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Is quantum mechanics equivalent to a classical stochastic process?

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The authors analyze the connection between the theory of stochastic processes and quantum mechanics. It is shown that quantum mechanics is not equivalent to a Markovian diffusion process as claimed in recent papers. The origin of a possible confusion about this question is clarified. The authors further demonstrate that there does not even exist a non-Markovian process equivalent to quantum mechanics.

I. INTRODUCTION

Since the early days of quantum mechanics there have been attempts to explain the quantum phenomena within the framework of classical mechanics. One approach, which also dates back to the 1930s,[1] relates the motion of a quantum particle to a classical stochastic process. This approach was brought to our attention again by Nelson[2] a decade ago, and since then the theory has been developed further in a still-increasing number of papers,[3] including a recent review[4] on this subject.

In these papers it is claimed that the quantum-mechanical motion of a particle governed by Schrödinger's equation can be equally well understood as a classical Brownian motion of the particle in a vacuum which acts upon the particle as does a heat bath in the theory of irreversible processes. In more technical terms, the quantum-mechanical process is claimed to be equivalent to a classical Markovian diffusion process. In this work we shall show that quantum mechanics has in fact little to do with a stochastic process and that the contrary conclusions of others[1-4] originate in an erroneous use of the theory of Markovian processes.

In Sec. II we present a particularly short "derivation" of the Schrödinger equation from the Fokker-Planck equation of a diffusion process. The Fokker-Planck equation thereby associated with the Schrödinger equation is identical with the Fokker-Planck equation introduced by others.[2-4] We show that the Fokker-Planck operator is not a linear operator on the space of single-event distributions $\rho(x,t)$ whence it does not characterize a Markovian process.

Since the origin of erroneous conclusions in earlier works is not always obvious, in Sec. III we follow their line of argumentation more closely. Especially, we investigate the properties of the backward Markovian process introduced in this context, and show that this process is in fact not Markovian, although it has many of the properties of a Markovian process.

In Sec. IV we explicitly determine the Fokker-Planck equation for an example and calculate the position correlation function on the basis of the claimed equivalence with quantum mechanics. This correlation function is in clear contradiction to the true quantum-mechanical correlation.

Finally, we investigate in Sec. V the question whether we can describe quantum mechanics within the framework of non-Markovian processes, and we are led to the conclusion that the title question must be answered in the negative.

II. STOCHASTIC DERIVATION OF THE SCHRÖDINGER EQUATION

Let us assume that the quantum-mechanical motion of a particle in a potential $V(x,t)$ in n-dimensional space can be described by a continuous Markovian process[5] with constant isotropic diffusion Q. Then the probability distribution $\rho(x,t)$ of the position x of the particle at time t obeys a Fokker-Planck equation of the form

$$\dot{\rho}(x,t) = -\nabla\mu(x,t)\rho(x,t) + \tfrac{1}{2}Q\Delta\rho(x,t)\,. \tag{2.1}$$

The drift $\mu(x,t)$ and the diffusion constant Q will be specified below. If we define a function $\mu^*(x,t)$ by

$$\mu^*(x,t) = \mu(x,t) - Q\nabla\ln\rho(x,t)\,, \tag{2.2}$$

Eq. (2.1) can be transformed into

$$\dot{\rho}(x,t) = -\nabla\mu^*(x,t)\rho(x,t) - \tfrac{1}{2}Q\Delta\rho(x,t)\,. \tag{2.3}$$

By adding the two Fokker-Planck-type equations (2.1) and (2.3), we obtain the continuity equation

$$\dot{\rho}(x,t) = -\nabla v(x,t)\rho(x,t)\,, \tag{2.4}$$

where we have introduced the velocity of the prob-

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ability current

$$v(x,t)=\tfrac{1}{2}[\mu(x,t)+\mu^*(x,t)] \\ =\mu(x,t)-\tfrac{1}{2}Q\nabla\ln\rho(x,t)\,. \quad (2.5)$$

