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Correlation functions of the two-mode ring laser

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The intensity and amplitude correlation functions of the optical field of a ring laser at line center are calculated, under conditions when the two pump parameters corresponding to the counter-rotating traveling wave modes are not necessarily equal. This represents a generalization of earlier treatments by M-Tehrani and Mandel (1978) and Hioe (1978). The laser is assumed to be at rest. A perturbative technique is used to express the correlation functions for a small difference ϵ of pump parameters in terms of the solutions for $\epsilon = 0$. It is found that the cross correlations are unchanged to the first order in ϵ, whereas the autocorrelations both of the light amplitude and of the light intensity are modified. Curves are presented to illustrate the behavior.

I. INTRODUCTION

The theory of the nonrotating two-mode ring laser,[1] which is a particular example of a general two-mode laser, has recently been developed sufficiently to yield expressions for the correlation functions[2] of the optical field. The analysis was based on the solution of the master equation for the laser field, which has the form of a Fokker-Planck equation for the probability distribution of the field, and was a generalization of an earlier calculation by Grossman and Richter.[3] The treatment has since been further generalized to an N-mode laser by Hioe.[4]

Unfortunately, all the time-dependent solutions of the multimode laser problem obtained so far have a common restriction: they are limited to equal pump parameters for all the laser modes. Although, superficially, it might seem that the two ring laser modes, which correspond to waves propagating clockwise and counterclockwise around the ring, would have similar losses, in practice slight asymmetries are generally present. As a result, the pump parameters of the two modes are usually slightly different also. In some recent experiments in which the light-intensity fluctuations of the two modes of a particular ring laser were investigated,[5] it was found that the two pump parameters differed by about 0.8, and that this difference remained approximately constant as the pump parameters were varied. It is therefore important to be able to generalize the theory for a laser with unequal pump parameters.

This turns out to be a nontrivial problem. As a first step towards its solution we use a perturbative technique to generate the solution for a small difference ϵ of pump parameters in terms of the previously obtained solutions for equal pump parameters. We show that, to the first order in ϵ, cross correlations between the two modes of the laser are independent of ϵ, but that the autocorrelation functions of the light amplitude and the light intensity both vary with ϵ. Curves are presented that illustrate the effect of the asymmetries.

II. FORMULATION OF PROBLEM

We start as in Ref. 2 from the coupled equations of motion for the slowly varying complex amplitudes $E_1(t)$ and $E_2(t)$ of the two laser modes. These equations, which were first derived by Lamb, Aronowitz, and others,[1] are supplemented by the introduction of random Langevin forces $q_1(t)$ and $q_2(t)$, corresponding to spontaneous-emission fluctuations, and take the form

$$\frac{dE_1}{dt} = (a_1 - |E_1|^2 - \xi|E_2|^2)E_1 + q_1(t)$$
$$\frac{dE_2}{dt} = (a_2 - |E_2|^2 - \xi|E_1|^2)E_2 + q_2(t) \; . \qquad (1)$$

Here a_1 and a_2 are the pump parameters of the two modes, which correspond to the two counter-rotating waves of the ring laser. The mode coupling constant ξ depends on the detuning $\Delta\omega$ of the cavity from the atomic line center, and is given by

$$\xi = 1/[1 + (\Delta\omega T_1)^2] \, , \qquad (2)$$

where T_1 is the lifetime of the atomic transition. The coupling constant ξ becomes unity at the line center. The ring laser itself is assumed to be nonrotating, and the frequencies of the two counter-rotating modes are taken to be equal. The Langevin noise terms are taken to be statistically independent, δ-correlated, Gaussian random processes with

$$\langle q_i^*(t) q_j(t')\rangle = 2\delta_{ij}\delta(t - t'), \quad i, j = 1, 2 \; . \qquad (3)$$

If we write

$$E_1=\sqrt{I_1}\, e^{i\phi_1}=x_1+ix_2, \quad E_2=\sqrt{I_2}\, e^{i\phi_2}=x_3+ix_4\,, \qquad (4)$$

then the laser field is described by the four-dimensional vector $\vec{x}(t)$.

One can write a Fokker-Planck equation for the probability density $p(\vec{x},t)$ of $\vec{x}(t)$, which corresponds to the two coupled Langevin equations (1) and takes the form

$$\frac{\partial p}{\partial t}=-\sum_{i=1}^{4}\frac{\partial}{\partial x_i}(A_i p)+\tfrac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\frac{\partial^2}{\partial x_i\partial x_j}(D_{ij}p)\,, \qquad (5)$$

with the drift vector $\vec{A}$ given by

$$\begin{aligned} A_1&=[a_1-(x_1^2+x_2^2)-\xi(x_3^2+x_4^2)]x_1\\ A_2&=[a_1-(x_1^2+x_2^2)-\xi(x_3^2+x_4^2)]x_2\,,\\ A_3&=[a_2-(x_3^2+x_4^2)-\xi(x_1^2+x_2^2)]x_3\,,\\ A_4&=[a_2-(x_3^2+x_4^2)-\xi(x_1^2+x_2^2)]x_4\,, \end{aligned} \qquad (6a)$$

and the diffusion tensor D_{ij} by

$$D_{ij}=2\delta_{ij}\,. \qquad (6b)$$

The steady-state solution $p_s(\vec{x})$ of Eq. (5) is readily found to be[2-4]

$$p_s(\vec{x})=(1/N)\exp\left(\tfrac{1}{2}a_1I_1-\tfrac{1}{4}I_1^2+\tfrac{1}{2}a_2I_2-\tfrac{1}{4}I_2^2-\tfrac{1}{2}\xi I_1I_2\right), \qquad (7a)$$

in which I_1 and I_2 are the instantaneous light intensities defined by Eqs. (4) and N is a constant that ensures the normalization of $p_s(\vec{x})$. In the special case $a_1=a_2=a$ and $\xi=1$ it reduces to the simple expression[2]

$$N=\pi^{5/2}[a\exp(\tfrac{1}{4}a^2)(1+\operatorname{erf}\tfrac{1}{2}a)+2/\sqrt{\pi}\,]\,. \qquad (7b)$$

