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Ground state of a spin-1/2 charged particle in a two-dimensional magnetic field

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We prove that a spin-1/2 charged particle moving in a plane under the influence of a perpendicular magnetic field has $(N-1)$ zero-energy states, where N is the closest integer to the total flux in units of the flux quantum. The $(N-1)$ independent wave functions are calculated explicitly. The result, which is extremely simple to prove, is an example of the Atiyah-Singer index theorem when applied to the Euclidean two-dimensional Dirac equation.

The movement of a charged spin-$\frac{1}{2}$ particle under the influence of homogeneous magnetic field is a well-known classic solvable problem.[1] This is not true for a general (x,y)-dependent field which points in the z direction. However, we have come across a remarkably simple result which seems to be unknown in this context—the ground state is exactly calculable and possesses a degeneracy related to the total flux. The results will be seen to hold for the nonrelativistic (Pauli) as well as the relativistic (Dirac) case which are governed by essentially the same equation.

Consider the Pauli Hamiltonian for a charged particle in a magnetic field, specialized to the case where the field points in the z direction and depends on x and y only ($\hbar = c = 2m = 1$):

$$H = (\Pi_\perp \sigma_\perp)^2 = \Pi_\perp^2 - eB\sigma_z \,, \tag{1}$$

where 2Π is the transverse-velocity operator:

$$\Pi_\perp = p_\perp - eA_\perp \tag{2}$$

and B is the magnetic field:

$$\frac{1}{ie}[\Pi_x, \Pi_y] = B(x,y) = \partial_x A_y(x,y) - \partial_y A_x(x,y)\,. \tag{3}$$

The vector potential can and will be chosen to be divergenceless:

$$\partial_x A_x + \partial_y A_y = 0\,. \tag{4}$$

We also define the total magnetic flux and relate it to a positive integer N:

$$\int dx\,dy\,B(x,y) \equiv \Phi = \frac{2\pi}{e}(N+\epsilon), \quad 0<\epsilon<1\,. \tag{5}$$

Clearly Φ may always be chosen to be positive by an appropriate redefinition of the z axis.

We now prove the following two theorems: *Theorem* 1: If $N+\epsilon>1$ the Hamiltonian H [Eq. (1)] has exactly $N-1$ zero-energy normalizable eigenstates whose spin has the same sign as the flux [positive according to the convention of Eq. (5)]. *Theorem* 2: All nonzero energy eigenstate are degenerate with respect to spin flip.

In order to prove theorem 1 observe that H may be rewritten as a product of two conjugate first-order differential operators:

$$H = \Pi_\perp^2 - eB\sigma_z = (\Pi_x - i\sigma_z\Pi_y(\Pi_x + i\sigma_z\Pi_y)\,, \tag{6}$$

where Eq. (3) has been used to evaluate the commutator $[\Pi_x \Pi_y]$. If ψ is a zero eigenstate of H with $\sigma_z = \sigma$ ($=\pm 1$) it must therefore be annihilated by $\Pi_x + i\sigma\Pi_y$:

$$[(1/i)(\partial_x + e\sigma A_y) + \sigma(\partial_y - e\sigma A_x)]\psi_\sigma = 0\,. \tag{7}$$

Note now that Eqs. (3) and (4) which define the magnetic field and the gauge condition imply that $A_\perp$ is a two-dimensional curl whose "potential" satisfies the Laplace equation with $B(xy)$ as a source:

$$A_x = -\partial_y\phi\,, \quad A_y = \partial_x\phi\,; \tag{8}$$

$$(\partial_x^2 + \partial_y^2)\phi = B\,. \tag{9}$$

Substitute now in Eq. (7),

$$\psi_\sigma = e^{-e\sigma\phi} f_\sigma\,, \tag{10}$$

which leads to

$$(\partial_x + i\sigma\partial_y) f_\sigma(x,y) = 0\,. \tag{11}$$

Equation (11) states that f_σ is an entire function of $x + i\sigma y$. Now, the Green's function of the two-dimensional Laplacian is $(2\pi)^{-1} \ln r/r_0$ so that as $r\to\infty$ the exponential in Eq. (10) behaves as