In order to connect this stochastic process with ordinary quantum mechanics we identify $\rho(x,t)$ with the correct quantum-mechanical probability density

$$\rho(x,t)=|\psi(x,t)|^2\,, \quad (2.6)$$

and $v(x,t)$ with the correct velocity of the quantum probability current

$$v(x,t)=(\hbar/2im)[\nabla\ln\psi(x,t)-\nabla\ln\psi^*(x,t)]\,. \quad (2.7)$$

If we substitute (2.6) and (2.7) into Eq. (2.4) and require that $\psi(x,t)$ obey a *linear* differential equation, we find after a few elementary manipulations that this differential equation must be of the form

$$i\hbar\dot{\psi}(x,t)=-(\hbar^2/2m)\Delta\psi(x,t)+V(x,t)\psi(x,t)\,, \quad (2.8)$$

where $V(x,t)$ is an as yet undetermined *real* function of x and t which we identify with the potential. For $Q=\hbar/m$ the Fokker-Planck equation (2.1) hereby associated with the Schrödinger equation (2.8) is identical with the Fokker-Planck equation introduced in recent papers.[2-4] It appears that this Fokker-Planck equation leads in a rather unambiguous way to the Schrödinger equation and that "the radical departure from classical physics produced by the introduction of quantum mechanics was unnecessary."[2] Here several comments are in order. The Fokker-Planck equation governing the time evolution of $\rho(x,t)$ is not uniquely determined. The above argumentation is completely independent of the choice of the diffusion constant Q. Further, the drift $\mu(x,t)$ is not a preassigned vector as in the theory of Markovian diffusion processes.[5,6] From Eqs. (2.2), (2.5), and (2.7) we find

$$\mu(x,t)=Q\nabla\ln|\psi(x,t)|+(\hbar/m)\nabla\arg\psi(x,t)\,, \quad (2.9)$$

which depends on the special process. This means that the Fokker-Planck equation takes a different form for every solution of the Schrödinger equation for a given initial state. This dependence on the initial state is typical for certain master equations in the theory of non-Markovian processes.[7] That type of master equation leads to a correct time evolution of the single-event distribution but to nothing else. Here, the Green's function $G(xt|ys)$ of the Fokker-Planck equation (2.1) also describes the propagation of $\rho(x,t)$ correctly by

$$\rho(x,t)=\int dy\,G(xt|ys)\rho(y,s)\,,\quad t\geq s\,. \quad (2.10)$$

Furthermore, the Green's function fulfills the Chapman-Kolmogorov equation

$$G(xt|ys)=\int dz\,G(xt|zu)G(zu|ys)\quad\text{for } t\geq u\geq s\,. \quad (2.11)$$

as does the conditional probability of a Markovian process. However, $G(xt|ys)$ depends on the chosen initial state in contrast to the conditional probability of a Markovian process, and it will be shown that correlation functions calculated by means of $G(xt|ys)$ have nothing to do with the true time correlations in a quantum system.

III. BACKWARD MARKOVIAN PROCESSES IN STOCHASTIC THEORIES OF QUANTUM MECHANICS

In this section we further examine the connection of stochastic processes and quantum mechanics, following the argumentation of some previous works.[2-4] The origin of erroneous conclusions drawn in these works will be clarified.

The usual concept of Markovian processes introduces an asymmetry in time which is reflected in the irreversible character of the process. This presents a difficulty if one wants to describe the quantum-mechanical evolution by a Markovian process. To overcome this difficulty Nelson and others[2-4] introduce for a given Markovian process a so-called backward Markovian process whose properties we shall discuss below.