The general time-dependent solution $p(\vec{x},t)$ can be shown to be of the form[2]

$$p(\vec{x},t)=\sum_{l,m,n,p} c_{lmnp}\sqrt{p_s(\vec{x})}\, g_{lmnp}(\vec{x})\exp(-\lambda_{lmnp}t)\,, \qquad (8)$$

in which the coefficients c_{lmnp} are constants determined by the initial conditions, and the $g_{lmnp}(\vec{x})$ and λ_{lmnp} are orthonormal eigenfunctions and eigenvalues of a self-adjoint differential operator. With the help of the change of variables

$$I_1+I_2=u,\quad 0\leq u,\quad (I_1-I_2)/(I_1+I_2)=v,\quad -1\leq v\leq 1\,, \qquad (9)$$

and

$$a_1+a_2=2a,\quad a_1-a_2=\epsilon\,,$$

or

$$a_1=a+\tfrac{1}{2}\epsilon\,,\quad a_2=a-\tfrac{1}{2}\epsilon\,, \qquad (10)$$

in terms of which the differential element becomes

$$d^4x=\tfrac{1}{8}u\,du\,dv\,d\phi_1\,d\phi_2\,, \qquad (11)$$

the differential equation for $g_{lmnp}(u,v,\phi_1,\phi_2)$ takes the form

$$[\mathcal{L}(u,v,\epsilon)+\epsilon\mathcal{L}_1(u,v)]g_{lmnp}=\lambda_{lmnp}g_{lmnp}\,. \qquad (12)$$

Here $\mathcal{L}$ and $\mathcal{L}_1$ are differential operators defined by

$$\mathcal{L}(u,v,\epsilon)\equiv-4u\frac{\partial^2}{\partial u^2}-8\frac{\partial}{\partial u}+2a-(3-\tfrac{1}{16}\epsilon^2)u+\tfrac{1}{4}u(a-u)^2+\frac{4}{u}\left(-(1-v^2)\frac{\partial^2}{\partial v^2}+2v\frac{\partial}{\partial v}-\frac{1}{2(1+v)}\frac{\partial^2}{\partial\phi_1^2}-\frac{1}{2(1-v)}\frac{\partial^2}{\partial\phi_2^2}\right) \qquad (13)$$

and

$$\mathcal{L}_1(u,v)\equiv\tfrac{1}{4}uv(a-u)\,. \qquad (14)$$

Once the general time-dependent solution is found, we also have the Green function of the process, which is the particular solution for which the initial probability density has the form of a δ function. The Green function is also the conditional probability density $G(\vec{x},t+\tau|\vec{x}',t)$ that the optical field is characterized by the vector $\vec{x}$ at time $t+\tau$ if it was $\vec{x}'$ at time t. This immediately allows us to write the joint probability density $p_2(\vec{x},t+\tau;\vec{x}',t)$ for the field at two times t and $t+\tau$, and we find, in the steady state,[2]

$$p_2(\vec{x},t+\tau;\vec{x}',t)=G(\vec{x},t+\tau|\vec{x}',t)p_s(\vec{x}')=[p_s(\vec{x})p_s(\vec{x}')]^{1/2}\sum_{l,m,n,p}g^*_{lmnp}(\vec{x})g_{lmnp}(\vec{x}')\exp(-\lambda_{lmnp}|\tau|)\,. \qquad (15)$$

With the help of $p_2(\vec{x}, t+\tau; \vec{x}', t)$ any two-time correlation function of the optical field can be evaluated, once the eigenfunctions $g_{lmnp}(\vec{x})$ and eigenvalues λ_{lmnp} are known.

In the special case of equal pump parameters $a_1 = a_2 = a$, or $\epsilon = 0$, and when the detuning $\Delta\omega = 0$, or $\xi = 1$, the problem simplifies. The eigenfunctions and eigenvalues then satisfy the simpler differential equation

$$\mathcal{L}(u, v, 0) g^{(0)}_{lmnp} = \lambda^{(0)}_{lmnp} g^{(0)}_{lmnp} , \qquad (16)$$

where the label (0) serves to remind us that $\epsilon = 0$. It can be shown[2,4] that in this case the variables u, v, ϕ_1, ϕ_2 are separable, so that $g^{(0)}_{lmnp}$ factorizes and may be written

$$g^{(0)}_{lmnp}(u, v, \phi_1, \phi_2) = S_{lnp}(v) R_{lmnp}(u) \times (e^{in\phi_1}/\sqrt{2\pi})(e^{ip\phi_2}/\sqrt{2\pi}) , \qquad (17)$$

in which each of the functions is separately normalized. If we put

$$S_{lnp}(v) = \sqrt{M_{lnp}}\,(1+v)^{n/2}(1-v)^{p/2} P_l^{(p,n)}(v) ,$$

where M_{lnp} is a normalizing factor, then $P_l^{(p,n)}$ is found to obey the differential equation

$$(1-v^2)\frac{d^2}{dv^2} P_l^{(p,n)}(v) + [n - p - (n+p+2)v]\frac{d}{dv} \times P_l^{(p,n)}(v) + [\beta_{lnp} - \tfrac{1}{4}(n+p)(n+p+2)] P_l^{(p,n)}(v) = 0 , \qquad (18)$$

which becomes identical with the differential equation satisfied by the Jacobi polynomials[6] $P_l^{(p,n)}(v)$ if we put

$$\beta_{lnp} = l(l+n+p+1) + \tfrac{1}{4}(n+p)(n+p+2) .$$

Accordingly we write[7]

$$g^{(0)}_{lmnp}(u, v, \phi_1, \phi_2) = (M_{lnp})^{1/2}(1-v)^{p/2}(1+v)^{n/2} P_l^{(p,n)}(v) R_{lmnp}(u) e^{in\phi_1} e^{ip\phi_2}/2\pi, \quad l,m = 0,1,2,\ldots,$$
$$\text{if } n = -l, -l+1, \ldots, -1, 0, 1, 2, \ldots, \quad p = 0, 1, 2, \ldots, \qquad (19a)$$

$$= (M_{lnp})^{1/2}(1-v)^{-p/2}(1+v)^{-n/2} P^{(-p,-n)}_{l+p+n}(v) R_{lmnp}(u) e^{in\phi_1} e^{ip\phi_2}/2\pi, \quad l,m = 0,1,2,\ldots,$$
$$\text{if } n = -l, -l+1, \ldots, -1, 0, 1, 2, \ldots, \quad p = -l, -l+1, \ldots, -2, -1, \text{ and } l+n+p \geq 0. \qquad (19b)$$

For convenience we use Eq. (19a) in the subsequent analysis, with the understanding that this is to be replaced by Eq. (19b) whenever p becomes a negative integer. We shall find that only the terms with $p = 0$ are actually needed. The normalizing factor is given by[6]

$$M_{lnp} = \frac{2l+n+p+1}{2^{n+p+1}} \frac{l!(l+n+p)!}{(l+n)!(l+p)!} . \qquad (20)$$

The functions $R_{lmnp}(u)$ and the corresponding eigenvalues $\lambda^{(0)}_{lmnp}$ are expressible in terms of the solutions of the one-dimensional Schrödinger equation

$$\frac{d^2\psi_{lmnp}(y)}{dy^2} + [\lambda^{(0)}_{lmnp} - V_{lnp}(y)]\psi_{lmnp}(y) = 0 , \qquad (21)$$

in which

$$\psi_{lmnp}(y) = \tfrac{1}{2} y^{3/2} R_{lmnp}(y^2) , \qquad (22)$$

and the "potential" $V_{lnp}(y)$ is given by

$$V_{lnp}(y) = [4l(l+n+p+1) + (n+p)(n+p+2) + \tfrac{3}{4}]/y^2 + 2a + (\tfrac{1}{4}a^2 - 3)y^2 - \tfrac{1}{2}ay^4 + \tfrac{1}{4}y^6 . \qquad (23)$$

