}+\tfrac{1}{2}\vec{k}|)f(|\vec{k}+\tfrac{1}{2}\vec{p}|)\\
&\quad-\gamma(-s_p)g(|\vec{p}+\tfrac{1}{2}\vec{k}|)g(|\vec{k}+\tfrac{1}{2}\vec{p}|),\\
k_{22}&=\alpha(-s_p)g(|\vec{p}+\tfrac{1}{2}\vec{k}|)g(|\vec{k}+\tfrac{1}{2}\vec{p}|)\\
&\quad-\gamma(-s_p)g(|\vec{p}+\tfrac{1}{2}\vec{k}|)f(|\vec{k}+\tfrac{1}{2}\vec{p}|).
\end{aligned} \tag{A4}$$

A first indication of the role of the weak low-energy scattering at the two dimensional "resonance" can be seen in the trace K_1 of Eq. (37):

$$K_1(s)=2\int d\vec{p}\,\frac{[k_{11}(\vec{p},\vec{p})+k_{22}(\vec{p},\vec{p})]}{\Delta(-s_p)(s+3p^2)}. \tag{A5}$$

Short of the dimer threshold ($A>0$) the small-momentum contribution in Eq. (A5) is finite except at $s=0$. At the threshold, the small argument approximations presented in Eqs. (33) and (34) show that the small-momentum contributions remain finite even at $s=0$.

For positive s ($s>0$), the Green's-function factor in Eq. (32) is nonsingular and the kernel is compact for $A\geq 0$. For negative A, with a dimer of energy $E_2=-s_2$, the kernel is noncompact for $s\to s_2$. This appears to be the same type of singularity as the one in Eq. (23) for $s=s_2$, which does not signal a finite accumulation point in the spectrum of the Faddeev kernel. The remaining question is the nature of the spectrum of the kernel for the case $A=s=0$, for which the Hilbert-Schmidt norm Eq. (38) is infinite.

We now outline the treatment of the kernel as a function of the parameter A ($A\geq 0$) for the case $s=0$. The first iterate of the kernel, $K^{(0)2}$, is shown to be compact even in the limit $A\to 0$, so that the kernel $K^{(0)}$ has no nonzero finite accumulation point in its spectrum even for the case $A=s=0$.

The singular contribution to the Hilbert-Schmidt norm $K_2(0)$, Eq. (38), comes from the small-momentum dependence of the kernel K_{ij}, which we now isolate in the form

$$K_{ij}^{(0)}=S_{ij}+\delta K_{ij}^{(0)} \tag{A6}$$

with

$$S_{ij}=k_{ij}(0,0)G(k,p) \tag{A7}$$

and the singular "scalar" part of the kernel

$$G(k,p)=2\pi/\Delta(-\tfrac{3}{4}p^2)[(k^2+p^2)^2-k^2p^2]^{1/2}. \tag{A8}$$

The Hilbert-Schmidt norm for the first iterate is

$$K_2^2(0)=\mathrm{Tr}K^{(0)}K^{(0)}K^{(0)\dagger}K^{(0)\dagger}. \tag{A9}$$

By power-counting in the small-momentum portions of the multiple integrals implicit in Eq. (A9), we find the norm is finite if the "singular piece"

$$K_2^s=\mathrm{Tr}\vec{S}^2\vec{S}^{\dagger 2}=(\mathrm{Tr}\vec{k}^2\vec{k}^{\dagger 2})\,\mathrm{Tr}G^2G^{\dagger 2} \tag{A10}$$

is finite.

The contribution of the scalar kernel G in Eq. (A10) is bounded by

$$\mathrm{Tr}G^2G^{\dagger 2}\leq(\mathrm{Tr}GG^\dagger)^2 \tag{A11}$$

with

$$\begin{aligned}
\mathrm{Tr}GG^\dagger&=\pi^2\int_0^{\Lambda^2}dx\,(1/x)[\Delta(-\tfrac{3}{4}x)]^{-2}\\
&\quad\times\int_0^{\Lambda^2/x}dy\,[(1+y)^2-y]^{-1},
\end{aligned} \tag{A12}$$

where Λ is a large-momentum cutoff implicit in the separation used in Eq. (A6). For small positive values of A we find a proportionality

