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Kolmogorov entropy of a dynamical system with an increasing number of degrees of freedom

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Lyapunov characteristic numbers are used to estimate numerically the Kolmogorov entropy of an isolated one-dimensional self-gravitating system consisting of N plane parallel sheets with uniform density. It appears that the Kolmogorov entropy increases linearly when the number of degrees of freedom is greater than or equal to 2.

I. INTRODUCTION

In recent years much numerical work has been devoted to the investigation of the ergodic properties of classical dynamical systems.[1] Namely, while the extreme cases of near-integrable and of ergodic systems are presently at least partially understood in a rigorous mathematical context,[2] almost nothing is known theoretically for many models of physical interest, which are in fact very far from both integrability and ergodicity.

Numerical experiments, however, strongly indicate that these extreme situations are at least good "paradigms" for understanding the behavior of other dynamic systems. Indeed, it is often found that the phase space of a system decomposes (at least roughly) into two invariant components: an "ordered" region with integrablelike behavior and a "stochastic" region, with ergodiclike behavior.

On the one hand, relatively simple techniques are available for the study of systems with two degrees of freedom: namely, for such systems it is not too difficult to compute[3,4] an extensive quantity, i.e., the relative measure μ_s of the stochastic region, and just one intensive quantity, i.e., the maximal Lyapunov characteristic number[5] of the flow. Then, using Piesin's formula,[6] one can estimate[4] the Kolmogorov entropy, which is certainly a quantity very relevant to ergodic theory.[2]

On the other hand, for systems with more than two degrees of freedom the situation is not so simple. Indeed, computing μ_s is not straightforward and moreover, to estimate entropy it is necessary to compute all the positive Lyapunov characteristic numbers besides the maximal one.

In the present paper we are concerned with a system consisting of N parallel plane sheets, coupled by gravitational potential, i.e., a system with N degrees of freedom. This model, which is of astrophysical interest, has already been studied in Refs. 7 and 8. Froeschlé and Scheidecker have found[9] that, with increasing N, the relative measure μ_s tends rapidly to one. We now complete the study of the system, by computing all its Lyapunov characteristic numbers (LCN) and its entropy. For the LCN we use the very recent method introduced in Ref. 10.

In Sec. II we describe the model and recall some known results. In Sec. III, after recalling the necessary mathematical notions, we briefly describe the numerical technique. In Sec. IV we present and discuss our results.

II. MODEL

Let us consider as a model problem a one-dimensional dynamic system consisting of N plane parallel sheets of equal mass m per unit area. These sheets are of infinite extent and move perpendicularly to their plane along the x axis under the influence of their mutual gravitation. Their positions and velocities are indicated by $x_1, \ldots, x_n$ and $u_1, \ldots, u_n$, respectively. The sheets are allowed to pass freely through each other when they cross.

The system is described by the Hamiltonian

$$H(x,u)=\tfrac{1}{2}m\sum_{i=1}^{N}u_i^2+2\pi Gm^2\sum_{\substack{i,j=1\\ i>j}}|x_j-x_i|,$$
$$x=(x_1,\ldots,x_n),\quad u=(u_1,\ldots,u_n), \tag{1}$$

where G is the gravitational constant. It turns out that two uniform integrals exist:

$$\overline{u}(x,u)=\frac{1}{N}\sum_{i=1}^{N}u_i=U, \tag{2}$$

$$H(x,u)=E; \tag{3}$$

i.e., the velocity U of the center of mass and the total energy E are constant. By a trivial change of the frame of reference one can always take the center mass at rest in the origin, i.e.,

$$\overline{x}=\frac{1}{N}\sum_{i=1}^{N}x_i=0\ .$$

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TABLE I. Relative measure μ_s of the stochastic region, entropy density ρ, and Kolmogorov entropy h as functions of the number N of sheets.