Although all three indices l, n, p appear in this expression, it is clear that certain combinations of them give rise to the same potential, and therefore to the same eigenfunction R_{lmnp} and eigenvalue $\lambda^{(0)}_{lmnp}$. If we denote by L the combination

$$L \equiv 2l + n + p , \qquad (24)$$

the potential becomes

$$V_L(y) = [L(L+2) + \tfrac{3}{4}]/y^2 + 2a + (\tfrac{1}{4}a^2 - 3)y^2 - \tfrac{1}{2}ay^4 + \tfrac{1}{4}y^6 , \qquad (25)$$

and this clearly depends only on the one index L, which has a degeneracy $(L+1)^2$. Accordingly, we henceforth replace the three labels l, n, p on V_{lnp}, ψ_{lmnp}, R_{lmnp}, $\lambda^{(0)}_{lmnp}$ by the single label L, and write V_L, ψ_{Lm}, R_{Lm}, $\lambda^{(0)}_{Lm}$. As the Schrödinger equation is one-dimensional, we may take the eigensolutions $R_{Lm}(u)$ to be real functions from now on.

III. LASER WITH UNEQUAL PUMP PARAMETERS

We now consider the case when the pump parameters a_1, a_2 are not necessarily equal, but the difference $a_1 - a_2 \equiv \epsilon$ is small, so that terms in ϵ^2, ϵ^3, etc., may be neglected. To this degree of approximation the differential operator $\mathcal{L}(u, v, \epsilon)$ given by Eq. (13) is independent of ϵ, and will henceforth be denoted by $\mathcal{L}(u,v)$. If we assume that $g_{lmnp}(u,v,\phi_1,\phi_2)$ depends on ϕ_1, ϕ_2 through the oscillatory phase factors $\exp(in\phi_1)$ and $\exp(ip\phi_2)$ as before, then $\mathcal{L}(u,v)$ becomes

$$\mathcal{L}(u,v) = -4u\frac{\partial^2}{\partial u^2} - 8\frac{\partial}{\partial u} + 2a - 3u + \tfrac{1}{4}u(a-u) + \frac{4}{u}\left(-(1-v^2)\frac{\partial^2}{\partial v^2} + 2v\frac{\partial}{\partial v} + \frac{n^2}{2(1+v)} + \frac{p^2}{2(1-v)}\right). \tag{26}$$

We now make the ansatz that, for small ϵ,

$$g_{lmnp}(u,v,\phi_1,\phi_2) = g^{(0)}_{lmnp}(u,v,\phi_1,\phi_2) + \epsilon g^{(1)}_{lmnp}(u,v,\phi_1,\phi_2), \tag{27}$$

$$\lambda_{lmnp} = \lambda^{(0)}_{Lm} + \epsilon\lambda^{(1)}_{lmnp}, \tag{28}$$

where $g^{(1)}_{lmnp}$ and $\lambda^{(1)}_{lmnp}$ remain to be determined. On substituting Eqs. (27) and (28) into Eq. (12), using Eq. (16), and dropping terms of order ϵ^2, we obtain

$$(\mathcal{L} - \lambda^{(0)}_{Lm})g^{(1)}_{lmnp} = (\lambda^{(1)}_{lmnp} - \mathcal{L}_1)g^{(0)}_{lmnp}. \tag{29}$$

Next we multiply both sides of this equation on the left by $g^{(0)*}_{l'm'n'p'}$ and integrate. As $\mathcal{L} - \lambda^{(0)}_{Lm}$ is a self-adjoint differential operator, we have

$$\left(\int g^{(1)*}_{lmnp}(\mathcal{L} - \lambda^{(0)}_{Lm})g^{(0)}_{l'm'n'p'}\,d^4x\right)^* = \int g^{(0)*}_{l'm'n'p'}(\lambda^{(1)}_{lmnp} - \mathcal{L}_1)g^{(0)}_{lmnp}\,d^4x,$$

and, with the help of Eq. (16) and the orthonormality of the $g^{(0)}_{lmnp}$ functions, this becomes

$$(\lambda^{(0)}_{L'm'} - \lambda^{(0)}_{Lm})\int g^{(0)*}_{l'm'n'p'}g^{(1)}_{lmnp}\,d^4x = \lambda^{(1)}_{lmnp}\delta_{l'l}\delta_{m'm}\delta_{n'n}\delta_{p'p} - \int g^{(0)*}_{l'm'n'p'}\mathcal{L}_1 g^{(0)}_{lmnp}\,d^4x.$$

We now substitute the explicit forms of $g^{(0)}_{lmnp}$ and $\mathcal{L}_1$ given by Eqs. (19) and (14), respectively, into this equation, and obtain

$$\begin{aligned}(\lambda^{(0)}_{L'm'} - \lambda^{(0)}_{Lm})\int g^{(0)*}_{l'm'n'p'}g^{(1)}_{lmnp}\,d^4x &= \lambda^{(1)}_{lmnp}\delta_{l'l}\delta_{m'm}\delta_{n'n}\delta_{p'p} - (M_{l'np}M_{lnp})^{1/2}\int_{-1}^{1} dv(1-v)^{|p|}(1+v)^{|n|}P^{(|p||n|)}_{l'}(v)P^{(|p||n|)}_{l}(v)v\,\delta_{n'n}\delta_{p'p} \\ &\quad\times\int_0^\infty \tfrac{1}{4}u(a-u)R_{L'm'}(u)R_{Lm}(u)\tfrac{1}{8}u\,du.\end{aligned} \tag{30}$$

The v integral can be transformed with the help of the recurrence relations among Jacobi polynomials,[6]

$$\begin{aligned}vP^{(pn)}_l(v) &= \frac{2(l+n)(l+n+p)}{L(L+1)}P^{(pn)}_l(v) + \frac{2(l+1)(l+n+p+1)}{(L+1)(L+2)}P^{(pn)}_{l+1}(v) + \frac{2(l+n)(l+p)}{L(L+1)}P^{(pn)}_{l-1}(v) \\ &\quad + \left(\frac{2(l+1)(l+p+1)}{(L+1)(L+2)} - 1\right)P^{(pn)}_l(v).\end{aligned} \tag{31}$$