A usual Markovian-process $x(t)$ is generated from a conditional probability $R(xt|ys)$, $t\geq s$, which satisfies the Chapman-Kolmogorov equation

$$R(xt|ys)=\int dz\,R(xt|zu)R(zu|ys) \quad (3.1)$$

for $t\geq u\geq s$. This conditional probability is independent of the single-event distribution, which propagates in time according to

$$\rho(x,t)=\int dy\,R(xt|ys)\rho(y,s)\,,\quad t\geq s\,; \quad (3.2)$$

consequently, the relation (3.2) is *linear* in ρ. The multivariable probabilities of the Markovian process are given by

$$\rho^{(n)}(x_nt_n,\ldots,x_1t_1)=\prod_{i=2}^{n}R(x_it_i|x_{i-1}t_{i-1}) \\ \times\rho(x_1,t_1)\,, \quad (3.3)$$

where $t_n\geq t_{n-1}\geq\cdots\geq t_1$. For an arbitrary function $f(x(t),t)$ of the stochastic variables $x(t)$ and time t a mean forward derivative[8] is defined by

$$Df(x(t),t)=\lim_{s\to t^+}\frac{1}{s-t}\int dy\,R(ys|xt) \\ \times[f(y,s)-f(x,t)]\,. \quad (3.4)$$

The symbol t^+ means that s approaches t through larger values.

With the Markovian-process $x(t)$, one can associate a *backward Markovian process* which is generated from the backward conditional probability defined by[4,9]

$$R^*(ys\,|xt)\rho(x,t)=R(xt\,|ys)\rho(y,s) \tag{3.5}$$

or

$$R^*(ys\,|xt)=\frac{R(xt\,|ys)\rho(y,s)}{\int dy\,R(xt\,|ys)\rho(y,s)}\,. \tag{3.6}$$

Note that the backward conditional probability explicitly depends on the single-event probability $\rho(y,s)$. While the (forward) conditional probability of a Markovian process generates *a whole class* of stochastic process for all possible initial probabilities, the backward conditional probability depends on the special process chosen. Such a dependence is typical for a non-Markovian process. Nevertheless, the backward Markovian process has many properties of a Markovian process with opposite time direction. For example, $R^*(ys\,|xt)$ is normalized and fulfills the Chapman-Kolmogorov equation

$$R^*(ys\,|xt)=\int dz\,R^*(ys\,|zu)R^*(zu\,|xt) \tag{3.7}$$

for $s\leqslant u\leqslant t$. The multivariate probability $\rho^{(n)}(x_nt_n,\ldots,x_1t_1)$ can be written

$$\rho^{(n)}(x_nt_n,\ldots,x_1t_1)=\prod_{i=2}^{n}R^*(x_{i-1}t_{i-1}\,|x_it_i)\times\rho(x_n,t_n)\,, \tag{3.8}$$

where $t_1\leqslant t_2\leqslant\cdots\leqslant t_n$, and the single-event probability develops (backwards) in time according to

$$\rho(y,s)=\int dx\,R^*(ys\,|xt)\rho(x,t)\,,\quad s\leqslant t\,. \tag{3.9}$$

These properties follow from the corresponding properties of $R(xt\,|ys)$ and Eq. (3.5). However, (3.9) is *not* linear in ρ, since R^* itself depends on the single-event distribution, in contrast to the corresponding equation (3.2) for a true Markovian process.

A *mean backward derivative* of a function $f(x(t),t)$ can be defined by[8]

$$D^*f(x(t),t)=\lim_{s\to t^-}\frac{1}{t-s}\int dy\,R^*(ys\,|xt)\times[f(x,t)-f(y,s)]\,. \tag{3.10}$$

This derivative also depends on the special process chosen, in contrast to the mean forward derivative, which is the same for the whole class of stochastic processes generated by the Markovian (forward) conditional probability.