N	μ_s	ρ	h
2	(integrable)		
3	0.04	0.02	0.0008
4	0.86	0.04	0.034
5	0.99	0.073	0.072
6	1	0.108	0.108
7	1	0.13	0.13
8	1	0.18	0.18
10	1	0.24	0.24

For $N=2$ the model is integrable. It has been used for the study of the stochasticity of dynamic systems when the number of degrees of freedom increases.[9]

From the particular form of the potential it follows that the motion on any surface of constant energy reproduces on scale the motion on any other surface of constant energy. Namely, let $z=(x,u)$ and ϕ^t denote the Hamiltonian flow, i.e., $\phi^t(z)$ is the trajectory with initial point z. Here Γ_E is the surface of constant energy E, which turns out to be compact. A normalized measure μ_L on Γ_E (Liouville measure) which is absolutely continuous with respect to the Lebesque measure on it is preserved by ϕ^t. The application Ψ_λ: $\Gamma_E \to \Gamma_{\lambda^2 E}$ given by $\Psi_\lambda(x,u)=(\lambda^2 x, \lambda u)$ satisfies $\phi^{\lambda t}\circ\Psi_\lambda=\Psi_\lambda\circ\phi^t$.

The character of the trajectories is then not affected by Ψ_λ. In particular, Ψ_λ is measure preserving, so that the relative measure μ_s of the stochastic region is the same on Γ_E and $\Gamma_{\lambda^2 E}$, i.e., it does not depend on energy. This allows us to obtain an estimate of μ_s by means of a Monte Carlo procedure.[9] It was found that μ_s increases very rapidly with the number N of sheets, as shown by Table I.

To decide whether a point belongs to the stochastic region, the simple qualitative criterion of divergence of nearby trajectories was used. A precise and quantitative definition of stochasticity can be given by introducing the Lyapunov characteristic numbers, to which the next section is devoted.

III. LYAPUNOV CHARACTERISTIC NUMBERS

A. Divergence of trajectories and LCN's

It is well known that nearby trajectories of integrable systems diverge linearly. It has been made clear by many numerical experiments[1,3] that the stochastic region is characterized by exponential-like divergence of trajectories. To give a precise quantitative definition of exponential divergence, and thus of stochasticity, one is naturally led to consider the spectral properties of a linear operator. That is, let M be an n-dimensional compact differentiable manifold, μ a normalized measure on it, and ϕ^t a measure-preserving flow, i.e., a one-parameter group of measure-preserving diffeomorphisms $M\to M$ with composition law $\Phi^{t+s}=\phi^t\circ\phi^s$. In the framework of ergodic theory the collection (M,μ,ϕ^t) is called a classical dynamic system.[2] Let $z\in M$ and denote by T_zM the space tangent to M at z and by $D\phi_z^t$ the tangent (or linearized) mapping, which maps T_zM onto $T_{\phi^t(z)}M$.

In the particular case of a periodic orbit of period t_0, $D\phi_z^{t_0}$ is a mapping of T_zM onto itself. Suppose there are n independent eigenvectors $e_1,\ldots,e_n$, with eigenvalues $\lambda_1,\ldots,\lambda_n$, with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. Let $\chi_i=t_0^{-1}\ln|\lambda_i|$. Then one clearly has

$$||D\phi_z^{kt_0}(e_i)||/||e_i||=e^{\chi_i k t_0},\quad i=1,\ldots,n. \tag{4}$$

where $||\ \ ||$ denotes the Euclidean norm on T_zM. For a vector w with a nonvanishing component along e_1 it follows asymptotically for large t that

$$||D\phi_z^t(w)||/||w|| \simeq e^{\chi_1 t}, \tag{5}$$

in the sense that

$$\lim_{t\to\infty} t^{-1}\ln||D\phi_z^t(w)||/||w||=\chi_1. \tag{6}$$

One has then asymptotic exponential divergence for almost all tangent vectors as far as $\chi_1>0$. The periodic orbit is in this case unstable, and with an improper language usage one frequently says that nearby orbits exponentially diverge from it.

