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Order-$\alpha^4 R_\infty$ contributions to positronium hyperfine structure from radiative corrections to two-photon virtual annihilation

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We present the details of our calculation of radiative corrections affecting the two-photon virtual-annihilation contributions to the positronium hyperfine splitting in order $m\alpha^6(\alpha^4 R_\infty)$. Our result, which is obtained in analytic form, is $\Delta E^A_{2\gamma}(\alpha^4 R_\infty) = -13.13$ MHz. We point out the gauge invariance of the result and demonstrate the cancellation of potential infrared contributions which appear at intermediate stages of the calculation.

I. INTRODUCTION

Accurate measurements of the hyperfine splitting of positronium have been available since 1975. These experiments give[1,2]

$$\Delta\nu_{t-s} = 203\,384.9 \pm 1.2 \text{ MHz},$$

$$\Delta\nu_{t-s} = 203\,387.0 \pm 1.6 \text{ MHz}.$$

The theoretical calculations, as has frequently been the case in low-energy quantum electrodynamics, are not as yet of comparable accuracy.

While calculations of the hyperfine splitting[3] and the fine structure[4] to the first nonleading order $m\alpha^5$ were performed and verified by experiment[5,6] some time ago, the situation is quite different for the next terms in the perturbation expansion, which produce corrections of order $m\alpha^6 \ln\alpha^{-1}$ and $m\alpha^6$. Various contributions to order $m\alpha^6 \ln\alpha^{-1}$ have been calculated and confirmed.[7-12]

As far as contributions to order $m\alpha^6$ are concerned, only a few terms in the perturbation expansion have been evaluated.[11-16] The fact that $\frac{1}{2}m\alpha^6 \simeq 9.3$ MHz makes the calculation of the $m\alpha^6$ term mandatory if a sensible comparison between theory and experiment is to be made.

In an effort to remedy this situation, we have undertaken a systematic calculation of order-α^2 virtual-annihilation corrections to the hyperfine splitting. In the first stage of this program we evaluated the contributions coming from three-photon virtual annihilation.[15,16] In this paper, we present the details of our calculation[17] of all the first-order radiative corrections to the two-photon virtual-annihilation process. In the next section, the relevant diagrams are displayed and briefly discussed. In particular, we argue that the set of diagrams is gauge invariant (to order $m\alpha^6$). In Secs. III and IV we present the evaluation of the contributions from vacuum polarization and electron self-energy insertion, while Secs. V and VI are devoted to the study of vertex insertion and of the diagrams involving the exchange of a photon in the momentum-transfer (t) channel.

II. ANALYSIS OF DIAGRAMS

Four typical two-photon virtual-annihilation diagrams, representing all those relevant to order $m\alpha^6$, are shown in Fig. 1. To this order, the calculation of the energy shift associated with radiative corrections to the two-photon virtual annihilation of parapositronium (the $^1S_0$ state) is equivalent to computing the corrections to the electron-positron forward scattering amplitude at threshold when the intermediate state consists of two photons. However, as will be discussed in Sec. V, special care is needed in the study of diagram 1(d), which we will call the "box" diagram. By using the charge-conjugation[16] operator,[18] performing appropriate momentum relabeling, and suitably changing variables, we can demonstrate that diagrams with the crossed photon configuration (not illustrated in Fig. 1) give a contribution equal to those diagrams with the corresponding uncrossed configuration. Moreover, the contribution is the same no matter where we attach the radiative correction. (More details in this regard will be given when the individual insertions are discussed.)

In order to keep track of the low-momentum singularities which appear at intermediate stages of the calculation, we give the photon an infinitesimal mass λ. The box diagram requires particular attention in this respect [see the discussion following Eq. (21)].

