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Energy spectra of certain randomly-stirred fluids

C. DeDominicis* and P. C. Martin
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
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The velocity correlations of an incompressible fluid governed by the Navier-Stokes equations are studied in steady states maintained by random-white-noise stirring forces with varying spatial correlations. The asymptotic properties of the long-wavelength fluctuations are deduced by field-renormalization-group techniques. The results of Forster, Nelson, and Stephen are recovered for the random-force spectra these authors discuss, and a Kolmogorov spectrum is obtained when the force correlations have equal strength at all wave numbers, that is, when the force correlations behave as k^{-d} in d dimensions and $d > 2$. Although the derivation is valid to all orders in the anomalous dimension, it implicitly assumes that there is no crossover in operator dimensionality.

INTRODUCTION

The large-distance, long-time behavior of velocity correlations generated by the Navier-Stokes equations for various regular forcing functions has been extensively studied with renormalization-group methods by Forster, Nelson, and Stephen (FNS).[1] The purpose of this comment is to discuss the singular case of a random stirring force in which equal weight is given to all wave vectors, i.e., a force characterized by a noise correlation essentially proportional to k^{-d}. We shall show that this stirring force yields a Kolmogorov spectrum. This derivation of the Kolmogorov spectrum depends on a special noise force and does not address the central issue of why such a spectrum, or one that does not deviate greatly from it, is found in experiments on strong turbulence. Nevertheless, the model might provide a concrete starting point for quantitatively studying discrepancies from the Kolmogorov predictions, and how universally they apply.

To avoid uninteresting infrared divergences, we take a white-noise random force f whose only nonvanishing cumulant is, in momentum space,

$$\langle ff\rangle \simeq D_0 k^{4-d}(m_0^2+k^2)^{-y/2}\,; \tag{1}$$

m_0^{-1} is a stirring length (infrared cutoff). We shall focus our attention on the asymptotic domain in which

$$m_0 \ll k \ll \Lambda\,, \tag{2}$$

and will eventually let the ultraviolet cutoff Λ tend to infinity. In this limit we shall see that for any $y \leqslant 4$ (and $d > 2$), the FNS result for the energy spectral function may be generalized to

$$E(k) \simeq k^{1-2y/3}\,. \tag{3}$$

The limiting case $y=4$ yields the Kolmogorov behavior. The conclusion holds only if there is no crossover of operator dimensionality as y is varied between 0 and 4.

GENERATING FUNCTIONAL FOR NAVIER-STOKES CORRELATIONS

The Navier-Stokes equation for an incompressible fluid may be written

$$\frac{\partial}{\partial t} v_\alpha = \nu_0 \nabla^2 v_\alpha + \lambda_0 \tau_{\alpha\beta}(\vec{v}\cdot\vec{\nabla}) v_\beta + f_\alpha\,, \tag{4}$$

where $\tau_{\alpha\beta}$ is a projection operator that eliminates longitudinal components, and f_α is a noise source whose only nonvanishing cumulant is transverse and proportional to Eq. (1). The velocity correlation functions may be generated[2-4] as the Taylor coefficients[5] of the quantity $\hat{Z}(l)$, defined by

$$\hat{Z}(l) = \int Dv\, D\hat{v}\, \exp\Big(\mathcal{L}[v,\hat{v}] + \int dt\, d^d x\; l_\alpha(xt) v_\alpha(xt)\Big)\,, \tag{5}$$

with

$$\mathcal{L} = \int dt\, d^d x \left[-i\hat{v}_\alpha\left(\frac{\partial}{\partial t} - \nu_0\nabla^2\right) v_\alpha - \lambda_0 i\hat{v}_\alpha \tau_{\alpha\beta}\vec{v}\cdot\vec{\nabla} v_\beta + i\hat{v}_\alpha \langle f_\alpha f_\beta\rangle i\hat{v}_\beta\right]. \tag{6}$$

More explicitly, the unperturbed propagators are

$$\langle i\hat{v}_\alpha v_\beta\rangle_0 = [-i\omega + \nu_0 k^2]^{-1}\delta_{\alpha\beta}\,, \tag{7}$$

$$\langle v_\alpha v_\beta\rangle_0 = \frac{2D_0 k^{4-d}}{|-i\omega+\nu_0 k^2|^2}(m_0^2+k^2)^{-y/2}\tau_{\alpha\beta}(k)\,, \tag{8}$$

and

$$\tau_{\alpha\beta}(k) = \delta_{\alpha\beta} - k_\alpha k_\beta/k^2\,. \tag{9}$$

From an inspection of the perturbation series, we may conclude that the bare propagators take the form

$$\langle i\hat{v}, v\rangle = \nu_0^{-1} F_B(\omega/i\nu_0, k; g_0, \Lambda, m_0) \tag{10}$$

and

$$\langle vv\rangle \equiv C(k,\omega) = (D_0/\nu_0^2) G_B(\omega/i\nu_0, k, g_0, \Lambda, m_0)\,, \tag{11}$$

with

$$\lambda_0^2 D_0 \nu_0^{-3} \equiv g_0 \Lambda^y . \tag{12}$$

RENORMALIZED NAVIER-STOKES THEORY

The asymptotic behavior in region (2) may be obtained by standard renormalization-group techniques[6]; we let $\Lambda \to \infty$, and look for the ultraviolet behavior (with respect to m_0) of the correlation functions. For this purpose, we write Eq. (6) in terms of renormalized fields. This involves the following steps:

(i) Count powers:

$$[\hat{v}] = d - 1 + \tfrac{1}{2}y\,, \tag{13}$$

$$[v] = 1 - \tfrac{1}{2}y\,, \tag{14}$$

$$[\lambda_0] = \tfrac{1}{2}y\,. \tag{15}$$

The last equation shows that the coupling is marginal for $y=0$.