The problem is how to generalize this construction to nonperiodic orbits. Suppose that one is able to identify naturally the tangent spaces at different points of M; then $D\phi_z^t$ becomes a linear operator on the n-dimensional Euclidean space and its asymptotic spectral properties can be studied. One idea is to study directly the behavior of the eigenvalues $\lambda_1^t,\ldots,\lambda_n^t$ of $D\phi_z^t$, in order to see whether $\lim_{t\to\infty}t^{-1}\ln|\lambda_i^t|$ exists. This idea is the basis of the numerical computations of Ref. 11. However, the existence in general of the above limit is not theoretically guaranteed. Nevertheless, one can prove under rather general hypotheses that the limit

$$\lim_{t\to\infty} t^{-1}\ln||D\phi_z^t(w)||/||w||=\chi(z,w) \tag{7}$$

exists for almost all initial data z and all nonzero vectors $w\in T_zM$.[5]

The χ's are called Lyapunov characteristic numbers and allow us to extend the quantitative definition of exponential divergence to the case of nonperiodic orbits.

B. Theoretical results on LCN's

Lyapunov characteristic numbers can be defined for rather general flows (or mappings) on compact manifolds. For simplicity we shall restrict ourselves to the case of our Hamiltonian system of N degrees of freedom. Let Γ_E be a compact surface of constant energy which does not contain equilibrium points for $E>0$; Γ_E has dimension $n=2N-1$.

From the general results of Ref. 5 one has in our case the following theorems:

(a) For almost all $z=(x,u)\in\Gamma_E$ and all nonzero vectors $w\in T_z\Gamma_E$, the limit

$$\lim_{t\to\infty} t^{-1}\ln\|D\phi_z^t(w)\| = \chi(z,w) \tag{8}$$

exists and is finite (M being compact, all equivalent norms give the same limit). Here $\chi(z,w)$ depends only on the orbit and on the direction of w, in the sense that $\chi(\phi^s(z),D\phi_z^s(cw))=\chi(z,w)$ for any s and any $c\neq 0$.

(b) As w varies in $T_z\Gamma_E$, $\chi(z,w)$ takes at most n distinct values $\chi_1(z),\ldots,\chi_n(z)$, which we suppose ordered by decreasing values. There exists at least one basis $(e_1\ldots e_n)$ of $T_z\Gamma_E$ such that

$$\lim_{t\to\infty} t^{-1}\ln\|D\phi_z^t(e_i)\| = \chi_i(z). \tag{9}$$

For any vector $w=\sum_{i=k}^n c_ie_i$, with $c_k\neq 0$, we have $\chi(z,w)=\chi_k(z)$. Thus for almost all vectors $\chi(z,w)=\chi_1(z)$.

(c) Let $w^1,\ldots,w^k$, $1\leq k\leq n$, be the parallelepiped generated by the linearly independent vectors $w^1,\ldots,w^k$ belonging to $T_z\Gamma_E$. We denote the corresponding k-dimensional volume by $V^k(w_1,\ldots w_k)$ (as in proposition (a), the metric is irrelevant). The limit

$$\lim_{t\to\infty} t^{-1}\ln V^k(D\phi_z^t(w^1),\ldots,D\phi_z^t(w^k)) = \chi^k(z,w^1,\ldots,w^k) \tag{10}$$

exists and is finite for almost all $z\in\Gamma_E$. The χ^k are LCN's of order k (previously defined LCN's were of order 1).

(d) For almost all vectors $w^1,\ldots,w^k$ belonging to $T_z\Gamma_E$, one has

$$\chi^k(z,w^1,\ldots,w^k)=\sum_{i=1}^k \chi_i(z),\quad k=1,\ldots,n. \tag{11}$$

This relation is formulated explicitly in Ref. 10, but implicitly contained in Ref. 5. It is at the basis of the computational technique that we shall shortly recall in Sec. III D.

(e) As our system is Hamiltonian, we have in addition[10]

$$\chi_i(z)=-\chi_{n-i+1}(z),\quad i=1,\ldots,n.$$

The spectrum of LCN's at point z is then

$$\{\chi_1(z),\ldots,\chi_{N-1}(z),0,-\chi_{N-1}(z),\ldots,-\chi_1(z)\}.$$

One has $\chi(z,w)=0$ for w in the direction of the flow, as Γ_E is compact and does not contain equilibrium points.