In their attempt to describe a quantum system by a stochastic process, Nelson and others[2-4] assume that the position x of a quantum-mechanical particle in a potential $V(x,t)$ undergoes a Markovian diffusion process characterized by the Fokker-Planck equation (2.1) with a diffusion constant

$$Q=\hbar/m \tag{3.11}$$

inversely proportional to the mass. To determine the drift $\mu(x,t)$ they demand for the process a generalized time-symmetrical form of Newton's law,

$$\tfrac{1}{2}m(DD^*+D^*D)x(t)=-\nabla V(x,t)\,. \tag{3.12}$$

For a diffusion process the forward derivative D is easily expressed in terms of the drift μ and the diffusion constant Q. However, since the backward derivative D^* also depends on the single-event distribution $\rho(x,t)$, the requirement (3.12) does not determine the drift $\mu(x,t)$ as a preassigned vector essentially determined by the potential, but rather the drift will explicitly depend on $\rho(x,t)$, i.e., on the special process chosen.

We shall not give the somewhat cumbersome evaluation of (3.12),[2,4] since the final expression for the drift coincides with the expression derived in Sec. II. Clearly, by the dependence of $\mu(x,t)$ on $\rho(x,t)$ the stochastic process so determined is *not* Markovian, and the erroneous conclusions drawn in some previous works[2-4] can in part be attributed to the author's overlooking of the non-Markovian character of the so-called backward Markovian process.

IV. EXAMPLE: THE HARMONIC OSCILLATOR

In this section we illustrate the general analysis by a simple example. Consider a harmonic oscillator initially at time $t=0$ in the coherent state[10]

$$\psi(x,0)=(\alpha^{1/2}/\pi^{1/4})e^{-1/2\alpha^2(x-x_0)^2}, \tag{4.1}$$

where

$$\alpha=(m\omega/\hbar)^{1/2}\,. \tag{4.2}$$

This state centers about the position $x=x_0$. Its change in time is governed by the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t)=\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\tfrac{1}{2}m\omega^2x^2\right)\psi(x,t)\,, \tag{4.3}$$

and the state at time t is found to be[11]

$$\psi(x,t)=(\alpha^{1/2}/\pi^{1/4})\exp[-\tfrac{1}{2}\alpha^2(x-x_0\cos\omega t)^2 - \tfrac{1}{2}i\omega t - i\alpha^2(xx_0\sin\omega t - \tfrac{1}{4}x_0^2\sin 2\omega t)]\,. \tag{4.4}$$

Hence the position probability is

$$\rho(x,t)=\psi(x,t)\psi^*(x,t) = (\alpha/\pi^{1/2})e^{-\alpha^2(x-x_0\cos\omega t)^2}\,. \tag{4.5}$$

Let us now consider the stochastic process associated with this quantum-mechanical dynamics in the work of Nelson and others.[2-4] With (2.9), (3.11), and (4.4) the Fokker-Planck equation (2.1) reads

$$\frac{\partial}{\partial t}\rho(x,t)=-\frac{\partial}{\partial x}[-\omega x-\omega x_0(\cos\omega t-\sin\omega t)]\rho(x,t) + \frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\rho(x,t)\,. \tag{4.6}$$

The explicit dependence of the drift on x_0 clearly shows that this equation is not a linear equation in $\rho(x,t)$, rather, it only holds for the special $\rho(x,t)$ given in Eq. (4.5). Nevertheless, *formally* we might consider the drift as an assigned time-dependent vector and look upon Eq. (4.6) as a linear equation in $\rho(x,t)$. Then Eq. (4.6) characterizes a nonstationary Gauss-Markov process, and the conditional probability of this process is obtained[12] as the Green's-function solution of (4.6):

$$R(xt\,|\,ys)= \exp\left(-\frac{[x-\xi(x,t,s)]^2}{2\sigma(t-s)}\right)\Big/[2\pi\sigma(t-s)]^{1/2}\,, \tag{4.7}$$

where

$$\xi(y,t,s)=ye^{-\omega(t-s)}+x_0(\cos\omega t-e^{-\omega(t-s)}\cos\omega s)\,,$$
$$\sigma(\tau)=(\hbar/2m\omega)(1-e^{-2\omega\tau})\,. \tag{4.8}$$