(ii) Observe that for $y=0$, Galilean invariance assures that all other couplings are irrelevant (see below).

(iii) Introduce one renormalization function for each type of vertex that can appear in $\mathcal{L}$; the standard parametrization is

$$v = Z^{1/2} v_R\,, \tag{16}$$

$$\hat{v} = Z^{1/2}\hat{v}_R/\hat{Z}\,, \tag{17}$$

$$\lambda_0 = \lambda\mu^{y/2} Z_\lambda Z^{-3/2}\,, \tag{18}$$

$$\nu_0 = \nu Z_\nu\,, \tag{19}$$

$$D_0 = D Z_D\,, \tag{20}$$

where μ is an arbitrary wave vector (we could use m_0, but it is less confusing to take $\mu \gg m_0$). Given that only five independent primitive divergences exist, we may take

$$\hat{Z} = 1\,. \tag{21}$$

(iv) Observe that the ω derivative of $\langle i\hat{v}v\rangle$ does not diverge, and therefore that

$$Z = \hat{Z} = 1\,; \tag{22}$$

the k^2 derivative of $\langle \hat{v}v\rangle$ diverges, but Z_ν accounts for this divergence.

(v) Observe that $\langle vv\rangle$ is logarithmically divergent for $d=2$ (this divergence is accounted for by Z_D) and convergent for $d>2$.

(vi) Observe, finally, that the λ_0 coupling (imposed by Galilean invariance) leads to the Ward identity[1,7]

$$\langle A(t)B(x_1t_1)C(x_2t_2)\rangle = \vec{h}\cdot\vec{\nabla}_1\langle B(x_1t_1)C(x_2t_2)\rangle \quad t_1 < t < t_2$$

$$= 0\,, \quad \text{otherwise}\,, \tag{23}$$

where

$$A(t) = \int d^dx\, \hat{v}_\alpha(xt)\tau_{\alpha\beta}(\vec{h}\cdot\vec{\nabla})v_\beta(xt)\,, \tag{24}$$

and B and C are operators constructed from v and $\hat{v}$. Equation (23) tells us in particular that the divergence that appears in the truncated vertex $\langle \hat{v}_\alpha \tau_{\alpha\beta}(\vec{v}\cdot\nabla)v_\beta\rangle$ is the same as the one in $(\partial/\partial i\omega)$ $\langle i\hat{v}v\rangle$, or, in other words, that

$$Z_\lambda = 1\,. \tag{25}$$

ASYMPTOTIC BEHAVIOR

The renormalized form of the correlation function (11),

$$C(k,\omega) = (D/\nu^2) G_R(\omega/i\nu, k, g, \mu, m_0)\,, \tag{26}$$

satisfies a renormalization group equation

$$\left(\mu\frac{\partial}{\partial\mu} + W_g\frac{\partial}{\partial g} + 2\eta_\nu - \eta_D + \eta_\nu\omega\frac{\partial}{\partial\omega}\right)C(k,\omega) = 0\,, \tag{27}$$

with anomalous exponents given by the derivatives

$$\eta_j = \mu\frac{\partial}{\partial\mu}\ln Z_j\big|_B \quad (j=\nu, D) \tag{28}$$

for fixed values of the bare parameters. The Wilson function

$$W_g = \mu\frac{dg}{d\mu}\bigg|_B \tag{29}$$

is obtained by differentiating Eq. (12) and using Eqs. (18)–(25) and (28):

$$W_g = -g(y + \eta_D - 3\eta_\nu)\,. \tag{30}$$

Solving Eq. (27) in the asymptotic region (2),

$$m_0^2 \ll k^2\,, \quad \omega \ll \nu\mu^2\,, \tag{31}$$

we obtain the scaling form

$$C(k\omega) \simeq k^{-y-d+2\eta_\nu-\eta_D} G_R\left[\left(\frac{k}{\mu}\right)^{\eta_\nu-2}\frac{\omega}{i\nu}, \mu, g^*, \mu, 0\right]. \tag{32}$$

Frequency integration leads to the equal-time correlation function

$$C(k) \simeq k^{-y-d+2+\eta_\nu-\eta_D}. \tag{33}$$

The corresponding spectral function is

$$E(k) \simeq k^{-y+1+\eta_\nu-\eta_D}. \tag{34}$$

The values η_ν and η_D are to be computed at the infrared stable fixed point of Eq. (30); this excludes $g^*=0$. Corrections in k/Λ to the scaling form (32) are governed by the exponent

$$\omega_\nu = g\frac{\partial}{\partial g}(3\eta_\nu - \eta_D)\big|_{g=g^*}. \tag{35}$$