C. LCN's and Kolmogorov entropy

Piesin's formula gives the precise connection between Kolmogorov entropy and LCN's. Properly speaking, it is not guaranteed that this formula can be applied to our flow, because it is not sufficiently differentiable. The applicability of Piesin's formula has then to be considered as an assumption.

Denote by $\rho(z)$ the sum of all positive LCN's, i.e.,

$$\rho(z)=\sum_{i=1}^{N-1}\chi_i(z)$$

in our case. Piesin's formula states that one has

$$h(E)=\int_{\Gamma_E}\rho(z)\,d\mu_L, \tag{12}$$

where $h(E)$ denotes the Kolmogorov entropy of the flow on Γ_E. The quantity $\rho(z)$ consequently defines a density of Kolmogorov entropy.

For our model the E dependence of h is easily worked out. Namely, the linear mapping ψ_λ: $\Gamma_E\to\Gamma_{\lambda^2E}$ (Sec. II) induces a mapping

$$D\psi_\lambda:\ T_z\Gamma_E\to T_{\psi_\lambda(z)}\Gamma_{\lambda^2E},$$

such that

$$D\phi^{\lambda t}_{\psi_\lambda}(z)\circ D\psi_\lambda = D\psi_\lambda\circ D\phi_z^t.$$

When the new norm $\|w\|_\lambda=\|D\psi_\lambda(w)\|$ is introduced, it easily follows from the definition of LCN's that

$$\chi(z,w)=\lambda\chi(\psi_\lambda(z),D\psi_\lambda(w)).$$

As ψ_λ is measure preserving, one has finally

$$h(E)=\lambda h(\lambda^2E). \tag{13}$$

This equality also follows from the definition of entropy, if a suitable correspondence of partitions on the different energy surfaces is made.

Before explaining the numerical technique used to compute the LCN's, we must make one more theoretical remark. As we shall see in Sec. III D, practically one does not work on the restriction of ϕ^t to Γ_E, but on the whole $2N$-dimensional phase space. Precisely, let Ω be a region of phase space limited by two energy surfaces. Lebesgue measure on Ω is well known to be preserved by ϕ^t. Consider the $2N$-dimensional tangent space $T_z\Omega$, of which $T_z\Gamma_E$ is a subspace, and the linear mapping $D\phi_z^t$: $T_z\Omega \to T_{\phi(z)}\Omega$. The previously defined tangent mapping was a restriction of $D\phi_z^t$ on $T_z\Gamma_E$. Theorems analogous to those given in Sec. III B hold, of course, with n replaced by $\hat{n}=n+1=2N$. Indicating LCN's by $\hat{\chi}_1, \ldots, \hat{\chi}_{2N}$, we can write their spectrum

$$\{\hat{\chi}_1(z), \ldots, \hat{\chi}_{N-1}(z), 0, 0, -\hat{\chi}_{N-1}(z), \ldots, -\hat{\chi}_1(z)\}.$$

Moreover, one has $\hat{\chi}_i(z) \doteq \chi_i(z)$, $i=1,\ldots,N-1$.

D. Numerical technique for computing LCN's

The general computational method is explained in Ref. 10. Previous computations of the largest LCN's χ_1 can be found in Ref. 4. In principle the LCN's of any order k could be obtained by choosing randomly k vectors in $T_z\Omega$ and applying definition (10). Practically, naive application of the definition is not possible, because in general, in the stochastic region, the vectors become too large and the angles between their directions too small to allow a numerical computation of volumes. The procedure which follows overcomes these difficulties.

Choose $w^1, \ldots, w^k$ orthonormal and fix at not-too-large time τ. The idea is to replace, at regular time intervals τ, the evolved vectors by new orthonormal vectors, using the Gram-Smith procedure. Precisely, denoting $v_0^i = w^i$, $i=1,\ldots,k$, one defines and computes recursively

$$\begin{aligned} \tilde{v}_l^i &= D\phi^\tau_{\phi^{(l-1)\tau}(z)}(v_{l-1}^i) \\ \alpha_l^i &= \|(\tilde{v}_l^i)_\perp\|, \\ v_l^i &= (\tilde{v}_l^i)_\perp/\alpha_l^i , \end{aligned} \tag{14}$$