When we substitute this into Eq. (30) and make use of Eq. (20) together with the orthogonality of the Jacobi polynomials, we find

$$\begin{aligned}(\lambda^{(0)}_{L'm'} - \lambda^{(0)}_{Lm})\int g^{(0)*}_{l'm'n'p'}g^{(1)}_{lmnp}\,d^4x &= \lambda^{(1)}_{lmnp}\delta_{l'l}\delta_{m'm}\delta_{n'n}\delta_{p'p} \\ &\quad -2\left(-\tfrac{1}{2} - \frac{l(l+n)}{L} + \frac{(l+1)(l+n+1)}{L+2}\right)D_{Lm'Lm}\delta_{l'l}\delta_{n'n}\delta_{p'p} \\ &\quad -2\left(\frac{(l+1)(l+n+p+1)(l+p+1)(l+n+1)}{(L+1)(L+2)^2(L+3)}\right)^{1/2}D_{L+2\,m'Lm}\delta_{l'\,l+1}\delta_{n'n}\delta_{p'p} \\ &\quad -2\left(\frac{l(l+n+p)(l+p)(l+n)}{L^2(L-1)(L+1)}\right)^{1/2}D_{L-2\,m'Lm}\delta_{l'\,l-1}\delta_{n'n}\delta_{p'p},\end{aligned} \tag{32}$$

in which we have used the abbreviation

$$D_{L'm'Lm} \equiv \int_0^\infty \tfrac{1}{4}u(a-u)R_{L'm'}(u)R_{Lm}(u)\tfrac{1}{8}u\,du. \tag{33}$$

Equation (32) allows us to obtain the quantity $\lambda^{(1)}_{lmnp}$ immediately. We simply put $l'=l$, $m'=m$, $n'=n$, $p'=p$ on both sides of the equation, and find

$$\lambda^{(1)}_{lmnp} = \left(-1 - \frac{2l(l+n)}{L} + \frac{2(l+1)(l+n+1)}{L+2}\right)D_{LmLm}. \tag{34}$$

Therefore, $\lambda^{(1)}_{lmnp}$ is given in terms of the same eigenfunctions $R_{Lm}(u)$ that were encountered for the symmetric ring laser. In practice, in order to evaluate the most important correlation functions of the laser field, it suffices that $\lambda^{(1)}_{lmnp}$ is known for $n=0=p$ and for $n=1$, $p=0$ or $n=0$, $p=1$, as we shall see. In the first case, when $n=0=p$, it follows immediately from Eq. (34) that

$$\lambda^{(1)}_{lm00}=0\,, \tag{35a}$$

so that, from Eq. (28)

$$\lambda_{lm00}=\lambda^{(0)}_{Lm}+O(\epsilon^2)\,, \tag{35b}$$

and the eigenvalues are unchanged from the symmetric ring laser. For the other cases it is generally sufficient that the factors D_{LmLm} are known for a few small values of L, m. Some values of D_{LmLm} are given in Table I below.

From Eq. (32) we also find, when $m\neq m'$,

$$\begin{aligned}A^{l'm'n'p'}_{lmnp}\equiv\int g^{(0)}_{l'm'n'p'}g^{(1)}_{lmnp}d^4x=&-\left(-\tfrac{1}{2}-\frac{l(l+n)}{L}+\frac{(l+1)(l+n+1)}{L+2}\right)\frac{2D_{Lm'Lm}}{\lambda^{(0)}_{Lm'}-\lambda^{(0)}_{Lm}}\delta_{l'l}\delta_{n'n}\delta_{p'p}\\&-\left(\frac{(l+1)(l+n+p+1)(l+p+1)(l+n+1)}{(L+1)(L+2)^2(L+3)}\right)^{1/2}\frac{2D_{L+2m'Lm}}{\lambda^{(0)}_{L+2\,m'}-\lambda^{(0)}_{Lm}}\delta_{l'\,l+1}\delta_{n'n}\delta_{p'p}\\&-\left(\frac{l(l+n+p)(l+p)(l+n)}{L^2(L-1)(L+1)}\right)^{1/2}\frac{2D_{L-2m'Lm}}{\lambda^{(0)}_{L-2\,m'}-\lambda^{(0)}_{Lm}}\delta_{l'\,l-1}\delta_{n'n}\delta_{p'p}\,.\end{aligned}\tag{36}$$

This gives us a "representation" of $g^{(1)}_{lmnp}$ in terms of the complete set $g^{(0)}_{lmnp}$. For with the help of the completeness property

$$\sum_{l',m',n',p'}g^{(0)}_{l'm'n'p'}(\vec{x})g^{(0)*}_{l'm'n'p'}(\vec{x}')=\delta^4(\vec{x}-\vec{x}')\,, \tag{37}$$

we find immediately, on multiplying both sides of the equation by $g^{(1)}_{lmnp}(\vec{x}')$ and integrating,

$$g^{(1)}_{lmnp}(\vec{x})=\sum_{l',m',n',p'}A^{l'm'n'p'}_{lmnp}g^{(0)}_{l'm'n'p'}(\vec{x})\,, \tag{38}$$

in which the coefficients $A^{l'm'n'p'}_{lmnp}$ are just the quantities given by Eq. (36), so long as $m\neq m'$. We note that the nonvanishing contributions to $A^{l'm'n'p'}_{lmnp}$ are all real. When $m=m'$ and $l=l'$ the corresponding coefficients cannot be obtained in the same way. However, it follows from Eq. (27), when we take the squared modulus of both sides and integrate, and then make use of the normalization of both the g_{lmnp} and $g^{(0)}_{lmnp}$ functions, that

$$A^{lmnp}_{lmnp}=O(\epsilon^2)\,, \tag{39}$$

so that A^{lmnp}_{lmnp} may be taken as zero to the usual degree of approximation. Equations (36) and (39) together suffice to determine the principal correlation functions of the laser field, as we now show. Indeed, we shall find that the only contributing coefficients $A^{l'm'n'p'}_{lmnp}$ are those for which l, l', n, n', p, $p'=0$, 1 and that the others are not needed. A drastic simplification occurs in Eq. (36) when $n'=0=n$, $p'=0=p$, and $l=l'$, in which case we find

$$A^{lm'00}_{lm00}=0\quad\text{for all } l, m', m\,. \tag{40}$$

IV. INTENSITY CORRELATIONS

From Eq. (9) it follows immediately that the two-time intensity (auto- or cross-) correlation function of the field is given by

$$\langle I_{k'}(t)I_k(t+\tau)\rangle=\int\tfrac{1}{2}u'[1-(-1)^{k'}v']\tfrac{1}{2}u[1-(-1)^kv]p_2(\vec{x},t+\tau;\vec{x}',t)d^4x\,d^4x'\,,\quad\text{with } k,k'=1,2,$$

and with the help of Eqs. (15), (27), and (38) this becomes

$$\begin{aligned}\langle I_{k'}(t)I_k(t+\tau)\rangle=&\int\tfrac{1}{4}uu'[1-(-1)^{k'}v'][1-(-1)^kv]\sum_{l,m,n,p}[p_s(\vec{x})p_s(\vec{x}')]^{1/2}\\&\times\Big(g^{(0)*}_{lmnp}(\vec{x})g^{(0)}_{lmnp}(\vec{x}')+\epsilon g^{(0)*}_{lmnp}(\vec{x})\sum_{l',m',n',p'}A^{l'm'n'p'}_{lmnp}g^{(0)}_{l'm'n'p'}(\vec{x}')\\&+\epsilon g^{(0)}_{lmnp}(\vec{x}')\sum_{l',m',n',p'}A^{l'm'n'p'}_{lmnp}g^{(0)*}_{l'm'n'p'}(\vec{x})\Big)\exp(-\lambda_{lmnp}|\tau|)d^4x\,d^4x'\,.\end{aligned}\tag{41}$$