$$\mathrm{Tr}GG^\dagger\propto 1/A. \tag{A13}$$

The trace of the matrix factor in Eq. (A10) is

$$\mathrm{Tr}\vec{k}^2\vec{k}^{\dagger 2}=(k_{11}+k_{22})^2\,\mathrm{Tr}\vec{k}\vec{k}^\dagger, \tag{A14}$$

where we have used the fact that the matrix $k_{ij}(0,0)$ has zero determinant. Hence, using Eq. (34) the term shown in Eq. (A10) is finite even for $A\to 0$ and the kernel $K^{(0)}$ has no nonzero finite accumulation point in its spectrum for $s\geq 0$ and $A\geq 0$.

This argument can be generalized to many-term separable potentials. The small-argument behavior of the function $\Delta(-x)$, governing the dimer binding, is again of the form in Eq. (33). The matrix k_{ij} again has the special form

$$k_{ij}(0,0)=a_i\,b_j, \tag{A15}$$

which assures that it has at most one nonzero eigenvalue. The strength of the singular scalar kernel in Eq. (A5) is proportional to the zero-energy on-shell t matrix, which vanishes at resonance; the corresponding statement for the many-term potential would lead to the matrix k_{ij} having only zero eigenvalues at resonance and a net finite value for the norm K_2^s.

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[1]L. H. Thomas, Phys. Rev. 47, 903 (1935).
[2]R. A. Minlos and L. D. Faddeev, Zh. Eksp. Teor. Fiz. 41, 1850 (1961) [Sov. Phys. JETP 14, 1315 (1962)]; G. A. Skornyakov and K. A. Ter-Martirosyan, Zh. Eksp. Teor. Fiz. 31, 775 (1956) [Sov. Phys. JETP 4, 648 (1957)].
[3]V. N. Efimov, Phys. Lett. B 33, 563 (1970); Yad. Fiz. 12, 1080 (1970) [Sov. J. Nucl. Phys. 12, 589 (1971)]; in *Few Body Dynamics*, edited by A. N. Mitra, I. Slaus, V. S. Bhasin, and V. K. Gupta (North-Holland, Amsterdam, 1976).
[4]R. D. Amado and J. V. Noble, Phys. Rev. D 5, 1992 (1972).
[5]For a review of information on monolayers of quantum gases, see the chapter by M. Schick and J. G. Dash, in *The Physics of Liquid and Solid Helium*, Vol. 2, edited by K. H. Bennemann and J. B. Ketterson (Wiley, New York, 1978).
[6]L. W. Bruch, Physica A 93, 95 (1978) and references contained therein.
[7]L. R. Dodd, J. Math. Phys. 11, 207 (1970).
[8]F. Cabral (unpublished). This work is included in a Ph.D. thesis to be submitted to the University of Wisconsin-Madison.
[9]The Efimov type of state apparently occurs for the ^{4}He trimer in three dimensions: T. K. Lim, K. Duffy, and W. C. Damert, Phys. Rev. Lett. 38, 341 (1977); H. S. Huber and T.K. Lim, J. Chem. Phys. 68, 1006 (1978).
[10]E. W. Schmid and H. Ziegelmann, *The Quantum Mechanical Three-Body Problem* (Pergamon, Oxford, 1974).
[11]A. G. Sitenko and V. F. Kharchenko, Usp. Fiz. Nauk 103, 469 (1971) [Sov. Phys. Usp. 14, 125 (1971)].
[12]T. K. Lim and K. Duffy, J. Chem. Phys. 68, 655 (1978).
[13]N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space* (Ungar, New York, 1961).
[14]The form factors are taken to be real; there is no increase in generality here by allowing them to become complex.
[15]A. Ahmadzadeh and J. A. Tjon, Phys. Rev. 139, B1085 (1965).
[16]M. L. Cramer, L. W. Bruch, and F. Cabral, J. Chem. Phys. 67, 1442 (1977) and references contained therein.
[17]R. D. Amado, in *Elementary Particle Physics and Scattering Theory* (Brandeis University 1967 Summer Institute), edited by M. Chrêtien and S. S. Schweber (Gordon and Breach, New York, 1970).
[18]R. D. Amado and F. C Greenwood, Phys. Rev. D 7, 257 (1973).