Classical theory of the free-electron laser in a moving frame

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This paper presents a fully classical theory of the free-electron laser. The theory is formulated in a moving frame where the pseudoradiation field due to a static wiggler magnet is in resonance with the laser field imposed by an optical cavity. The basic amplification process is then one of elastic scattering. In this frame both fields are treated classically, as is the electron motion. When the laser and wiggler fields are taken to be constant during the interaction period, it is shown that the laser operation can be described by the classical pendulum equation. This is used to evaluate the unsaturated gain and momentum distribution. For the saturated situation numerical results are given, which have been obtained by the use of the Jacobi elliptic functions. The new features found are singularities in the momentum distribution and their evolution with intensity, saturation enhancement of the gain and the possibilities of bistability and hysteresis entailed by it, and the possibility of discussing and interpreting the various features in a unified way. Finally, a detailed discussion of the approximations and their validity is given.

I. INTRODUCTION

A free electron in vacuum cannot radiate energy. Indeed such a process would not conserve both energy and momentum, which is necessary in every physical process.

On the contrary, when the electron travels through a static periodic structure, such as a static magnetic field, emission of radiation can occur, provided that the periodic structure provides the necessary momentum. In the laboratory frame, this process can be described as a scattering process of a quasiphoton of the periodic field into the radiation field.

Recent experimental results[1] were obtained by utilizing a relativistic electron beam traversing a periodic magnetic field (wiggler) to obtain coherent radiation in the infrared region. The use of relativistic electrons is expedient because then a periodic structure of macroscopic dimensions (centimeters) can be used to obtain optical or infrared frequencies.

This process can be regarded in the laboratory frame as magnetic bremsstrahlung from the periodic field. In a series of papers[2] Hopf *et al.* have written a classical Vlasov equation for the electron cloud, and used this to solve for the basic properties of the scattering process. Relativistic considerations of the motion of the electrons have been given.[3]

A different approach has been used by the present authors.[4] In a properly chosen reference frame the wiggler field has been treated as a radiation field (referred to as pseudoradiation field). This approximation, known as the Weizsäcker-Williams approximation, turns out to be valid for ultrarelativistic electrons. Then the process has been described as a stimulated scattering process from the pseudoradiation field into a true radiation field traveling in the same direction as the electron beam. For a physically appealing description of the process, a moving frame has been chosen in such a way that the periodic structure transforms into a (pseudo) radiation field whose frequency coincides with the frequency of the stimulating field. Thus the process is essentially a laser process, where the source of radiation comes from a scattering process, rather than a stimulated emission process as in previous operating laser devices. To be consistent with a classical picture of the fields, the stimulating (laser) field has been assumed to be large, so that fluctuations can be neglected.

There are several reasons to choose that reference frame for the description of the laser process. First, the physical processes of scattering from one field to the other and vice versa become apparent. Indeed, in that frame, the two fields are treated on the same step, although they are quite different in the laboratory frame. Second, relativistic calculations can be avoided. In fact, in that frame, the electrons have nonrelativistic velocity, and the momentum exchanged with the fields is not sufficient to give the electrons a relativistic velocity. Third, in the limit in which the laser operation can be described in terms of ensemble averages over independent single particles, it becomes possible to follow the history for each electron in the field with simple equations of motion. These equations, for a wide range of operations reduce to the pendulum equations, which are not Lorentz invariant and are valid in that frame only.

In this work we describe the free-electron laser (FEL) as a single-particle process, in contradistinction to the approach chosen in Ref. 2. We have shown elsewhere[5] how the two approaches can be reconciled starting from a quantum-mechanical density-matrix description of the system.

Recent theoretical results by the Stanford group[6] present an extensive investigation of the properties of a free-electron laser. They use a different approach to evaluate the laser interaction; where we have independently derived the same properties, there seems to be full agreement between our results.

In this paper we formulate the fundamental model to be used in describing the FEL. We use the model to obtain the basic operating characteristics of the laser to show the agreement with earlier work; the new results of the paper concern the strong signal behavior of the FEL. It is shown that the present model can be used to calculate the gain and final momentum distribution of the operating laser. The latter is of great importance for the operation of a real device, because the laser field causes a momentum dispersion greatly exceeding the gain, over most of the range of operation of the laser.

In Sec. II the moving frame is defined in some detail; this part of our paper gives the basis also for the work presented in Refs. 4 and 5. The Hamiltonian to be used in the completely classical model is given in Sec. III and also its basic conservation laws. The corresponding quantum-mechanical expressions are discussed in Ref. 5. Laser operation well above threshold is considered. In this condition the fields are assumed to be classical, so that their fluctuations can be ignored. Therefore we can expect that each electron contributes only little to the total laser intensity and follows the fields adiabatically. Under this approximation, which is introduced in Sec. IV, the behavior of the FEL can be described by the pendulum equations. A perturbation approach is used in Sec. V to obtain the small signal gain and the lowest-order momentum spread. These results agree with those earlier derived in Ref. 2. In Sec. VI we show how the electron equation of motion can be solved exactly in terms of the Jacobi elliptic functions. These expressions can be used to evaluate the behavior of the system numerically for any laser intensity. The results of such calculations are presented in Sec. VII and some implications for the laser operation are discussed.

II. FRAME OF REFERENCE

In formulating the fully classical theory of the FEL we assume the laser field to be large enough to allow a classical description. This excludes operation very close to threshold where quantum fluctuations of the laser field are important. The field is chosen to be a circularly polarized mode of a ring cavity. Thus we can assume a traveling wave eigenmode with wavelength λ_L.

The static magnetic field provided by the wiggler magnet, with spatial period $\lambda_q = 2\pi k_q^{-1}$ is taken to be circularly polarized with the components

$$B[B_0 \cos(k_q z), B_0 \sin(k_q z), 0] \,. \tag{2.1}$$

Our assumption that B_0 is a constant can be valid over a limited region only because the field must satisfy the relations

$$\vec{\nabla}\cdot\vec{B}=0, \quad \vec{\nabla}\times\vec{B}=0\,. \tag{2.2}$$

It can be shown[7] that assuming the electron trajectories to be confined to a region much smaller than λ_q around the wiggler axis, we can neglect the spatial variation of B_0. A schematic diagram of the experiment is shown in Fig. 1.

The particles of an ultrarelativistic electron beam traverse the wiggler with an axial velocity very close to the velocity of light c. In the rest frame of the electron the wiggler field transforms into an electric as well as a magnetic component with the ratio close enough to c to justify treating it as a radiation field propagating towards the electron beam. This will be equivalent to a pseudoradiation (PR) field, which in the laboratory frame has wavelength

$$\lambda^* = \lambda_q(1+\beta_e)\,, \tag{2.3}$$

where β_e is the ratio of the axial velocity of the electron to the velocity of light. For an ultrarelativistic beam, $\beta_e \sim 1$ and

$$\lambda^* = 2\lambda_q\,. \tag{2.4}$$

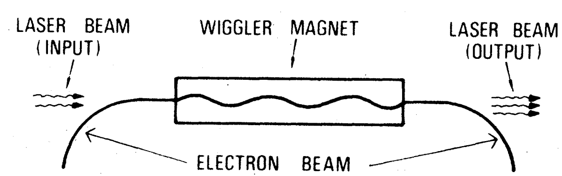

FIG. 1. Experimental layout.