The position correlation function of this process is given by

$$\langle x(t)x(s)\rangle=\int dx\,dy\,xyR(xt\,|\,ys)\rho(y,s)\,. \tag{4.9}$$

If we insert the position probability (4.5) and the conditional probability (4.7) we find

$$\langle x(t)x(s)\rangle=(\hbar/2m\omega)e^{-\omega(t-s)}+x_0^2\cos\omega t\cos\omega s\,. \tag{4.10}$$

For $t=s$ this correlation function gives the *correct* second moment

$$\langle x^2(t)\rangle=(\hbar/2m\omega)+x_0^2\cos^2\omega t \tag{4.11}$$

of the harmonic oscillator. However, for $t>s$ the correlation function (4.10) has nothing in common with the true correlations of the quantum oscillator. For instance, if we choose $x_0=0$ the oscillator is initially in its ground state. In this case we obtain from (4.10)

$$\langle x(t)x(s)\rangle=(\hbar/2m\omega)e^{-\omega(t-s)}\,. \tag{4.12}$$

Clearly, these exponentially damped correlations do not describe a reversible quantum-mechanical oscillator. Here we see explicitly that the Markovian diffusion process characterized by the Fokker-Planck equation (2.1) does not describe adequately the motion of a quantum-mechanical particle.

V. NON-MARKOVIAN PROCESSES AND QUANTUM MECHANICS

In this section we investigate the more general problem of deciding whether or not there exists a non-Markovian process equivalent to the Schrödinger evolution. A general stochastic process $x(t)$ is characterized by the set of all multitime moments $\langle f_n(t_n)\cdots f_1(t_1)\rangle$ of arbitrary function $f(t)\equiv f(x(t))$. The two-time moments (correlation functions) satisfy the following conditions.

(I) Compatibility:

$$\langle f(t_2)1(t_1)\rangle=\langle f(t_2\rangle\,, \tag{5.1}$$

where 1 denotes the unit function $1(x)=1$.

(II) Linearity:

$$\langle g(t_2)[\alpha_1 f_1(t_1)+\alpha_2 f_2(t_1)]\rangle = \alpha_1\langle g(t_2)f_1(t_1)\rangle+\alpha_2\langle g(t_2)f_2(t_1)\rangle\,. \tag{5.2}$$

(III) Symmetry:

$$\langle f_2(t_2)f_1(t_1)\rangle=\langle f_1(t_1)f_2(t_2)\rangle\,. \tag{5.3}$$

(IV) Positivity:

$$f_1\geq 0\,,\quad f_2\geq 0 \Rightarrow \langle f_2(t_2)f_1(t_1)\rangle\geq 0\,. \tag{5.4}$$

The correlation functions of classical statistical mechanics fulfull these conditions. There is no unique extension of the classical correlations to quantum systems, and various definitions of quantum-mechanical correlations have been introduced in different contexts.

A definition widely accepted in quantum optics,[13] reads[14]

$$\langle f_2(t_2)f_1(t_1)\rangle=\langle\psi\,|\hat{f}_2(t_2)\hat{f}_1(t_1)\,|\psi\rangle\,, \tag{5.5}$$

where $|\psi\rangle$ is the Schrödinger wave function at time $t=0$, and where

$$\hat{f}(t)=e^{(i/\hbar)\hat{H}t}f(\hat{x})e^{-(i/\hbar)\hat{H}t} \tag{5.6}$$

is the Heisenberg operator which corresponds to the classical observable $f(x(t))$.

This definition of the quantum-mechanical correlation function does not fulfill the symmetry and positivity requirements (III) and (IV), because the operators $\hat{f}_1(t_1)$ and $\hat{f}_2(t_2)$ generally do not commute for different times and their product is not a Hermitian operator.