We now introduce the explicit expressions for $p_s(\vec{x})$ and $g_{lmnp}(\vec{x})$ given by Eqs. (8) and (19), respectively, and obtain, after using Eq. (11) and integrating with respect to the four phase angles, to the first order in ϵ,

$$\langle I_{k'}(t)I_k(t+\tau)\rangle = \frac{\pi^2}{64N}\iint_0^\infty du\,du'\,u^2u'^2\exp(\tfrac{1}{4}au-\tfrac{1}{8}u^2+\tfrac{1}{4}au'-\tfrac{1}{8}u'^2)\iint_{-1}^1 dv\,dv'\,[1-(-1)^{k'}v'][1-(-1)^kv]$$
$$\times\sum_{l,m}\Big(M_{l00}P_l^{(0,0)}(v)P_l^{(0,0)}(v')R_{LM}(u)R_{LM}(u')[1+\tfrac{1}{8}\epsilon(uv+u'v')]$$
$$+\epsilon\sum_{l',m'}(M_{l00}M_{l'00})^{1/2}A_{lm00}^{l'm'00}[P_l^{(0,0)}(v)P_{l'}^{(0,0)}(v')R_{LM}(u)R_{L'm'}(u')$$
$$+P_l^{(0,0)}(v')P_{l'}^{(0,0)}(v)R_{L'm'}(u)R_{LM}(u')]\Big)\exp(-\lambda_{lm00}|\tau|)\,. \tag{42}$$

In arriving at this expression we have expanded the exponential $\exp(\epsilon uv)$ to the first order in ϵ, which requires that ϵa be small, since u is of order a for large positive a. Now $M_{l00}=l+\frac{1}{2}$, and $P_l^{(0,0)}(v)=P_l(v)$, which is the Legendre polynomial. Also, from the properties of the Legendre polynomials we have[6]

$$\int_{-1}^1[1-(-1)^kv]P_l(v)dv=2\delta_{l0}-\tfrac{2}{3}(-1)^k\delta_{l1}\,, \tag{43a}$$

and

$$\int_{-1}^1[1-(-1)^kv]vP_l(v)dv=\tfrac{2}{3}[\delta_{l1}-(-1)^k(\delta_{l0}+\tfrac{2}{5}\delta_{l2})]\,. \tag{43b}$$

When this is inserted in Eq. (42) and we make use of the orthonormality of the Legendre polynomials, together with Eq. (35b), we find

$$\langle I_{k'}(t)I_k(t+\tau)\rangle=\tfrac{1}{32}\pi^2\sum_m\Big[(K_{0m}^{(2)})^2\exp(-\lambda_{0m}^{(0)}|\tau|)+\tfrac{1}{3}(-1)^{k+k'}(K_{2m}^{(2)})^2\exp(-\lambda_{2m}^{(0)}|\tau|)$$
$$+\epsilon\sum_{m'}\Big(2K_{0m}^{(2)}K_{0m'}^{(2)}A_{0m00}^{0m'00}-\frac{(-1)^k+(-1)^{k'}}{\sqrt{3}}K_{0m}^{(2)}K_{2m'}^{(2)}A_{0m00}^{1m'00}\Big)\exp(-\lambda_{0m}^{(0)}|\tau|)$$
$$+\epsilon\sum_{m'}\Big(\tfrac{2}{3}(-1)^{k+k'}K_{2m}^{(2)}K_{2m'}^{(2)}A_{1m00}^{1m'00}-\frac{(-1)^k+(-1)^{k'}}{\sqrt{3}}K_{2m}^{(2)}K_{0m'}^{(2)}A_{1m00}^{0m'00}\Big)\exp(-\lambda_{2m}^{(0)}|\tau|)$$
$$-\tfrac{1}{24}\epsilon[(-1)^k+(-1)^{k'}][K_{0m}^{(2)}K_{0m}^{(3)}\exp(-\lambda_{0m}^{(0)}|\tau|)+K_{2m}^{(2)}K_{2m}^{(3)}\exp(-\lambda_{2m}^{(0)}|\tau|)]\Big]\,, \tag{44}$$

in which we have introduced the following abbreviation for the u integrals:

$$K_{Lm}^{(r)}\equiv\frac{1}{\sqrt{N}}\int_0^\infty u^r\exp(\tfrac{1}{4}au-\tfrac{1}{8}u^2)R_{Lm}(u)\,du\,. \tag{45}$$

However, from Eq. (40) it follows that several of the coefficients A vanish, so that finally, with the help of Eq. (36), we have

$$\langle I_{k'}(t)I_k(t+\tau)\rangle=\tfrac{1}{32}\pi^2\sum_{m=0}^\infty\Big[(K_{0m}^{(2)})^2\exp(-\lambda_{0m}^{(0)}|\tau|)+\tfrac{1}{3}(-1)^{k+k'}(K_{2m}^{(2)})^2\exp(-\lambda_{2m}^{(0)}|\tau|)+\tfrac{1}{3}\epsilon[(-1)^k+(-1)^{k'}]$$
$$\times\sum_{m'=0}^\infty\Big(K_{0m}^{(2)}K_{2m'}^{(2)}\frac{D_{2m'0m}}{(\lambda_{2m'}^{(0)}-\lambda_{0m}^{(0)})}\exp(-\lambda_{0m}^{(0)}|\tau|)+K_{2m}^{(2)}K_{0m'}^{(2)}\frac{D_{0m'2m}}{(\lambda_{0m'}^{(0)}-\lambda_{2m}^{(0)})}\exp(-\lambda_{2m}^{(0)}|\tau|)\Big)$$
$$-\tfrac{1}{24}\epsilon[(-1)^k+(-1)^{k'}][K_{0m}^{(2)}K_{0m}^{(3)}\exp(-\lambda_{0m}^{(0)}|\tau|)+K_{2m}^{(2)}K_{2m}^{(3)}\exp(-\lambda_{2m}^{(0)}|\tau|)]\Big]\,. \tag{46}$$