where $(\tilde{v}_l^i)_\perp$ stands for the component of $\tilde{v}_l^i$ orthogonal to all the (already orthonormal) v_l^j with $j<i$, i.e.,

$$\begin{aligned} (\tilde{v}_l^i)_\perp &= \tilde{v}_l^i, \quad i=1, \\ (\tilde{v}_l^i)_\perp &= \tilde{v}_l^i - \sum_{j=1}^{i-1} \langle v_l^j, \tilde{v}_l^i \rangle v_l^j, \quad i>1, \end{aligned} \tag{15}$$

where $\langle\ \rangle$ is the Euclidean scalar product on $T_z\Omega$. It is then not difficult to prove, using the linearity of $D\phi_z^t$ and relation (11), that one has

$$\hat{\chi}_i(z) = \lim_{L\to\infty} \frac{1}{L\tau} \sum_{l=1}^{L} \ln\alpha_l^i. \tag{16}$$

IV. RESULTS

A. Numerical integration

The particular form of the Hamiltonian allows us to compute the trajectory by an "exact" numerical method (i.e., only approximation errors are present). Each sheet has a constant acceleration between two crossings. The times necessary for crossings of neighboring sheets are computed by solving ordinary second-degree equations. Using the shortest of these times we compute new positions and velocities of all sheets.

Once one knows $\phi^t(z)$, it is not difficult to compute $D\phi_z^t(w)$ by an exact numerical method. So doing, we let the system have a crossing at time t_1 in $\phi^{t_1}(z)$, and let t_2 be the time of the next crossing. As $\phi^t(z)$ is nondifferentiable with respect to z at the crossing times, $D\phi_z^t(w)$ is not continuous at t_1 and t_2. Denote by $t_1^\pm$ and $t_2^\pm$ the instants immediately before and after crossing. Let ξ_i and η_i, $i=1,\ldots,N$, be the components of w in the system of coordinates naturally induced on $T_z\Omega$ by coordinates (x,u) in Ω. If at t_2 the rth and sth sheets are crossing each other, it follows that

(a) between t_1^+ and t_2^-,

$$\xi_i(t_2^-) = (t_2-t_1)\eta_i(t_1^+), \quad \eta_i(t_2^-) = \eta_i(t_1^+), \tag{17}$$

for $i=1,\ldots,N$, and

(b) between t_2^- and t_2^+, for $i=1,\ldots,N; i\neq r,s$,

$$\begin{aligned} &\xi_i(t_2^+) = \eta_i(t_2^-), \quad \eta_i(t_2^+) = \eta_i(t_2^-), \\ &\eta_r(t_2^+) = \eta_r(t_2^-) = 4\pi Gm(\xi_r - \xi_s)/|u_r-u_s|, \\ &\eta_s(t_2^+) = \eta_s(t_2^-) + 4\pi Gm(\xi_r - \xi_s)/|u_r-u_s|. \end{aligned} \tag{18}$$

The above equations can be obtained by applying the definition of tangent vector and tangent mapping. A quicker way is the following. In general, from the differential equations

$$\dot{x}_i = u_i, \quad \dot{u}_i = \frac{1}{m}\frac{\partial V(x)}{\partial x_i}, \quad i=1,\ldots,N, \tag{19}$$

the so-called variational equation is easily deduced for w:

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} 0 & I \\ A(x) & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{20}$$

where I is the identity and $A(x)$ is the matrix given by

$$A_{ij}(x) = -\frac{1}{m}\frac{\partial^2 V(x)}{\partial x_i \partial x_j}.$$

For our model one has

$$A_{ij} = -4\pi Gm\delta(x_i - x_j) \quad \text{for } i \neq j,$$

$$A_{ii} = 4\pi Gm \sum_{\substack{j=1 \\ j\neq 1}}^{N} \delta(x_i - x_j), \tag{21}$$

where $\delta(x)$ is Dirac's function. It follows that

$$\dot{\xi}_i = \eta_i, \quad \dot{\eta}_i = 4\pi Gm \sum_{\substack{j=1 \\ j\neq i}}^{N} (\xi_i - \xi_j)\delta(x_i - x_j). \tag{22}$$

Equations (17) and (18) are then trivially obtained.