$$e^{-e\sigma\phi} \underset{r\to\infty}{\simeq} \left(\frac{r_0}{r}\right)^{e\sigma\Phi/2\pi}, \tag{12}$$

where Φ is the total flux. Since an entire function cannot go to zero in all directions at infinity, a necessary condition for ψ_σ to be normalizable is

$$\sigma\Phi > 0\,. \tag{13}$$

Moreover, in order that ψ_σ be square-integrable we need

$$\lim_{r\to\infty} r^2 |f_\sigma|^2 \left(\frac{r_0}{r}\right)^{2e\sigma\Phi/2\pi} = 0\,. \tag{14}$$

Hence, f_σ must be a polynomial whose degree is no larger than N-1. The N-1 independent (though not orthogonal in general) zero modes are thus (for $\Phi > 0$)

$$\psi_j = (x+iy)^j e^{-e\phi}, \quad j = 0, 1, \ldots, N-1, \tag{15}$$

where N is defined by Eq. (5). Equations (13)–(15) obviously constitute theorem 1. Theorem 2 is even simpler to prove and follows almost trivially from the decomposition Eq. (6). In fact, if χ_σ is an eigenstate of H then clearly a *degenerate* eigenstate with an opposite sign of σ_z is given by

$$\chi_{-\sigma} = (\Pi_x + i\sigma\Pi_y)\chi_\sigma. \tag{16}$$

It should be remarked at this stage that our two theorems are well known in another context. The operator $(\Pi_\perp \sigma_\perp)$ whose square is the Pauli Hamiltonian is the Euclidean (imaginary time) continuation of the one-space–one-time massless Dirac operator. The degeneracy (16) is then due to γ_5 (Chiral) invariance, while the number of zero-energy states is governed by the Atiyah-Singer index theorem.[2,3] We also observe that the two theorems hold for the Dirac Hamiltonian. In fact, it is readily verified that the upper and lower components of the Dirac spinor (in a representation where γ_0 is diagonal) satisfy

$$(\sigma\Pi)^2 U = (E^2 - m^2)U, \tag{17}$$

while U_{up} and U_{down} are connected by

$$\begin{aligned}(E+m)U_{\text{down}} &= \sigma\Pi\, U_{\text{up}}, \\ (E-m)U_{\text{up}} &= \sigma\Pi\, U_{\text{down}}.\end{aligned} \tag{18}$$

Hence, the positive (negative) energy solutions with $E^2 = m^2$ are given by U_{up} (U_{down}) equal to ψ_j [Eq. (15)] and U_{down} (U_{up}) = 0.

We end by indicating a possible speculative application of the zero modes. Consider a two-dimensional low-temperature electron gas in circumstances where the interelectron Coulomb potential may be neglected. Imagine further a distribution of fluxons and antifluxons such that $(N_+ - N_-) \neq 0$. Our theorem then states that $(N_+ - N_-)q/e - 1$ of the electrons [where $q = 2\pi$ (flux quantum)$^{-1}$] will "condense" into the available zero modes, thus decreasing the Fermi level. We have not investigated this idea in any detail so that its relevance is purely speculative at the moment.

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[1] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, London, 1958).

[2] M. F. Atiyah and I. M. Singer, Bull. Am. Meteorol. Soc. 69, 422 (1963).

[3] The index theorem relates the number of zero modes of a particle moving in an external gauge field to the topological winding number (in our case the line integral of $A_\perp$ at infinity). A well-known application occurs in four-dimensional Yang-Mills theory. [For example, L. S. Brown, R. D. Carlitz, and C. Lee, Phys. Rev. D 16, 417 (1977)].