TABLE I. Some selected values of D_{LmLm}.

	a	D_{L0L0}	D_{L1L1}	D_{L2L2}	D_{L3L3}	D_{L4L4}
$L=0$	0	−1.00	−1.31	−1.82	−2.34	−2.86
	2	−1.00	−0.89	−1.33	−1.80	−2.26
	4	−1.00	−0.36	−0.62	−1.09	−1.52
	6	−1.00	−0.04	0.52	−0.20	−0.58
	8	−1.00	−0.01	1.55	0.70	0.82
	10	−1.00	−0.01	1.26	1.13	2.98
$L=1$	0	−1.29	−1.62	−2.11	−2.62	−3.14
	2	−1.18	−1.18	−1.58	−2.04	−2.50
	4	−1.09	−0.61	−0.82	−1.28	−1.72
	6	−1.04	−0.17	0.30	−0.28	−0.73
	8	−1.03	−0.05	1.27	1.14	0.56
	10	−1.01	−0.03	1.15	2.01	2.47
$L=2$	0	−1.59	−1.92	−2.39	−2.90	−3.41
	2	−1.38	−1.45	−1.83	−2.28	−2.74
	4	−1.21	−0.85	−1.04	−1.47	−1.91
	6	−1.11	−0.33	0.06	−0.38	−0.86
	8	−1.06	−0.12	0.99	1.28	0.44
	10	−1.04	−0.07	1.07	2.95	1.78

The first two terms correspond to the solution that was obtained previously[2,4] for the symmetric ring laser, whereas the remaining terms represent a correction when the pump parameters are unequal. If we are concerned with the cross correlation of the two mode intensities, so that $k \neq k'$, then the remaining terms vanish because $(-1)^k + (-1)^{k'} = 0$. Hence the asymmetry has no effect, to the first order in ϵ, on the cross correlation of the light intensities. However, it does modify the autocorrelation functions. The constant N is given by Eq. (7b) to the first order in ϵ.

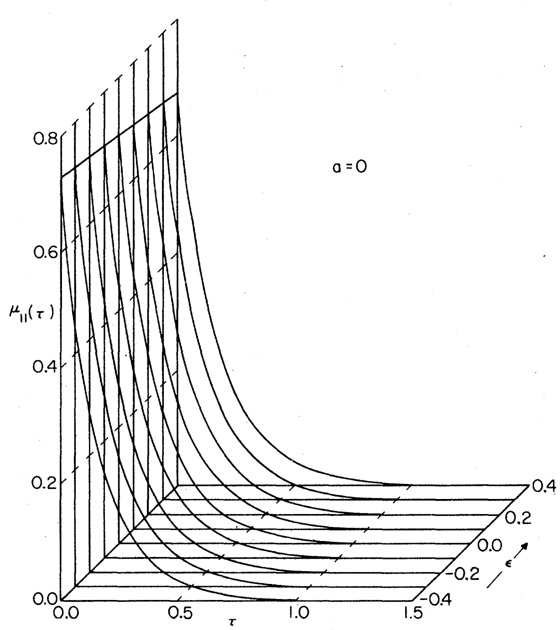

FIG. 1. Normalized intensity correlation function $\mu_{11}(\tau)$ as a function of τ for several values of ϵ, with average pump parameter $a = 0$.

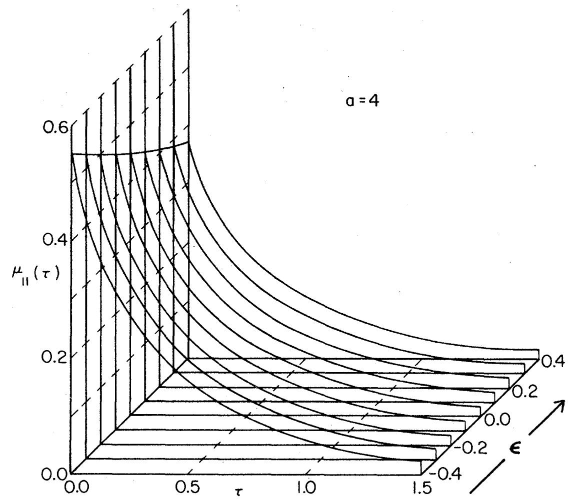

FIG. 2. Normalized intensity correlation function $\mu_{11}(\tau)$ as a function of τ for several values of ϵ, with average pump parameter $a = 4$.

We may obtain some useful approximations, both for the eigenvalues $\lambda_{L0}^{(0)}$ and for the coefficients K_{L0} and $D_{l'0l0}$ appearing in the leading term in Eq. (46) for large pump parameters a, by noting that the eigenfunctions $R_{L0}(u)$ are found not to depend too much on L for large positive a, and to be approximately equal to $R_{00}(u)$. Now the zeroth order eigenfunction $R_{00}(u)$ is proportional to the square root of the steady-state solution given by Eq. (8) (with $\xi = 1$),

$$R_{00}(u) = (2^{3/2}\pi/N^{1/2}) \exp(\tfrac{1}{4}au - \tfrac{1}{8}u^2), \qquad (47)$$

which is strongly peaked in the neighborhood of $u = a$. However, from Eqs. (21) and (25), $\psi_{Lm}(y)$, which is simply related to $R_{Lm}(u)$ $(u = y^2)$, satisfies the differential equation

$$\frac{d^2\psi_{L0}}{dy^2} + \left(\lambda_{L0}^{(0)} - \frac{L(L+2)}{y^2} + f(a, y)\right)\psi_{L0} = 0,$$

whereas ψ_{00} satisfies a similar differential equation with eigenvalue $\lambda_{00}^{(0)} = 0$,

$$\frac{d^2\psi_{00}}{dy^2} + f(a, y)\psi_{00} = 0.$$

The similarity of ψ_{L0} and ψ_{00} for large a then suggests that $\lambda_{L0}^{(0)}$ should be well approximated by

$$\lambda_{L0}^{(0)} \approx L(L+2)/a \qquad (48)$$

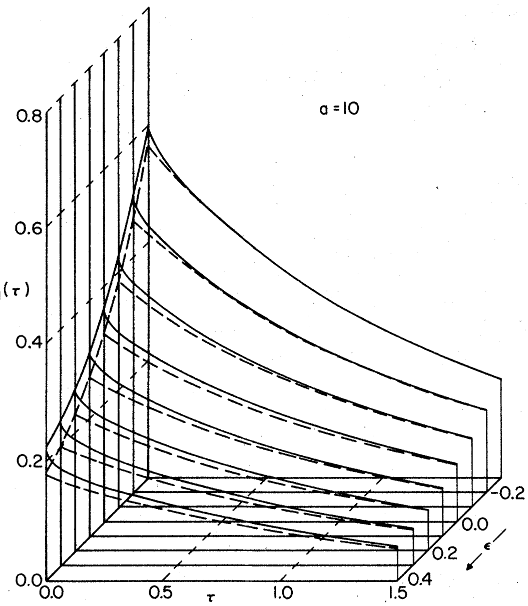

FIG. 3. Normalized intensity correlation function $\mu_{11}(\tau)$ as a function of τ for several values of ϵ, with average pump parameter $a = 10$. The broken curves are derived from Eq. (51).