Symmetry holds if one uses the definition

$$\langle f_2(t_2)f_1(t_1)\rangle = \tfrac{1}{2}\langle\psi\,|\hat{f}_2(t_2)\hat{f}_1(t_1)+\hat{f}_1(t_1)\hat{f}_2(t_2)\,|\psi\rangle\,, \qquad (5.7)$$

which often occurs in quantum-statistical thermodynamics.[15] However, this definition also does not fulfill the positivity requirement (IV). An explicit counterexample to (IV) is given in the Appendix.

Another definition of the quantum-mechanical correlation function comes from the theory of the measuring process and takes into account the reduction of the wave function at the time of the first observation.

For $t_2 > t_1$ the correlation of two arbitrary functions of the position x is defined by[16]

$$\begin{aligned}\langle f_2(t_2)f_1(t_1)\rangle &\equiv \langle f_1(t_1)f_2(t_2)\rangle\\ &= \mathrm{tr}\hat{f}_2(t_2-t_1)\hat{f}_1(0)\hat{W}^R(t_1^+)\,,\end{aligned} \qquad (5.8)$$

where tr denotes the trace, and where $\hat{W}^R(t_1^+)$ is the density matrix of the reduced ensemble produced by the position measurement at time t_1.[17] The measurement changes the situation from one corresponding to a pure state described by the density matrix

$$\hat{W}(t_1^-) = e^{-(i/\hbar)\hat{H}t}\,|\psi\rangle\langle\psi\,|e^{(i/\hbar)\hat{H}t} \qquad (5.9)$$

to a situation corresponding to a mixture which is described by $\hat{W}^R(t_1^+)$.

The correlation function (5.8) fulfills the symmetry and positivity conditions (III) and (IV). In order not to violate the linearity condition we must put

$$\langle f_2(t_2)1(t_1)\rangle = \mathrm{tr}\hat{f}_2(t_2-t_1)\hat{W}^R(t_1^+)\,, \quad t_2 > t_1 \qquad (5.10)$$

which does not coincide with

$$\langle f_2(t_2)\rangle = \mathrm{tr}\hat{f}_2(t_2-t_1)\,\hat{W}(t_1^-)\,. \qquad (5.11)$$

Consequently, compatibility (I) and linearity (II) cannot both hold. The reason for this is that the evolution of a quantum-mechanical system is influenced by measurements in contrast to the evolution of a classical system.

VI. CONCLUSIONS

In this paper we have analyzed the relations between the theory of stochastic processes and the statistical interpretation of quantum mechanics. We have shown that the Schrödinger evolution is not equivalent to a Markovian process, as claimed in several papers. Possible relations to a non-Markovian process have been investigated, and we have shown that the various correlation functions used in quantum theory do not have the properties of the correlations of a classical stochastic process. This leads to the conclusion that quantum mechanics has little if anything to do with the theory of stochastic processes.

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APPENDIX

Consider the harmonic oscillator of Sec. IV. Heisenberg's equation of motion for the position operator $\hat{x}$ has the solution

$$\hat{x}(t) = \hat{x}\cos\omega t + (\hat{p}/\omega m)\sin\omega t\,, \qquad (A1)$$

where $\hat{p} = -i\hbar\,\partial/\partial x$ is the momentum operator.

We now determine the symmetrical correlation (5.7) for the positive observables e^{cx} and x^2 in the ground state $|0\rangle$. An elementary calculation yields

$$\begin{aligned}&\tfrac{1}{2}\langle 0\,|e^{c\hat{x}(t)}\hat{x}(0)^2 + \hat{x}(0)^2 e^{c\hat{x}(t)}\,|0\rangle\\ &\quad = (\hbar/2m\omega)e^{\hbar c^2/4m\omega}[1+(\hbar c^2/2m\omega)\cos 2\omega t]\,,\end{aligned} \qquad (A2)$$

which for $\hbar c^2 > 2m\omega$ is not positive for all t.

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