From Eqs. (17) and (18) it follows that the spectrum of the LCN's contains two additional zeros. Namely, the vectors $w=(\xi,\eta)$, with $\xi_i = a$ and $n_i = b$, form a two-dimensional subspace (to which the direction of the flow does not belong) for which one has $\chi(z,w)=0$. The spectrum of the LCN's is then

$$\{\hat{\chi}_1(z), \ldots, \hat{\chi}_{N-2}(z), 0, 0, 0, 0, -\hat{\chi}_{N-2}(z), \ldots, -\hat{\chi}_1(z)\}.$$

This property is common to all systems conserving momentum.

B. Results

As in Ref. 9, our units are such that the total mass Nm is equal to unity and $4\pi G$ is equal to unity. Practically, we found it convenient to orthonormalize the vectors not at fixed intervals of time, but after a fixed number of crossings. We set it typically equal to 5, but this value was largely irrelevant. Having fixed the initial point z and k tangent vectors, we computed the quantities

$$\gamma_i(t,z) = \frac{1}{t}\sum_{l=1}^{L} \ln\alpha_l^i, \quad i=1,\ldots,k \leq 2N, \tag{23}$$

where L is the number of orthonormalizations performed up to time t. With increasing time, these quantities approach more or less well-defined limit values, which we identified with the LCN's $\chi_i(z)$. As expected, by making different choices of the initial vectors, we obtained a difference in the values of each $\gamma_i(t,z)$ which decreases rather rapidly with time (like t^{-1}). Also the property $\hat{\chi}_i(z) = -\hat{\chi}_{n-i+1}(z)$ was very well satisfied; precisely, we always found that the sum $\gamma_i(z,t)+\gamma_{n-i+1}(z,t)$ decreases regularly with time (as t^{-1}). In the following we shall then consider only the first N LCN's. The qualitative results of Ref. 9, from which our estimate of μ_s is taken, have been fully confirmed by the computation of the LCN's.

Figure 1 shows on a log-log scale the behavior of the γ_i's as a function of time for $N=4$ and an initial point taken in the ordered region. All the γ_i's appear to be decreasing functions of time (the two largest are not too different from t^{-1}), and

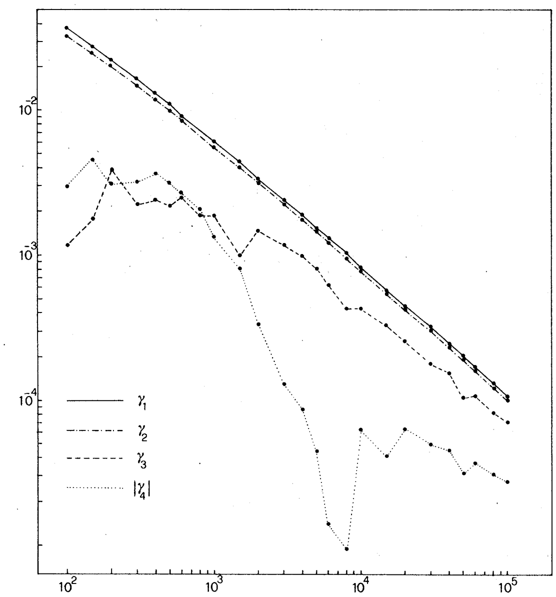

FIG. 1. Variation of the γ_i's (the limits of which are identified with the Lyapunov characteristic numbers) as functions of time for $N=4$ and initial point taken in the ordered region: the limit values clearly vanish.

one is allowed to take zero as limit value.

For $N=5$, Fig. 2(a) shows the behavior of the first three γ_i's for three different stochastic orbits. One can observe how, by increasing time, the curves can be progressively differentiated when approaching their limit values. The impression is that the three orbits have the same LCN's. As only these are positive for $N=5$, their sum gives the entropy density $\rho(z)$. Upper curves on figure 2(a) show the behavior of the sum $\gamma_1(t,z)+\gamma_2(t,z)+\gamma_3(t,z)$; one can observe that a rather well-defined limit value ($\simeq 0.073$) is approached. As explained in Sec. IV A, χ_4 and χ_5 have to vanish for $N=5$. The curves for the corresponding χ_i's are reported in Fig. 2(b) for the same initial conditions as in Fig. 2(a). The limit value clearly vanishes.