The factor of 2 in Eq. (2.4) can be understood in a very natural way. In the rest frame the wiggler passes the electron with nearly the velocity of light, and the interaction time is $\tau = L/c$, where

$$L = L_0(1-\beta_e^2)^{1/2}, \tag{2.5}$$

and L_0 is the length of the wiggler in the laboratory frame. Transformed back to the laboratory frame the wiggler field appears as a squared pulse of duration

$$\tau_0 = (L_0/c)(1+\beta_e) \approx 2L_0/c. \tag{2.6}$$

Because the number of periods along the wiggler is an invariant, Eq. (2.6) shows that the effective frequency is halved which directly gives (2.4).

In the laboratory frame, the electron beam propagates towards the PR field and scatters its energy into the laser field which propagates in the same direction as the electrons. From the point of view of the PR field, this is backward scattering. The laser field has a wavelength much shorter than the wiggler field, but their frequency difference is compensated by the Doppler shift due to the motion of the electrons.

It proves expedient to formulate the theory in such a frame that the scattering of radiation occurs at resonance. We perform a Lorentz transformation to a frame moving with velocity V in the direction of the electron beam. The relativistic Doppler effect transforms the laser wavelength into

$$\lambda_L' = (1+V/c)\lambda_L\gamma \approx 2\lambda_L\gamma, \tag{2.7}$$

and the wavelength of the counterpropagating PR field into

$$\lambda' = \frac{\lambda^*}{(1+V/c)\gamma} \approx \tfrac{1}{2}\lambda^*\gamma^{-1} \approx \lambda_q\gamma^{-1}, \tag{2.8}$$

where we have assumed $V \approx c$ and

$$\gamma^{-2} = 1-(V/c)^2. \tag{2.9}$$

Scattering of energy from one field to the other appears at resonance if we choose the frame such that

$$\lambda' = \lambda_L', \tag{2.10}$$

which by (2.7) and (2.8) gives

$$V = c(1-2\lambda_L/\lambda_q)^{1/2}. \tag{2.11}$$

This is the basic relation defining our frame of reference. The velocity V of this frame is determined solely by the ratio of the laser wavelength to the period of the static wiggler field and *not* by the velocity of the electron. When we tune the laser frequency by changing the external cavity, the frame must be changed, but it is always possible to find one where (2.11) is satisfied assuming

$$\lambda_L < \tfrac{1}{2}\lambda_q, \tag{2.12}$$

which is well satisfied for optical frequencies. In this paper, we make the additional assumption that V is close enough to the velocity of the electrons, to allow a nonrelativistic treatment of the problem in the moving frame chosen.

We will find that in order to achieve amplification of the laser field through stimulated Thomson scattering the electrons must move faster than the reference frame, i.e., their velocity in the moving frame must be positive. Consequently the factor β_e of Eqs. (2.3) and (2.5) must satisfy

$$\beta_e > V/c = \gamma^{-1}(\gamma^2-1)^{1/2}. \tag{2.13}$$

Relating the laser wavelength to that of the wiggler, we find in the laboratory frame the relation

$$\lambda_L = \lambda_q/2\gamma^2, \tag{2.14}$$

where Eqs. (2.10), (2.7), and (2.8) have been used. The spontaneous emission by the electron will take place at the frequency determined by the wiggler field in the rest frame of the electron; the spontaneous emission in the laboratory will be centered at wavelength

$$\lambda_{\text{spont}} = (1-\beta_e)\lambda_q \approx \tfrac{1}{2}\lambda_q(1-\beta_e^2) < \lambda_L \tag{2.15}$$

by the use of (2.13) and (2.14). If the velocity of the electrons entering the wiggler is kept constant the transition from spontaneous to stimulated radiation can be recognized by a red shift of the emerging radiation.

In the rest of this paper we will discuss the theory of the FEL in the frame moving with velocity V. In this frame the radiation scatters elastically and the wave vectors have the same modulus

$$k = \omega/c = 2\pi/\lambda' = 2\pi/\lambda_L', \tag{2.16}$$

but the two fields travel in opposite directions. In Appendix A we present a table where we express the physically interesting quantities for the laboratory frame in terms of their values calculated in the moving frame.

III. HAMILTONIAN

In the reference frame described in the previous paragraph, we assume the validity of a nonrelativistic description of the amplification process in the FEL. The axial velocity V of the electron in this frame is much smaller than c, but its transverse velocity can become very large (even relativistic) because of the large wiggler field. This effect can be taken into account in our nonrelativistic formulation by simply redefining the value of the mass m, which enters the equations. In place of electron rest mass m_0 we have a mass parameter which depends on the parameters of the static magnetic field. The detailed justification of this replacement is given in Appendix B.

Our nonrelativistic approach in the moving frame is justified for laser action throughout the ir region of the spectrum; in the optical and especially the near-uv region its validity must be reconsidered. For extremely large magnitudes of the laser or wiggler fields additional relativistic effects may become important.

We describe the interaction between the electron beam and the radiation fields in the Coulomb gauge, where the vector potential $\vec{A}$ is transverse. In the geometry of the FEL this property is invariant with respect to Lorentz transformations along the direction of the beam. The static Coulomb potential φ can be set equal to zero, when we neglect the space-charge effects due to electrostatic interaction within the electron beam.

In this situation it is possible to write a completely classical single-particle Hamiltonian for the system

$$H=\frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2+\tfrac{1}{2}(\vec{P}_L'^2+\omega^2\vec{Q}_L'^2)+\tfrac{1}{2}(\vec{P}_W'^2+\omega^2\vec{Q}_W'^2). \tag{3.1}$$

The radiation fields are represented as harmonic oscillators in terms of the canonical variables $\vec{Q}'$ and $\vec{P}'$ of the laser (L) and wiggler (W) fields. The first term of the Hamiltonian contains the energy of the electron and its interaction with the fields. The mass parameter is assumed to depend on the static field.

Expressing the vector potential $\vec{A}$ in terms of the canonical variables of the fields we write

$$\vec{A}=\frac{1}{k}\left(\frac{4\pi}{V}\right)^{1/2}[\omega(\vec{Q}_L'+\vec{Q}_W')\cos kz-(\vec{P}_L'-\vec{P}_W')\sin kz], \tag{3.2}$$

where the circular polarization of the fields are introduced and the frequencies ω and wave vectors k of the fields coincide in the frame chosen; see (2.16). Both fields are taken to consist of plane waves propagating in opposite directions. Hence the Hamiltonian does not depend on the transverse coordinates x and y, and the corresponding momenta p_x and p_y are constant of the motion. An initial transverse momentum of the electrons entering the wiggler will be conserved and make the electron drift out from the axis of the wiggler while traversing the interaction region. Then the electron would experience a wiggler field different from (2.1). Hence precautions have to be taken, that the electrons are injected into the wiggler properly. We shall assume p_x and p_y to be exactly zero throughout the interaction period. The only nonzero component of the momentum is p_z, and consequently the electron motion can be treated as one dimensional.

The two circularly polarized oppositely traveling fields, namely, the laser and wiggler fields, can be expressed in terms of the transverse x and y components as

$$\begin{aligned}\vec{P}'&=(1/\sqrt{2})(P\hat{x}-\omega Q\hat{y})\\ \vec{Q}'&=(1/\sqrt{2})[Q\hat{x}+(1/\omega)P\hat{y}]\,.\end{aligned} \tag{3.3}$$

This description holds for both fields.