The above equations include the results of FNS for the two models discussed in their paper: In model A, $y=2-d$, the fluctuation dissipation relation holds, and $\eta_D=\eta_\nu$. From Eq. (30) we then obtain (for $d<2$),

$$\eta_\nu = \tfrac{1}{2}y \text{ and } \omega_\nu \simeq y \text{ for small } y. \tag{36}$$

In model B, $y=4-d$, $Z_D=1$, and $\eta_D=0$. We therefore have

$$\eta_\nu = \tfrac{1}{3}y \text{ and } \omega_\nu \simeq y \text{ for small } y. \tag{37}$$

INFRARED BEHAVIOR FOR SINGULAR NOISE CORRELATIONS

In the region (31), when y is small and $d>2$, we find

$$\eta_D = 0 \text{ to all orders in } y, \tag{38}$$

$$\eta_\nu = y/3, \tag{39}$$

and

$$E(k) \simeq k^{-2y/3+1}. \tag{40}$$

These values do not depend upon how small the parameter y is, since the Ward identity that guarantees Eq. (30) is an exact result. As y grows and the noise correlations given by k^{4-d-y} become more singular, the dimensionality of the field operators changes.

In particular, as $y=2$, the dimension of v approaches zero and the counter terms that were irrelevant threaten to become important. To extend the arguments to larger values of y, we call upon the limitations imposed by Galilean invariance. These restrict the possible operators to combinations of (i) time-independent products of velocity operators

$$f_1 A_{\alpha\alpha'\beta\beta'\gamma\gamma'}\ldots = \nabla_\alpha \hat{v}_{\alpha'} \nabla_\beta v_{\beta'} \nabla_\gamma v_{\gamma'}\ldots, \tag{41}$$

where A is a dimensionless tensor and the dimension of f_1 is

$$[f_1] = (p-1)(\tfrac{1}{2}y-2); \tag{42}$$

and (ii) time-dependent products

$$f_2 B_{\alpha\beta\gamma}\ldots \hat{v}_\alpha\left(\frac{\partial}{\partial t} - \lambda_0 \vec{v}\cdot\vec{\nabla}\right)v_\beta\left(\frac{\partial}{\partial t} - \lambda_0\vec{v}\cdot\vec{\nabla}\right)v_\gamma \ldots, \tag{43}$$

where

$$[f_2] = (p-1)(\tfrac{1}{2}y-3). \tag{44}$$

These combinations all remain irrelevant for values of y that are less then 4. When $y=4$, the mode-coupling term $\lambda_0 \hat{v}_\alpha \vec{v}\cdot\vec{\nabla}v_\alpha$ behaves as Λ^2 ($[\lambda_0]=2$), all counter terms of form (41) become marginal, and the remaining terms are irrelevant. Thus the Kolmogorov behavior for the spectral function

$$E_K(k) \simeq k^{-5/3}$$

is approached as y approaches four from below, for the region which is ultraviolet with respect to m_0 and infrared with respect to Λ. This estimate for y is based on the assumption that there is no crossover in the full (naive plus anomalous) dimensionality of the field operators.

At the Oji Seminar, Kyoto, July 1978, we learned that the result reported in this paper had also been derived recently by F. Tanaka and T. Nakano. They also use the techniques of Refs. 2–4 and deduce the Ward identity described in step (vi).

Although the result has been derived by two groups, some questions remain. Specifically, for $y>3$, the individual lowest-order terms in a *self-consistent* perturbation expansion are infrared divergent. The divergences cancel in the equation for the energy transfer. Whether divergences in individual terms in such an equation signify that a new operator becomes relevant for $y \geq 3$ (implying a crossover in operator dimensionality at $y=3$) and invalidates our conclusions for $3 \leq y < 4$, requires further study.

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*Present address: Service de Physique Theorique, C.E.N. de Saclay, BP2, 91190 Gif-sur Yvette, France.

[1]D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A 16, 732 (1977). We also call the reader's attention to J. D. Fournier and U. Frisch, Phys. Rev. A 17, 747 (1978), and to M. Lücke, Phys. Rev. A 18, 282 (1978).

[2]P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev.

A 8, 423 (1973).
[3]C. DeDominicis, J. Phys. (Paris) C 1, 247 (1976); H. K. Janssen, Z. Phys. B 23, 377 (1976); R. Bausch, H. K. Janssen, and H. Wagner, Z. Phys. B 24, 113 (1976).
[4]C. DeDominicis and L. Peliti, Phys. Rev. Lett. 38, 505 (1977); Phys. Rev. B 18, 353 (1978).
[5]We omit a Jacobian whose only role at this stage is to suppress tadpole terms.
[6]E. Brezin, J. C. LeGuillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1975), Vol. VI.
[7]B. I. Halperin, P. C. Hohenberg, and E. D. Siggia, Phys. Rev. B 13, 1299 (1976).