for large a, and that, from Eq. (45),

$$K^{(r)}_{L0} \approx K^{(r)}_{00} = \frac{2^{3/2}}{\pi}\,\frac{\int_0^\infty u^r \exp[-\frac{1}{4}(u-a)^2]du}{\int_0^\infty u \exp[-\frac{1}{4}(u-a)^2]du}$$

$$\approx \frac{2^{3/2}}{\pi} a^{r-1} \quad \text{for } a \gg 1\,,\ r \geq 1\,. \qquad (49)$$

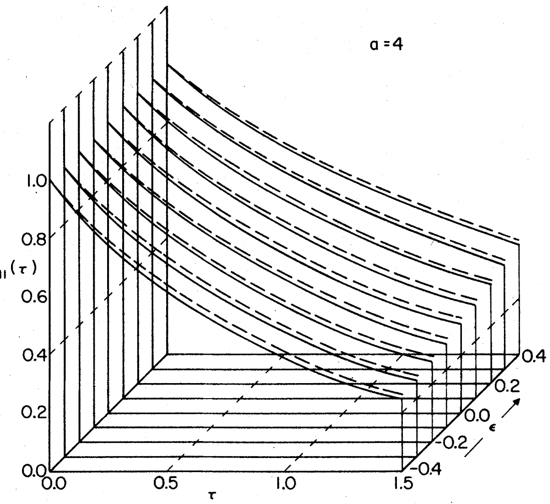

FIG. 5. Normalized correlation function $\gamma_{11}(\tau)$ of the complex field amplitude as a function of τ for several values of ϵ, with average pump parameter $a = 4$. The broken curves are derived from Eq. (56).

Also, for large a, the coefficient $D_{L'0L0}$ that appears in the leading term in Eq. (46) can be approximated by

$$D_{L'0L0} \approx D_{0000}$$

$$= \frac{\frac{1}{4}\int_0^\infty u^2(a-u)\exp[-\frac{1}{4}(u-a)^2]\,du}{\int_0^\infty u \exp[-\frac{1}{4}(u-a)^2]du} = -1\,. \qquad (50)$$

Thus the leading term in the expansion (46) for

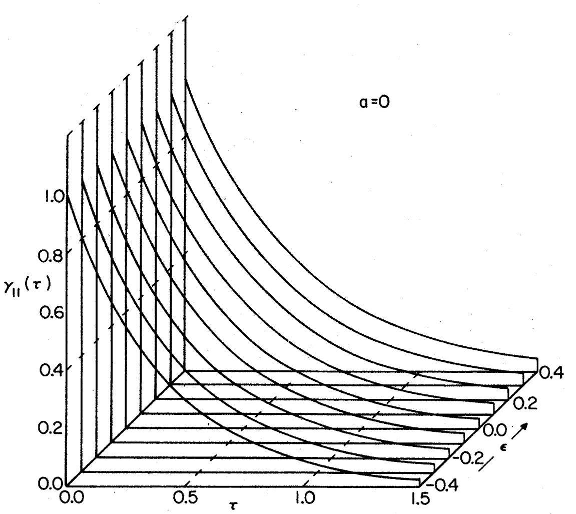

FIG. 4. Normalized correlation function $\gamma_{11}(\tau)$ of the complex field amplitude as a function of τ for several values of ϵ, with average pump parameter $a = 0$.

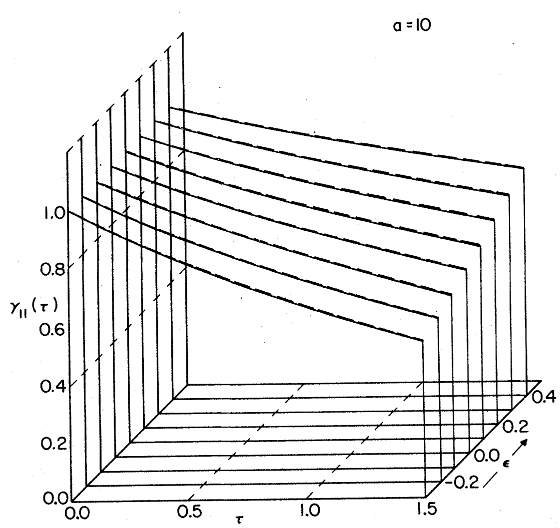

FIG. 6. Normalized correlation function $\gamma_{11}(\tau)$ of the complex field amplitude as a function of τ for several values of ϵ, with average pump parameter $a = 10$. The broken curves are derived from Eq. (56).

the autocorrelation function yields, for large a but with ϵa still small,

$$\langle I_{\frac{1}{2}}(t) I_{\frac{1}{2}}(t+\tau)\rangle \approx \tfrac{1}{4} a^2 [1 \pm \tfrac{1}{6}\epsilon a + \tfrac{1}{3}\exp(-8\tau/a) + \cdots] . \tag{51}$$

Figures 1–3 show the variation of the normalized autocorrelation function $\mu_{11}(\tau) \equiv \langle I_1(t) I_1(t+\tau)\rangle / \langle I_1\rangle^2 - 1$ as a function of τ, for several values of the average pump parameter a and the asymmetry parameter ϵ. Increasing asymmetry produces a relatively small effect near threshold, but a much larger effect well above threshold, as is to be expected from the fact that the light in the stronger mode becomes coherent and that in the weaker mode becomes incoherent. The approximation given by Eq. (51) is shown by dotted curves.

V. FIELD AMPLITUDE CORRELATIONS

We now calculate the autocorrelation function of the complex field of one of the laser modes, whose Fourier transform yields the spectral distribution. The cross correlation function $\langle E_1^*(t) E_2(t+\tau)\rangle$ vanishes, because the phases of the randomly fluctuating Langevin forces were assumed to be independent.