The classical Hamiltonian system (3.1) has three degrees of freedom, and implies a system of six first-order equations of motion. It is possible to find simple and physically significant integrals of the motion by the use of a contact transformation of the Hamiltonian. The new dynamic variables of the fields are the actions I and the corresponding phases ϕ.

For each field the transformation is generated by the function

$$F(Q,\phi,t)=\tfrac{1}{2}\omega Q^2\,\mathrm{cotg}(\omega t+\phi)\,. \tag{3.4}$$

Making use of the standard transformation formulas relating the old and new coordinates, we write

$$\begin{aligned}P&=\frac{\partial F}{\partial Q}=\omega Q\,\mathrm{cotg}(\omega t+\phi)\,,\\ I&=-\frac{\partial F}{\partial\phi}=\frac{\omega}{2}Q^2\frac{1}{\sin^2(\omega t+\phi)}\,.\end{aligned} \tag{3.5}$$

Solving for the old coordinates, we find from (3.5) that

$$\begin{aligned}P&=(2I\omega)^{1/2}\cos(\omega t+\phi)\,,\\ Q&=(2I/\omega)^{1/2}\sin(\omega t+\phi)\,,\end{aligned} \tag{3.6}$$

and the Hamiltonian is given by

$$H'=H+\frac{\partial F}{\partial t}=\frac{p^2}{2m}+\frac{2\pi r_0c^2}{\omega V}[I_L+I_W+2(I_WI_L)^{1/2}\times\cos(\phi_L-\phi_W-2kz)]\,, \tag{3.7}$$

where r_0 is the classical radius of the electron (e^2/mc^2). From (3.6) it follows that the energy of the field

$$\tfrac{1}{2}(P^2+\omega^2Q^2)=\omega I \tag{3.8}$$

is expressed very simply in terms of the action. In the quantized field formulation,[5] the action I corresponds to the photon number. Henceforth the prime on the new Hamiltonian will be dropped.

The use of the new form of the Hamiltonian, (3.7) is exactly equivalent with (3.1) and introduces no approximations except those inherent in our basic model. The following noteworthy features are thus rigorous consequences of our basic approach. First, the Hamiltonian is time independent; hence energy is conserved during the interaction. Second, the phase variables of the fields and the electronic coordinate z appear only in the argument ψ of the trigonometric function, viz.,

$$\psi = \phi_L - \phi_W - 2kz\,. \tag{3.9}$$

This fact leads to additional conservation laws. The canonical equations of motion are

$$\dot{p} = -\frac{\partial H}{\partial z} = 2k\frac{\partial H}{\partial \psi}\,, \quad \dot{I}_L = -\frac{\partial H}{\partial \phi_L} = -\frac{\partial H}{\partial \psi}\,, \quad \dot{I}_W = \frac{\partial H}{\partial \psi}\,, \tag{3.10}$$

where (3.9) has been introduced. Combining these we find the relations

$$\mu \equiv p + k(I_L - I_W) = \text{const} \tag{3.11}$$

and

$$I_0 \equiv I_L + I_W = \text{const}. \tag{3.12}$$

Relation (3.11) expresses the conservation of longitudinal momentum and relation (3.12) the conservation of the transverse field energy in our frame where the scattering is elastic. These conservation laws have been discussed in the frame of a quantized theory by Bambini and Stenholm.[5]

Relation (3.11) shows that diverting energy from the PR field into the laser mode requires the electron to provide the necessary momentum. Relation (3.12) states that no energy is needed for this; we obtain gain for the laser field by losing on the wiggler.

We can use the constants of the motion to derive a single equation of motion for the longitudinal momentum of the electron. Denoting the conserved value of the energy (3.7) by E and using (3.10)–(3.12), we find

$$\dot{p}^2 = [(4\pi r_0 c/\omega V)]^2[(kI_0)^2 - (p-\mu)^2] - 4k^2[E - (p^2/2m + 2\pi r_0 c I_0/\omega V)]^2\,. \tag{3.13}$$

So far no approximations have been introduced; (3.13) is a rigorous consequence of (3.1). The values of the constants of the motion are defined by their initial values when an electron enters the wiggler at $t = t_0$ with $\psi = \psi_0$ and $p = p_0$. Integrating (3.13) with these initial values we get an exact trajectory for the electron through the wiggler. As the FEL operation involves an assembly of electrons, it is impossible to fix the initial phase variables and the laser gain must be evaluated as an ensemble average over the full range of the phase ψ_0. From (3.9) it follows that this is equivalent with introducing the electrons at random positions z_0 within the length of the wiggler, but keep their interaction time the same.

The average change of the electron momentum gives us directly a measure of the transfer of energy from the PR field to the laser field because of (3.11); the optical gain is directly proportional to the averaged momentum change.

IV. ADIABATIC APPROXIMATION

In the theory of the atomic laser, it is possible to assume that the field changes slowly enough that the electronic state can follow its instantaneous value adiabatically. A considerable simplification occurs in the theory of the FEL if we assume analogously that the fields do not change appreciably during the interaction time of one electron. Each one sees the wiggler field pass in the time $\tau = L/c$ (see 2.5), and it contributes little to the energy of the field once the laser is enough above threshold for the classical description to be valid. If the time constants of the laser system are considerably longer than τ we can assume the fields to remain constant throughout the interaction. This approximation assumes a large laser field, a low loss cavity, and a low density of electrons, but these are the assumptions needed for a simple description of the FEL operation any way. For the slowly varying amplitude of conventional atomic laser theories, the corresponding approximation is adopted almost without exceptions.

Dropping the conserved energy of the fields $(2\pi r_0 c^2/\omega V)(I_L + I_W)$ from Eq. (3.7) and assuming the product $I_L I_W$ to be a constant, we find the simplified Hamiltonian

$$H = p^2/2m + C\cos(\phi_L - \phi_W - 2kz)\,, \tag{4.1}$$

where

$$C = (4\pi r_0 c^2/\omega V)(I_L I_W)^{1/2} \tag{4.2}$$

is a constant.

We introduce the scaled momentum variable

$$W = (2k/m)p\,, \tag{4.3}$$

and the interaction frequency Ω defined by

$$\Omega^2 = (16\pi r_0/m)[(\omega I_L/V)(\omega I_W/V)]^{1/2}. \tag{4.4}$$

In terms of these variables the Hamiltonian (4.1) leads to the equations of motion

$$\frac{dW}{dt} = \Omega^2 \sin\psi\,, \quad \frac{d\psi}{dt} = -W \tag{4.5}$$

because the variable ψ in (3.9) is essentially the position coordinate canonical to W. The motion described by (4.5) corresponds to that of a classical pendulum, and forms the basis for our treatment of the FEL in the rest of this paper.

For small values of ψ the pendulum oscillates harmonically with frequency Ω. From (4.4) follows that this basic rate of change depends on the laser amplitude as $\Omega \propto E_L^{1/2}$ which provides the mechanism for saturation.

Figure 2 shows the well-known phase-space trajectories of the pendulum. Two types of motion occur, the closed paths with periodic motion in both W and ψ (region I) and those where W is still periodic but ψ increases steadily with time (region II). The two regions are separated by a separatrix on which the motion is aperiodic in both W and ψ. This depends on the value of the laser intensity through the parameter $\Omega^2 \alpha E_L$; when this increases the region I grows and occupies a larger area in the phase plane. Given an initial distribution of the electron ensemble over the phase plane, the behavior of the laser operation will change when this is transferred from region II into region I.