A good agreement between the qualitative criterium of the divergence of nearby trajectories and the computation of the LCN's has been found with different values of N and different initial conditions. In particular, for initial points in the stochastic region we always found $N-2$ positive limits. In addition, these limits seem to be independent of the initial point. This supports the conjecture that there is only one stochastic region, where the LCN's are constant.

Table I summarizes our results. For N from 2 to 10 we report the values of μ_s, the limit for the entropy density, and their product, i.e., the Kolmogorov entropy h. As remarked in Sec. III C,

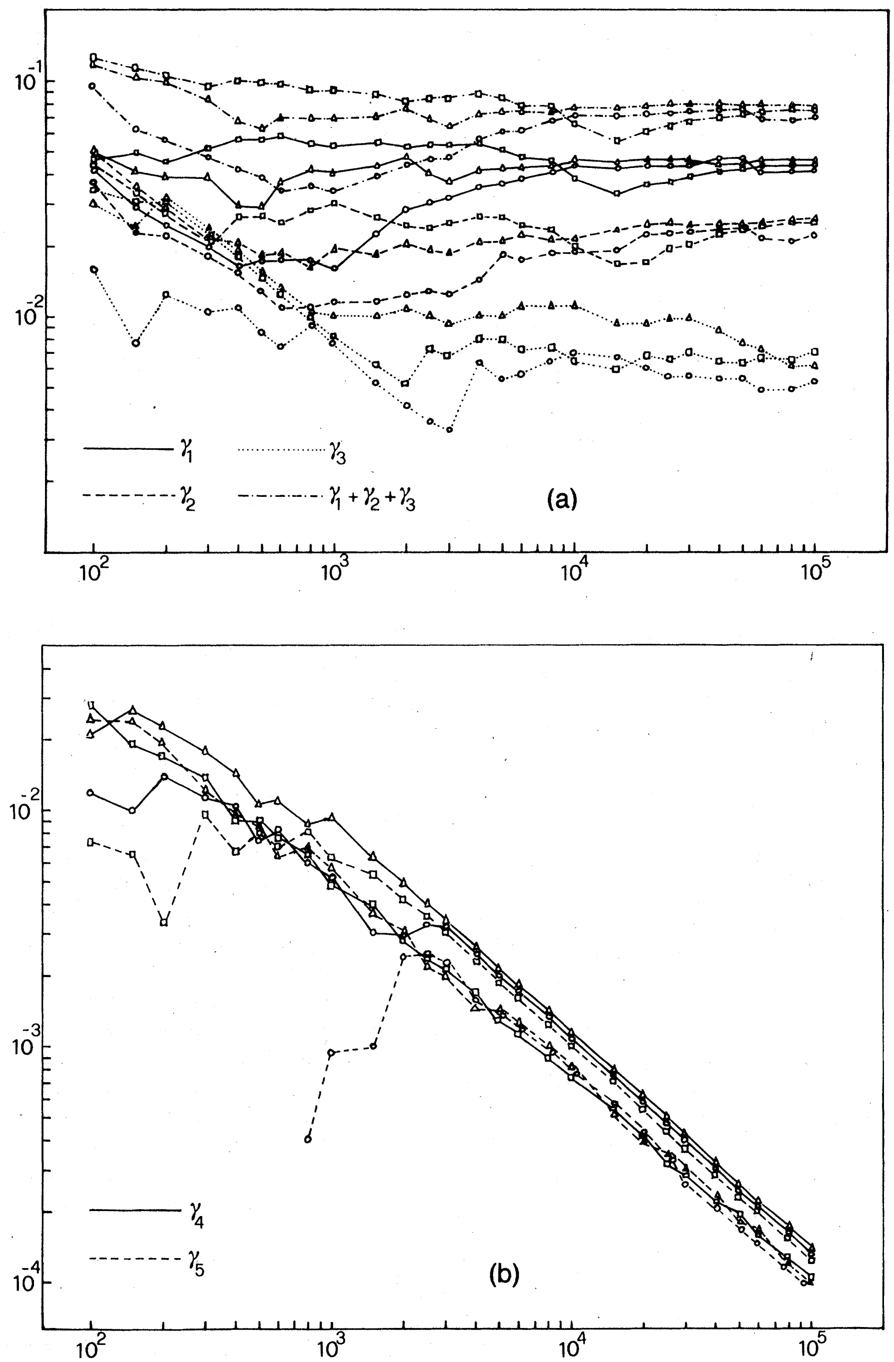

FIG. 2. Variation of the γ_i's as functions of time for $N=5$: (a) γ_1, γ_2, and γ_3 are plotted, as well as their sum, i.e., the entropy density stochastic orbits: the curves approach positive limit values. (b) γ_4 and γ_5 are plotted for the same initial conditions as in (a): their limit values clearly vanish.

the energy dependence of h is trivial. (The relation $h(E)=\lambda h(\lambda^2 E)$ was confirmed well by our numerical computations.) Table I refers to fixed specific energy $E/N=0.5$. We note that the values of μ_s for $N=8$ and 10 had not been computed in Ref. 9: by trivial extrapolation we assumed $\mu_s=1$. Figure 3 reports graphically the results of Table I. As a remarkable fact, our numerical experiment strongly suggests that at fixed specific energy the Kolmogorov entropy is a linear function of N.

In Ref. 9 few orbits were found with an "intermediate" behavior, i.e., in a few cases the qualitative criterium did not allow to decide clearly between ordered and stochastic motion. In some cases the computation of the LCN's shows that the

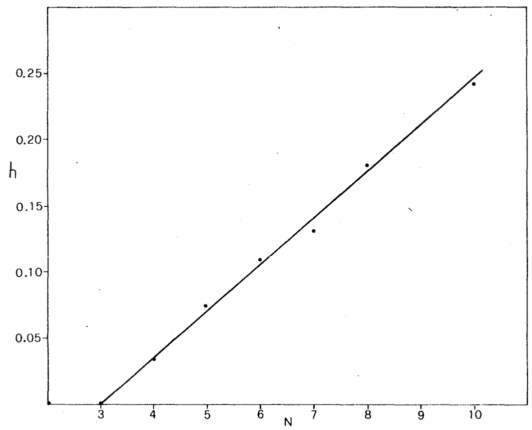

FIG. 3. Variation of the Kolmogorov entropy h with the number N of sheets; this figure reports the values of Table I. At fixed specific energy, Kolmogorov entropy seems to be a linear function of N.