From Eq. (9)

$$\langle E_1^*(t) E_1(t+\tau)\rangle = \int \tfrac{1}{2}[uu'(1+v)(1+v')]^{1/2} e^{i(\phi_1 - \phi_1')} p_2(\bar{x}, t+\tau; \bar{x}', t) d^4x\, d^4\bar{x}' ,$$

and, with the help of Eqs. (15), (27), and (38) as before, this becomes

$$\begin{aligned}\langle E_1^*(t) E_1(t+\tau)\rangle = \int \tfrac{1}{2}[uu'(1+v)(1+v')]^{1/2} e^{i(\phi_1-\phi_1')} \sum_{l,m,n,p} [p_s(\bar{x}) p_s(\bar{x}')]^{1/2} \\ \times\Big(g_{lmnp}^{(0)*}(\bar{x}) g_{lmnp}^{(0)}(\bar{x}') + \epsilon g_{lmnp}^{(0)*}(\bar{x}) \sum_{l',m',n',p'} A_{lmnp}^{l'm'n'p'} g_{l'm'n'p'}^{(0)}(\bar{x}') \\ + \epsilon g_{lmnp}^{(0)}(\bar{x}') \sum_{l',m',n',p'} A_{lmnp}^{l'm'n'p'} g_{l'm'n'p'}^{(0)*}(\bar{x})\Big) \exp(-\lambda_{lmnp}|\tau|) .\end{aligned}$$

On substituting for $p_s(\bar{x})$ and $g_{lmnp}^{(0)}(\bar{x})$ from Eqs. (8) and (19) and integrating over phase angles, we obtain

$$\begin{aligned}\langle E_1^*(t) E_1(t+\tau)\rangle = \frac{\pi^2}{32N} \iint_0^\infty du\, du' (uu')^{3/2} \exp(\tfrac{1}{4}au - \tfrac{1}{8}u^2 + \tfrac{1}{4}au' - \tfrac{1}{8}u'^2) \iint_{-1}^1 dv\, dv' (1+v)(1+v') \\ \times \sum_{l,m} \Big(M_{l10} P_l^{(0,1)}(v) P_l^{(0,1)}(v') R_{Lm}(u) R_{Lm}(u') [1 + \tfrac{1}{8}\epsilon(uv + u'v')] \\ + 2\epsilon \sum_{l',m'} (M_{l10} M_{l'10})^{1/2} P_l^{(0,1)}(v) P_{l'}^{(0,1)}(v') A_{lm10}^{l'm'10} R_{2l+1}(u) R_{2l'+1}(u')\Big) \exp(-\lambda_{lm10}|\tau|) .\end{aligned} \tag{52}$$

We now make use of Eq. (31) together with the orthonormality of the Jacobi polynomials expressed by

$$\int_{-1}^1 (1-v)^p (1+v)^n P_l^{(p,n)}(v) P_{l'}^{(p,n)}(v) dv = \frac{1}{M_{lnp}} \delta_{ll'} , \tag{53}$$

and choose $n=1, p=0, l'=0$. As $P_0^{(0,1)}(v) = 1$, the integrals in Eqs. (52) can be evaluated immediately, and we find with the help of Eqs. (36) and (39)

$$\langle E_1^*(t) E_1(t+\tau)\rangle = \tfrac{1}{16}\pi^2 \sum_{m=0}^\infty \Big((K_{1m}^{(3/2)})^2 + \tfrac{1}{12}\epsilon K_{1m}^{(3/2)} K_{1m}^{(5/2)} - \tfrac{2}{3}\epsilon \sum_{m'\neq m} \frac{D_{1m'1m}}{(\lambda_{1m'}^{(0)} - \lambda_{1m}^{(0)})} K_{1m'}^{(3/2)} K_{1m}^{(3/2)} \Big) \exp(-\lambda_{0m10}|\tau|) , \tag{54}$$

where $K_{Lm}^{(r)}$ is defined by Eq. (45) as before.

The first term agrees with the expression found previously[2,4] for the symmetric two-mode ring laser when $\epsilon = 0$, whereas the term in ϵ represents a correction when the two pump parameters are unequal. (Note that the eigenvalue λ_{0m10} was denoted by λ_{1m10} in Ref. 2). We emphasize, however, that the new eigenvalues λ_{0m10} differ from the corresponding eigenvalues $\lambda_{1m}^{(0)}$ for the symmetric ring laser, as is clear from Eqs. (28) and

(34). We find for large pump parameter a, with the help of the approximation given by Eqs. (48) and (50), for the first eigenvalue in the expansion,

$$\lambda_{0010} = \lambda_{10}^{(0)} + \epsilon\,\lambda_{0010}^{(1)} \approx (3/a)(1 - \tfrac{1}{9}\,\epsilon a)\,, \qquad (55)$$

provided ϵa is small. With the help of Eqs. (49) and (50) the leading term in the expansion (54) then yields

$$\langle E_1^*(t)E_1(t+\tau)\rangle = \tfrac{1}{2}\,a(1 + \tfrac{1}{12}\,\epsilon a) \times \exp[-(3\tau/a)(1 - \tfrac{1}{9}\epsilon a)] + \cdots\,. \qquad (56)$$

Some curves illustrating the variation of the normalized correlation function

$$\gamma_{11}(\tau) \equiv \langle E_1^*(t)E_1(t+\tau)\rangle / \langle I_1\rangle$$

given by Eq. (54), with increasing difference ϵ of the pump parameters, are shown in Figs. 4–6. The other correlation function $\langle E_2^*(t)E_2(t+\tau)\rangle$ is of course given by a similar expression with ϵ replaced by $-\epsilon$. The effect of increasing asymmetry is most noticeable well above threshold, but is relatively small in the neighborhood of the threshold. The approximation given by Eq. (56) is shown by the dotted curves and is seen to be an excellent approximation for large pump parameters.

We have, therefore, solved the general problem of determining correlation functions for the field of a ring laser with unequal pump parameters, at least for small differences. In practice the difference is due largely to an asymmetry in the diffraction losses and is usually small.

ACKNOWLEDGMENT

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[1]Some of the principal early contributions to the theory were by F. Aronowitz, Phys. Rev. 139, A635 (1965) and Appl. Opt. 11, 2146 (1972); V. E. Privalov and S. A. Fridrikhov, Usp. Fiz. Nauk. 97 (1969) [Sov. Phys. Usp. 12, 153 (1969)]; and L. Menegozzi and W. E. Lamb, Jr., Phys. Rev. A 8, 2103 (1973). See also H. Haken, *Laser Theory* (Springer, Heidelberg, 1970); M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, Mass., 1974); and Refs. 2 and 4 for a more complete list of references.

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[6]See, for example, M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), Chap. 22.

[7]For the calculation of the most important correlation functions of the laser field, only the combinations for which $n = 0 = p$ and $n = 1$, $p = 0$ or $n = 0$, $p = 1$ are needed, and were treated in Ref. 2. For these cases the Jacobi polynomials are easily expressed in terms of Legendre polynomials. However, we find it convenient here to retain the more general notation with n, p arbitrary. The Jacobi polynomials $P_l^{(p,n)}$ are usually defined for non-negative integers n, p, but they can be generalized to negative integer values with some restrictions. This subject has been discussed in detail by H. Bateman in *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 2, p. 253.