We assume a sharp initial momentum distribution for the electrons centered around $W_0 = \overline{W}_0$ but a totally unspecified initial phase ψ_0. Then the properly normalized initial distribution of the assembly of electrons is given by

$$\rho(\psi_0, W_0) = (1/2\pi)\,\delta(W_0 - \overline{W}_0). \tag{4.6}$$

The gain is determined by the shift of the average momentum from $\overline{W}_0$ after the interaction time, namely,

$$G = -\iint d\psi_0\, dW_0\, \rho(\psi_0, W_0)\,[W(\tau, \psi_0, W_0) - W_0]$$
$$= -\int \frac{d\psi_0}{2\pi}[W(\tau, \psi_0, W_0) - W_0]\,. \tag{4.7}$$

Another quantity worth evaluating is the distribution of the final momentum over the electron assembly after the interaction has taken place. This can be obtained by summing the measure of all those values ψ_0 which lead to a given momentum W. This is achieved by writing

$$\rho(W, W_0, \tau) = \int \delta(W - \overline{W})\frac{d\psi_0}{2\pi}\,, \tag{4.8}$$

where

$$\overline{W} = W(\tau, \psi_0, W_0) \tag{4.9}$$

is the final momentum of the electron having the initial coordinates ψ_0 and W_0. Inverting this relation to give

$$\psi_0 = \psi_0(\tau, W_0, \overline{W}), \tag{4.10}$$

and changing the variable of integration in (4.8) we find

$$\rho(W, W_0, \tau) = \frac{1}{2\pi}\int \delta(W - \overline{W})\frac{d\psi_0}{dW}dW$$
$$= \frac{1}{2\pi}\left(\frac{d\psi_0}{d\overline{W}}\right) = \left(2\pi\frac{dW}{d\psi_0}\right)^{-1}, \tag{4.11}$$

where the initial phase is eliminated by the use of (4.10) with $\overline{W} = W$ to give the distribution function in W with W_0 and τ as parameters. Relation (4.11) has already been used to evaluate the distribution numerically[4]; the perturbation treatment of Sec. V allows us to obtain analytical expressions for these quantities of interest for the FEL.

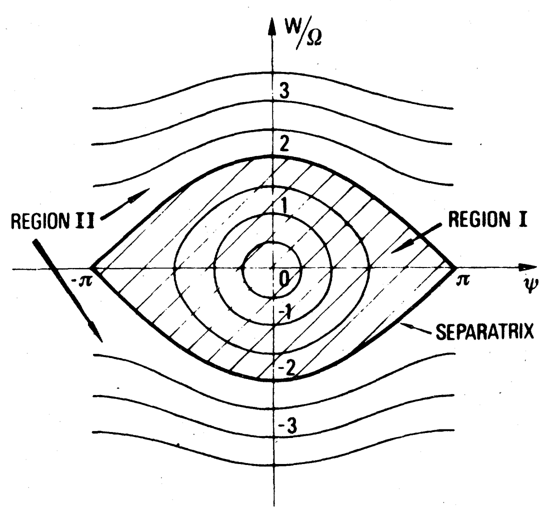

FIG. 2. Pendulum phase-space trajectories.

V. UNSATURATED MOMENTUM DISTRIBUTION

The adiabatic equations for the FEL describe the laser operation as a pendulum motion, which provides gain until the laser amplitude grows large enough to smear the electron distribution into a region where the losses counteract the gain and the operating point is stabilized.

When saturation effects are neglected we can obtain analytical expressions, which partly have been obtained earlier.[2-4] In this section we rederive these results partly to prove the consistency with earlier treatments but mainly because we can display how the ensemble average over the initial phase is carried out explicitly. It is also possible to see why the unsaturated gain remains much smaller than the momentum spread. The ensuing explicit expression for the momentum distribution has not been published earlier.

By introducing the initial phase ψ_0 explicitly into Eqs. (4.5), we write

$$\frac{dW}{dt} = \Omega^2 \sin(\psi - \psi_0)\,, \tag{5.1}$$

$$\frac{d\psi}{dt} = -W\,. \tag{5.2}$$

Using an expansion in the parameter Ω^2, we write

$$\psi = \psi^{(1)} + \psi^{(2)} + \cdots ,$$
$$W = W_0 + W^{(1)} + W^{(2)} + \cdots , \tag{5.3}$$

where W_0 is the initial value of the momentum. Equation (5.2) immediately gives

$$\psi^{(1)} = -W_0 t , \tag{5.4}$$

and (5.1) implies

$$\dot{W}^{(1)} = -\Omega^2 \sin(W_0 t + \psi_0) , \tag{5.5}$$

which with the proper initial condition integrates to

$$W^{(1)} = (\Omega^2/W_0)[\cos(W_0 t + \psi_0) - \cos\psi_0]$$
$$= -(2\Omega^2/W_0)\sin(\tfrac{1}{2}W_0 t + \psi_0)\sin(\tfrac{1}{2}W_0 t) . \tag{5.6}$$

To the order Ω^2/W_0 there occurs a modification of the momentum distribution, but when we average over ψ_0, we find

$$\overline{W}^{(1)} = \frac{1}{2\pi}\int_0^{2\pi} d\psi_0 \, W^{(1)} = 0 , \tag{5.7}$$

and no gain appears in this order; the distribution is symmetric. The maximum deviation in momentum is obtained for those values of ψ_0 which make $|\sin(\tfrac{1}{2}W_0 t + \psi_0)| = 1$ and then

$$(\Delta W)_{\max} = (2\Omega^2/W_0)\sin\tfrac{1}{2}W_0 t . \tag{5.8}$$

In order to obtain the mean-square deviation we calculate the average

$$\langle \Delta W^2 \rangle_{av} = \langle (W - W_0)^2 \rangle_{av}$$
$$= [(2\Omega^2/W_0)\sin(\tfrac{1}{2}W_0 t)]^2$$
$$\times \langle \sin^2(\tfrac{1}{2}W_0 t + \psi_0) \rangle_{av}$$
$$= (2\Omega^4/W_0^2)\sin^2\tfrac{1}{2}W_0 t , \tag{5.9}$$

which gives the earlier known result[5]

$$(\langle \Delta W^2 \rangle_{av})^{1/2} = (\sqrt{2}\,\Omega^2/W_0)\sin\tfrac{1}{2}W_0 t . \tag{5.10}$$

Both measures of the width, (5.8) and (5.10), contain the factor $\sin\tfrac{1}{2}W_0 t$, but with different coefficients depending on the definition of the spread. The special feature of the classical model is that the momentum spread is exactly confined within the limits set by (5.8). This is in agreement with Ref. 5. The ratio of the widths is the constant factor

$$(\Delta W)_{\max}/(\langle \Delta W^2 \rangle)^{1/2} = \sqrt{2} , \tag{5.11}$$

which shows how much of the momentum spread resides near the center of the distribution.