above orbits are stochastic, but the stochasticity appears later. In some other cases the time necessary for the curves to approach a well-defined behavior is so long that an answer cannot be given. We cannot exclude the possibility that there exist very small stochastic regions, separated from the large one, where the LCN's assume different values. Actually, such a phenomenon has been found for other models.[12] In any case the presence of these regions would not change our results for the Kolmogorov entropy appreciably.

Our computations have been performed on an IBM370-168 with a precision of 15 digits. Energy was very well conserved (relative error $\simeq 10^{-8}$). Of course, due to the exponential divergence, it is not possible to follow exactly a given orbit in the stochastic region. Nevertheless, the computation of the LCN's can be considered reliable: for a discussion of the point see Ref. 13.

V. CONCLUSIONS

The computation of the LCN's allowed us to perform a quantitative study of the stochasticity of our system. Agreement is found with the qualitative criterium employed in Ref. 9, by means of which the behavior of μ_s as a function of N was obtained. For this particular dynamical system the Kolmogorov entropy turns out to be a linearly increasing function of the number of degrees of freedom, as far as the specific energy is kept constant. Other systems instead seem to behave completely differently.[14] We note that our model has two very particular features: (a) The energy dependence is trivial and (b) the so-called "connectance"[15] is maximal, i.e., each particle interacts with equal strength with any other particle. This property is probably responsible for the increasing stochasticity with N, but we are not able to explain the very regular linear behavior we have found.

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