In order to obtain an explicit expression for the momentum distribution we need the derivative $dW/d\psi_0$ according to Eq. (4.11). Using (5.6) we find

$$\left(\frac{dW^{(1)}}{d\psi_0}\right)^2 + (W^{(1)})^2 = \left(\frac{2\Omega^2}{W_0}\right)^2 \sin^2(\tfrac{1}{2}W_0 t)$$
$$= (\Delta W_{\max})^2 . \tag{5.12}$$

As the function W is periodic in ψ_0 each value of $(dW/d\psi_0)$ is reached twice and hence we multiply the derivative in (5.12) by 2 and introduce it into (4.11) to obtain

$$\rho(W, W_0, \tau) = (1/\pi)[(\Delta W_{\max})^2 - (W - W_0)^2]^{-1/2} . \tag{5.13}$$

This shows the exact nature of the end-point singularities obtained numerically in Bambini and Stenholm.[4] Integrating (5.13) over W from $-\Delta W_{\max}$ to $+\Delta W_{\max}$ we find that (5.13) is correctly normalized.

In order to obtain the lowest-order contribution to the gain we have to calculate $W^{(2)}$. From (5.2), we obtain

$$\dot{\psi}^{(2)} = (\Omega^2/W_0)[\cos\psi_0 - \cos(W_0 t + \psi_0)] , \tag{5.14}$$

which gives

$$\psi^{(2)} = \frac{\Omega^2}{W_0}\left(t\cos\psi_0 - \frac{1}{W_0}\sin(W_0 t + \psi_0) + \frac{\sin\psi_0}{W_0}\right) , \tag{5.15}$$

with the correct initial condition $\psi^{(2)}(0) = 0$. From Eq. (5.1) we find the perturbation relation

$$\frac{d}{dt}(W^{(1)} + W^{(2)}) = \Omega^2 \sin(\psi^{(1)} + \psi^{(2)} - \psi_0)$$
$$= \Omega^2[\sin(\psi^{(1)} - \psi_0) + \psi^{(2)}\cos(\psi^{(1)} - \psi_0)$$
$$+ O(\psi^{(2)})^2 + \cdots] . \tag{5.16}$$

Using Eq. (5.6) to eliminate the lower-order terms, we obtain

$$\dot{W}^{(2)} = \left(\frac{\Omega^4}{W_0} t\cos\psi_0 - \frac{1}{W_0}\sin(W_0 t + \psi_0) + \frac{\sin\psi_0}{W_0}\right)$$
$$\times \cos(W_0 t + \psi_0) . \tag{5.17}$$

This can be written

$$\dot{W}^{(2)} = \frac{\Omega^4}{W_0}\left(\cos\psi_0 \, t\cos(W_0 t + \psi_0) + \frac{\sin\psi_0 \cos(W_0 t + \psi_0)}{W_0} - \frac{\sin(2W_0 t + 2\psi_0)}{2W_0}\right) . \tag{5.18}$$

Integrating this we obtain

$$W^{(2)}=\frac{\Omega^4}{W_0^2}\left(\cos\psi_0\, t\,\sin(W_0t+\psi_0)+\frac{\cos\psi_0\cos(W_0t+\psi_0)}{W_0}+\frac{\sin\psi_0\sin(W_0t+\psi_0)}{W_0}+\frac{\cos2(W_0t+\psi_0)}{4W_0}-\frac{4+\cos2\psi_0}{4W_0}\right),\tag{5.19}$$

which has the correct initial value $W^{(2)}(0)=0$. Averaging this function as in (5.7) we use

$$\begin{aligned}\langle\sin\psi_0\sin(W_0t+\psi_0)\rangle_{av}&=\tfrac{1}{2}\cos W_0t\,,\\ \langle\cos\psi_0\sin(W_0t+\psi_0)\rangle_{av}&=\tfrac{1}{2}\sin W_0t\,,\\ \langle\cos\psi_0\cos(W_0t+\psi_0)\rangle_{av}&=\tfrac{1}{2}\cos W_0t\,,\end{aligned}\tag{5.20}$$

which gives the averaged momentum change

$$\langle W-W_0\rangle_{av}=\langle W^{(2)}\rangle_{av}=\frac{\Omega^4}{2W_0^3}[W_0t\sin W_0t-2(1-\cos W_0t)]\,.\tag{5.21}$$

When we insert this into the gain (4.7), we obtain

$$\begin{aligned}G=-\langle W^{(2)}\rangle_{av}&=\frac{\Omega^4}{2W_0^3}[2(1-\cos W_0t)-W_0t\sin W_0t]\\ &=-\frac{\Omega^4}{2}\frac{\partial}{\partial W_0}\frac{1-\cos W_0t}{W_0^2}\\ &=-\Omega^4\frac{\partial}{\partial W_0}\left(\frac{\sin\frac{1}{2}W_0t}{W_0}\right)^2\\ &=-\frac{1}{2}\frac{\partial}{\partial W_0}\langle\Delta W^2\rangle_{av},\end{aligned}\tag{5.22}$$

which agrees with results derived earlier.

We form the ratio

$$\begin{aligned}\frac{G}{(\langle\Delta W^2\rangle)^{1/2}}&=\frac{\partial}{\partial W_0}(\langle\Delta W^2\rangle)^{1/2}\\ &=\frac{\Omega^2}{2\sqrt{2}W_0^2}\frac{2(1-\cos W_0t)-W_0t\sin W_0t}{\sin\frac{1}{2}W_0t}\\ &=\sqrt{2}\frac{\Omega^2}{W_0^2}[\sin\tfrac{1}{2}W_0t-\tfrac{1}{2}W_0t\cos\tfrac{1}{2}W_0t]\,.\end{aligned}\tag{5.23}$$

If we introduce the dimensionless saturation parameter $(\Omega/W_0)^4$, we find that the ratio (5.23) is proportional to its square roots. Consequently, in the limit when our unsaturated expansion in Ω^2 is justified, the ratio (5.23) must remain small; the momentum spread dominates the gain. Using the relation (5.11) to express the result (5.23) in terms of the maximum momentum spread, we find that the present results are in agreement with the numerical computations of Ref. 4 and the analytical derivation of Refs. 2 and 3.

VI. NUMERICAL EVALUATION OF THE ELECTRON DISTRIBUTION

In the papers of Ref. 4, Eqs. (4.5) were directly interpreted numerically assuming given initial values ψ_0 and W_0. The gain (4.7) was then evaluated by an explicit summation over initial phases ψ_0. In the region where the amplification is strongly saturated it turns out to be more advantageous to replace the basic equations by their exact solution in terms of the Jacobi elliptic functions.[8]

Equations (4.5) give immediately

$$\ddot{W}=-\Omega^2W\cos\psi=-\Omega^2W[1-(\dot{W}/\Omega^2)^2]^{1/2}\,.\tag{6.1}$$

Multiplying by $\dot{W}$ and separating the variables we can integrate once and obtain

$$\dot{W}=\pm[\Omega^4-\tfrac{1}{4}(W^2-W_0^2+2\Omega^2\cos\psi_0)^2]^{1/2},\tag{6.2}$$

which can be transformed into

$$\begin{aligned}y\equiv\frac{W_0t}{2}&=\int_1^x\frac{dx}{[(a^2-x^2)(b^2+x^2)]^{1/2}}\\ &=-\int_x^a\frac{dx}{[(x^2+b^2)(a^2-x^2)]^{1/2}}+y_0\,,\end{aligned}\tag{6.3}$$

where

$$\begin{aligned}a&=[1+(4\Omega^2/W_0^2)\sin^2\tfrac{1}{2}\psi_0]^{1/2},\\ b&=[(4\Omega^2/W_0^2)\cos^2\tfrac{1}{2}\psi_0-1]^{1/2},\\ y_0&=\int_1^a\frac{dx}{[(x^2+b^2)(a^2-x^2)]^{1/2}}\,.\end{aligned}\tag{6.4}$$

Using Ref. 8 we can write this in terms of the Jacobi elliptic function cn as

$$x=a\,\mathrm{cn}[(a^2+b^2)^{1/2}y+\mathrm{cn}^{-1}(a^{-1}|m)|m]\,,\tag{6.5}$$

where

$$m=a^2/(a^2+b^2)\,,\tag{6.6}$$

$$(a^2+b^2)^{1/2}=4\Omega^2/W_0^2\,.\tag{6.7}$$

As a check on the validity of (6.5) we set $y=0$ and find, indeed, $x=1$ as required.

The parameter denoted by b^2 is positive only if

$$\cos^2\tfrac{1}{2}\psi_0>W_0^2/4\Omega^2\,.\tag{6.8}$$

This region is called region I and is characterized by closed orbits in the phase plane (see Fig. 2). Here the electron will change the sign of its momentum during the interaction, thus leading to partial absorption even for electrons with gain initially.

Those values of ψ_0 and W_0 which do not satisfy (6.8) fall within region II. Here the electronic momentum is not changed much but oscillates periodically with little decrease in gain.

The separatrix between the two regions is given by

$$|W_0| = 2\Omega \cos\tfrac{1}{2}\psi_0, \tag{6.9}$$

and hence an increase in intensity of the laser ($\Omega \sim E_L^{1/2}$) forces more and more electrons into region I where their contribution to the gain disappears. This is saturation in the laser.

In region II we write

$$b = [1 - (4\Omega^2/W_0^2)\cos^2\tfrac{1}{2}\psi_0]^{1/2}, \tag{6.10}$$

and find

$$y = \int_1^x \frac{dx}{[(a^2-x^2)(x^2-b^2)]^{1/2}}$$

$$= -\int_x^a \frac{dx}{[(a^2-x^2)(x^2-b^2)]^{1/2}} + y_0, \tag{6.11}$$

which in terms of elliptic functions gives

$$x = a\,\mathrm{dn}[ay + \mathrm{dn}^{-1}(1/a|m')|m'], \tag{6.12}$$

with

$$m' = (a^2 - b^2)/a^2. \tag{6.13}$$

The two results (6.5) and (6.12) can be used to evaluate the final momentum as a function of the initial phase. Because the behavior of the elliptic functions is well known it turns out that our choice of parametrization of the solution allows for considerably shorter computations than the direct integration of motion. We have evaluated the gain and final momentum distribution for various values of the intensity parameter $(\Omega/W_0)^2$. It is also possible to use the Jacobi elliptic functions to discuss the solution analytically; this will be carried out in a forthcoming publication.

VII. RESULTS AND DISCUSSION

As we have already mentioned it is possible to integrate the equation of motion for arbitrary intensity and interaction times, and we can obtain the quantities of interest for the laser operation. We define the dimensionless laser field amplitude

$$\epsilon = \frac{2C}{\frac{1}{2}p_0^2/m}\alpha E_L, \tag{7.1}$$

where p_0 is the initial longitudinal momentum of the electrons entering the wiggler and C is the magnitude of the interaction part of the Hamiltonian (4.1). Using (4.3) and (4.4) we write

$$\epsilon = (2\Omega/W_0)^2. \tag{7.2}$$

The dimensionless laser intensity is given by $\epsilon^2 \alpha I_L$ and can be used as the saturation parameter. The parameter used in Bambini-Stenholm[5] is given by

$$\eta^2 = \epsilon^2/2^5.$$

It is easy to understand this because the interaction width is determined by Ω, which is the frequency of small oscillations and W_0 is a measure of the initial momentum of the electron measured in frequency units determined by the corresponding Doppler shift, see Eq. (4.3). When the interaction strength starts to equal this Doppler frequency we have a strongly saturated laser. The unsaturated region is determined by the condition $\epsilon^2 \ll 1$.

We define a gain ratio by writing

$$g(\epsilon^2) = \frac{G}{\epsilon^2}\left(\lim_{\epsilon\to 0}\frac{G}{\epsilon^2}\right)^{-1}. \tag{7.3}$$

For low intensities this remains a constant, which is normalized to unity. Deviations from constancy signal the onset of saturation. For small enough intensity we can expand

$$g(\epsilon^2) = g_0 + \epsilon^2 g_1 + \cdots. \tag{7.4}$$

In Fig. 3 we show the value of g for the interaction period $W_0 t = 2.4$. Because the variation with ϵ^2 occurs over a large range of intensities, the scale is chosen logarithmically. We can see that g remains constant up to about $\epsilon^2 \cong 0.4$ ($\Omega \cong 0.4 W_0$) in agreement with our general considerations. After this point the gain decreases, but only very gently. The laser field at which $g = 0.5$ is around $\epsilon^2 \simeq 30$, namely, a very large one.

The steady-state operation of a free-electron laser can be described by Fig. 3 when we draw a

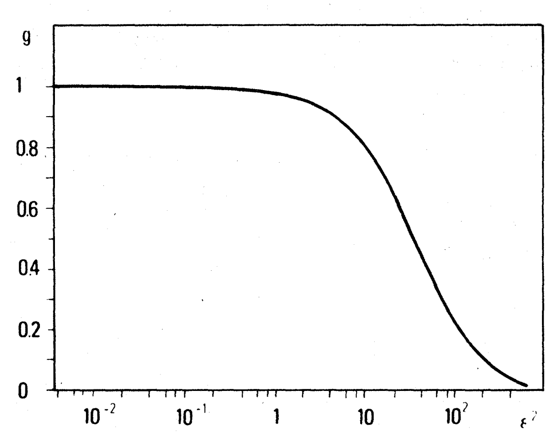

FIG. 3. Gain ratio vs laser intensity ($W_0 t = 2.4$).

horizontal line and solve the equation: $g(\epsilon^2)$ = losses (in suitable units). We see that there occurs an intersection between this line and the gain ratio as soon as the losses are smaller than unity in these units. This defines both a threshold and the steady-state operating intensity. It is, however, an essential feature of the FEL that interaction occurs for a finite time only. Consequently a real operating device must be described in a dynamic way. As the amplification time is of finite length much shorter than the repetition time of the pulses, it may be that time evolution during the interaction must be scrutinized more closely. Then these considerations can be used to obtain only qualitative information about the laser operation.

In Fig. 3 we have a saturation which decreases the gain, i.e., the coefficient g_1 of (7.4) is negative. When the interaction period becomes longer than $W_0 t = 3.7$ there is a change of sign in g_1 and saturation at first increases the gain. Such a case is illustrated in Fig. 4 for $W_0 t = 4.4$. The linear gain (5.22) varies rapidly in this neighborhood, and we find this to be near the position of maximum linear gain (see Bambini-Stenholm[4]). This appears to be a favorable operating point when large losses occur. For losses equaling the linear gain the small-signal theory can give no laser operation, but from Fig. 4 we see that there occurs a stable operating point at about $\epsilon^2 \cong 4$, which is a large intensity. For slightly larger losses there also occurs an unstable operating point, and hysteresis behavior is possible. It may hence be difficult to reach the stable operating point for large losses, but positive saturation contribution to the gain helps push the laser operating point from the region where saturation sets in around $\epsilon^2 \sim 0.4 - \epsilon^2 \sim 4$, viz., a factor of 10 in the laser intensity. In contrast to our expectations we find that saturation helps to increase the gain in this region. Unfortunately the (unsaturated) momentum spread is still about 80% of its maximum value when the gain is at its maximum. If refocusing causes difficulties the most advantageous operating point must be chosen elsewhere.

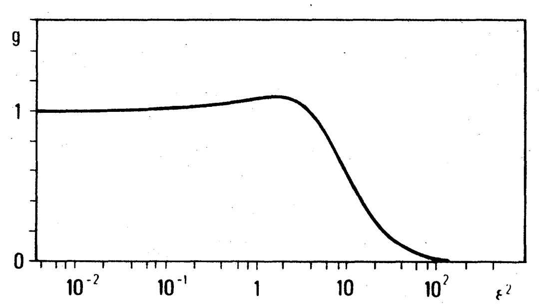

FIG. 4. Gain ratio vs laser intensity ($W_0 t = 4.4$).

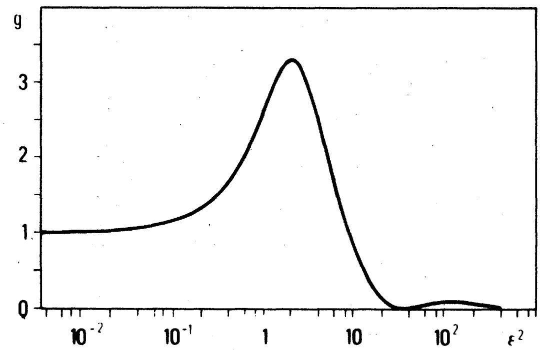

FIG. 5. Gain ratio vs laser intensity ($W_0 t = 5.8$).

It is possible to obtain an even larger saturation induced gain by going close to the region where the linear gain drops down. In Fig. 5 we show the gain ratio for $W_0 t = 5.8$, very near the point $W_0 t = 2\pi$ where the small signal gain vanishes. Here we have a gain peak of more than 3 times the unsaturated value. In this case saturation starts to increase the gain as early as $\epsilon^2 = 0.02$ but laser operation occurs mainly around $\epsilon^2 = 10$. This is the optimum region when one wants to observe saturation and bistability, because they are dominant effects here. The linear gain is small, only 43% of the maximum gain, and the momentum spread is about 24% of its maximum. Our program can be used also for periods $W_0 t > 2\pi$ but the results lack practical interest. There appears a repetition of the general behavior but with a lower value of the gain. Also large negative gains (losses) are possible.

It is also possible to use the program to evaluate

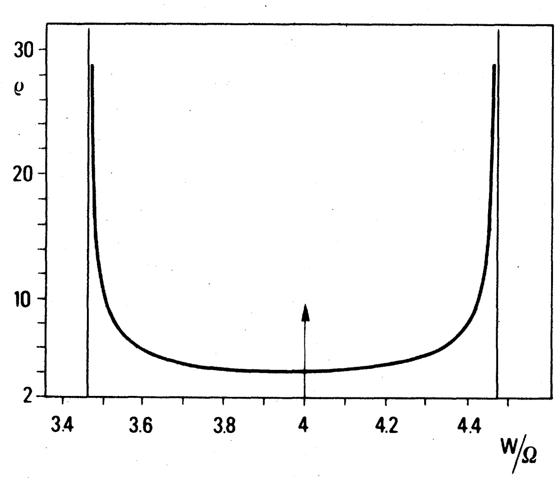

FIG. 6. Final momentum distribution (no saturation) $\epsilon^2 = 0.0625$, $W_0 t = 3.2$.

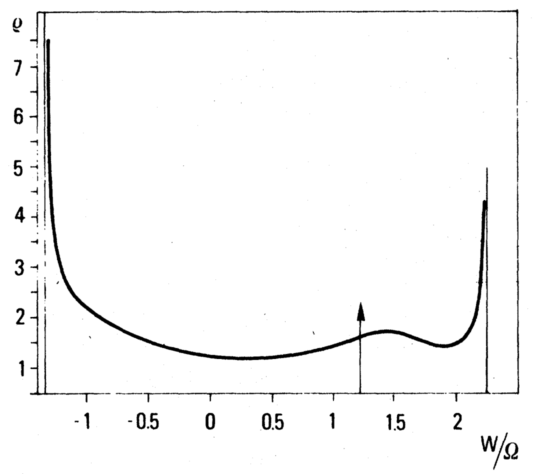

FIG. 7. Final momentum distribution (weak saturation) $\epsilon^2=7.56$, $W_0t=3.2$.

the distribution of final momentum by the method shown in Sec. V. In Figs. 6–8 we consider an interaction period $W_0t=3.2$ near the maximum width of the distribution [the vertical arrow indicates the position of the initial sharp momentum distribution, see Eq. (4.6)]. First, in Fig. 6, we look at the low-intensity result for $\epsilon^2=0.0625$. As we can see from Figs. 3 and 4 this is in the weakly saturated region and the general behavior of the distribution agrees with the result (5.13). The weakly singular end points are seen, as first reported in Bambini-Stenholm[4] and the distribution is nearly symmetric indicating only a very small gain.

The singularities in the momentum distribution are due to zeros of $dW/d\psi_0$, see Eq. (4.11). In this they are analogous to the van Hove singularities in solids. When the final momentum W, as a function of initial phase ψ_0, acquires more wiggles there appears new peaks in the momentum distribution. In Fig. 7 we have $\epsilon^2=7.56$ and we find from Figs. 3 and 4 that this is the saturation region. Here we can see both the emergence of a new maximum and a distinct asymmetry. For this intensity it is no longer time to say that the gain is much less than the momentum spread. When the intensity is increased further the peaks develop into real singularities, which however remain integrable. In Fig. 8 we have $\epsilon^2=15.9$ and there occurs one real singularity. The asymmetry is pronounced. We have evaluated the density of states for large intensities ($\epsilon^2>10$), where the gain ratio has dropped to about 0.5, see Figs. 3 and 4. The result is a series of several sharp singularities carrying most of the weight of the distribution. Their presence in a real laser must be considered doubtful because they will be easily smeared by technical perturbations.

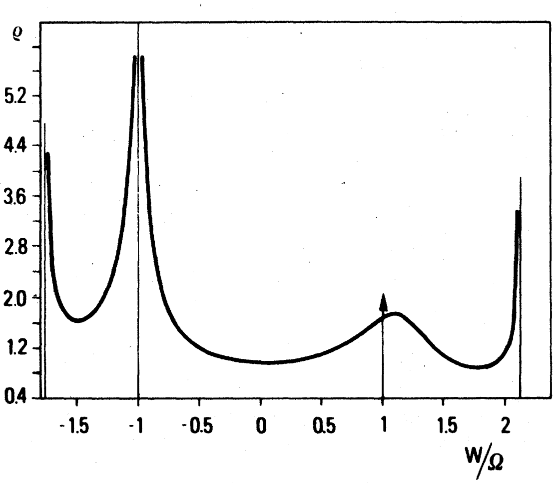

FIG. 8. Final momentum distribution (strong saturation) $\epsilon^2=15.9$, $W_0t=3.2$.

In the present section we have shown that the classical theory of the FEL operation is able to obtain both the strong signal gain and the momentum distribution for all parameter ranges of interest for the operation of a real system. In addition the possibility of saturation enhancement of gain has been discovered and the possibility of bistability in laser operation has been indicated. The latter property suggests a large number of interesting observable consequences (see Ref. 9). Because of the dynamical nature of the FEL operation it is, however, not clear if it is possible to measure these phenomena. Only the detailed design of the operating device will determine their observability.

VIII. CONCLUSIONS

In this paper we have presented a classical theory of the free-electron laser. By choosing a moving reference frame we can present the basic laser amplification process as resonant scattering of pseudoradiation energy from the wiggler field into the laser field. Both fields are described by classical harmonic oscillators, and the frequency of radiation is taken to be such that the electron can be described by nonrelativistic dynamics in the moving frame. By assuming the field amplitude to stay essentially unmodified during the interaction period we can introduce an adiabatic approximation, which reduces the problem to that of a classical pendulum equation. A straightforward perturbation expansion in the dimensionless quantity ϵ gives previously known results for the unsaturated region. The interpretation of ϵ is that it measures the ratio of the small-signaloscillation frequency (Rabi flipping) against the Doppler shift determined by the initial electron momentum. For strong signals it is possible to use the Jacobi elliptic functions to evaluate the FEL behavior exactly. Some detailed illustrations of the numerical work are given and commented upon.

The main new results of this paper compared with earlier theories[2,3] are the following.

(i) The definition of the moving frame which simplifies the treatment and can be useful also in other investigations. In fact this frame has been used in the papers.[4,5]

(ii) We can identify the dimensionless saturation parameter as (a power of) Ω/W_0, which has a simple physical interpretation.

(iii) The details of the final momentum distribution are clarified. Even the analytically given unsaturated result (5.13) has not been presented earlier. The numerical work shows that the momentum distribution can be obtained for arbitrary intensities and interaction times.

(iv) The gain is evaluated for arbitrary intensities and interaction times. It is found that saturation sets in very slowly, and can in a certain region be used to enhance the gain in contradistinction to the atomic laser. This also suggest the possibility of bistability and hysteresis as in the case of a laser with a saturable absorber. The practical possibility to utilize this feature of the system must await the realization of an operating device, whose technical details will determine what features are observable.

The basic approximations and limitations of the present approach are (a) The laser field is assumed strong enough so that only the scattering stimulated by the laser field itself is considered, while the spontaneous scattering in the laser and all other modes are ignored. (b) The quantized nature of the motion of the electron is neglected. As each step involves a change of electron momentum of magnitude $\hbar k$, we require this to be small compared with the electron momentum $p \sim p_0$, all in the moving frame. We obtain the condition

$$(\hbar k/p_0) \leq 1, \tag{8.1}$$

which is the condition obtained in Ref. 5 where a detailed estimate of the quantum corrections is given. From this condition we see that near the zero of the small-signal gain, $p_0 \approx 0$, it is not possible to use any classical model. As the momentum exchange of one basic interaction event $\hbar k$ is comparable to the electron momentum p_0, quantum fluctuations are dominant. If one wants to observe their influence, this region is favorable. In this situation we also have a strongly saturating system, see (7.2). (c) For very strong laser and wiggler fields the transverse motion may require a more detailed treatment than we have carried out here (Appendix B). (d) For laser frequencies in the blue or uv range the electron motion in the moving frame may need a relativistic treatment. The ensuing complications are, however, well understood. (e) If the dynamics of the fields during the interaction time must be considered, the adiabatic approximation of Secs. IV–VII cannot be used. It is then possible to utilize the exact constants of motion of Sec. III to follow the dynamics by Eq. (3.13). (f) Finally we want to point out the use of a pseudoradiation approximation (Weizäcker-Williams). If energy depletion or quantum fluctuations of this field become of considerable importance, we meet a phenomenon that cannot have any physical reality in the laboratory frame. It must hence be due entirely to our approximation, and the calculations must be reconsidered without this. For the large wiggler fields used in practice we do not expect any problems of this nature, but it is essential to remain aware of the possibility.

APPENDIX A

Lab frame	Moving frame
Laser wavelength	$\dfrac{\lambda'}{2\gamma}$
PR wavelength	$\lambda'\gamma$
Laser energy density	$\dfrac{\omega I_L}{V}(4\gamma^2)$
PR energy density	$\dfrac{\omega I_W}{V}\Big/2\gamma^2$
Interaction time	$\Delta t\gamma$
Electron energy	$mc^2\gamma\left(1+\dfrac{2p}{mc}\right)$
Electron density	$2\dfrac{N}{V}\gamma$

γ is defined by Eqs. (2.9) and (2.11); λ' is the frequency of the laser and PR fields in the moving frame; $\omega = 2\pi c/\lambda'$; I_L and I_W are the laser and PR field action; V is the mode volume; Δt is the interaction time in the moving frame; m is the relativistic electron mass in the moving frame (see Appendix B); p is the longitudinal electron momentum in the moving frame; N is the number of electrons in the mode volume V. It is worthwhile to note that we need some care in transforming the PR wavelength and energy density. We must remember that this field is not a true radiation field.

APPENDIX B

In the moving frame the transverse kinetic energy of the electron can be relativistic, if the wiggler magnetic field is sufficiently strong. Indeed, the constancy of the transverse canonical momentum of the electron implies that the kinetic transverse momentum is proportional to the vector potential $\vec{A}$, i.e., ignoring the laser field vector potential, to the wiggler field B_0. However, it is possible, in the moving reference frame, to account for relativistic corrections to the transverse motion of the electrons.

The relativistic Hamiltonian for the electron is given by

$$H_{\text{rel}} = \{[\vec{p} - (e/c)\vec{A}]^2 c^2 + m^2c^4\}^{1/2}. \tag{B1}$$

The transverse motion of the electron is caused by the strong static magnetic field of the wiggler; therefore we consider only the vector potential of that field. In the laboratory frame its components are

$$A_x^{(\text{lab})} = -(B_0/k_q)\sin k_q z^{(\text{lab})},$$
$$A_y^{(\text{lab})} = -(B_0/k_q)\cos k_q z^{(\text{lab})}. \tag{B2}$$

Transformation to the moving frame does not alter the amplitude of the transverse potential. Therefore, in that frame

$$A^2 = B_0^2/k_q^2. \tag{B3}$$

The longitudinal motion is still assumed to be nonrelativistic. Then, to the first order in p^2, the Hamiltonian (B1) takes the form

$$H_{\text{re}} \to H' \simeq \left(\frac{e^2B_0^2}{k_q^2} + m^2c\right)^{1/2}\left(1 + \frac{p^2}{2(e^2B_0^2/c^2k_q^2 + m^2)}\right)$$
$$= \text{const} + \frac{p^2}{(e^2B_0^2/c^4k_q^2 + m^2)^{1/2}} \quad . \tag{B4}$$

Therefore, we can account for a relativistic transverse motion of the electron by simply replacing the rest mass m of the electron by an "effective" mass

$$m_{\text{eff}} = (1 + K)^{1/2} m, \tag{B5}$$

where

$$K = e^2B_0^2\lambda_q^2/4\pi^2c^4m^2 = r_0/mc^2(\lambda_q B_0/2\pi)^2. \tag{B6}$$

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