Before examining the radiative corrections in detail, it is important that we observe that, to order $\alpha^4 R_\infty$, the contributions of two-photon virtual annihilation to the hyperfine structure of positronium are gauge independent. With due regard for

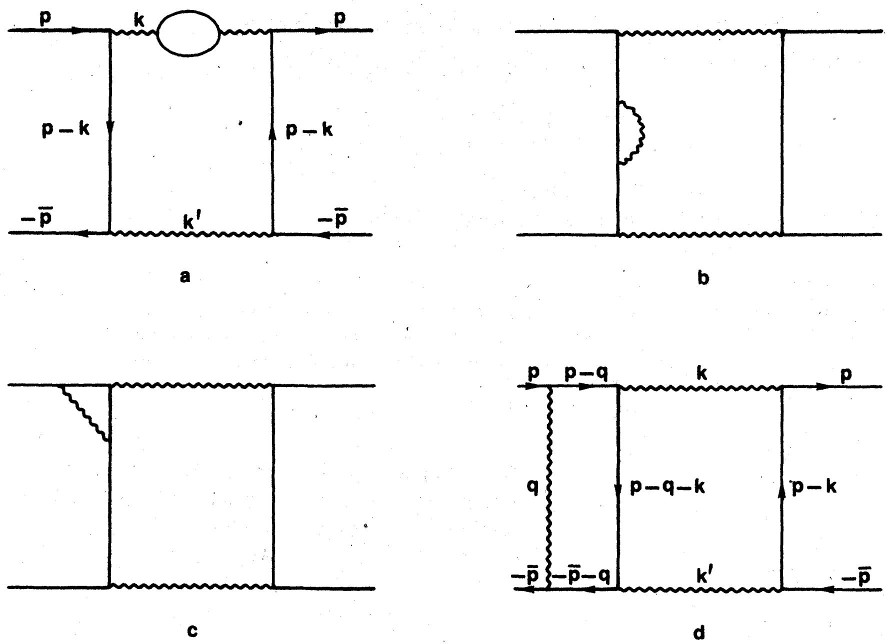

FIG. 1. Typical diagrams corresponding to first-order corrections to two-photon virtual annihilation.

the bound-state aspects of the problem,[19] gauge independence can be established by means of standard arguments.[20]

III. VACUUM POLARIZATION INSERTION

The amplitude associated with diagram 1(a) is

$$\bar{u}(p)\gamma^\mu S_F(p-k)\gamma^\nu v(\bar{p})\bar{v}(\bar{p})\gamma_\nu S_F(p-k) \times \gamma_\mu u(p)D(k)D_r(k'), \tag{1}$$

where $D(k)=k^{-2}$ and[21] $D_r(k')=k'^{-2}\Pi(k'^2)k'^{-2}$. We can reduce Eq. (1) to the form of a product of two traces by noting that, at threshold, we have

$$u_\alpha(p)\bar{v}_\beta(p) = -[(1-i\gamma_0)\gamma_5 2^{-3/2}]_{\alpha\beta}, \quad v_\alpha(p)\bar{u}_\beta(p) = [(1+i\gamma_0)\gamma_5 2^{-3/2}]_{\alpha\beta}. \tag{2}$$

Thus, after evaluation of the traces, the integrand associated with the four diagrams with the vacuum polarization insertion is

$$-16\bar{k}^2\Pi_f(k'^2)/k^2(k^2-2pk)^2, \tag{3}$$

where $\Pi_f(k^2)$ is given by Eqs. 9–65 of Ref. 18.

The integration of Eq. (3) can be performed with standard techniques. (To spot and correct eventual errors made in the analytical calculation, we used numerical methods to evaluate all Feynman parameter integrals throughout our work.] After completion of the integration of Eq. (3), we need the wave function and the Feynman rules factor

$$-i[(4\pi\alpha)^2/(2\pi)^4]\,|\phi(0)|^2$$

to obtain an expression for the energy $\Delta E_{2\gamma}^{\text{vac.pol.}}$.

Thus we obtain

$$\Delta E_{2\gamma}^{\text{vac.pol.}} = \frac{4\alpha^2|\phi(0)|^2}{m^2}\,\frac{\alpha}{\pi}\,\frac{\pi^2}{18}. \tag{4}$$

IV. ELECTRON SELF-ENERGY INSERTION

The amplitude associated with diagram 1(b) is, to order $m\alpha^6$,

$$\bar{u}(p)\gamma^\mu S_F'(p-k)\gamma^\nu v(\bar{p})\bar{v}(\bar{p})\gamma_\nu S_F(p-k)\gamma_\mu u(p), \tag{5}$$

where

$$S_F'(p-k) = S_F(p-k)\Sigma(p-k)S_F(p-k),$$

with

$$\Sigma(p) = A + (i\gamma p + m)B + (i\gamma p + m)^2\Sigma_f.$$

(A and B are infinite constants to be absorbed in the mass and wave-function renormalization.) The expression for Σ_f can be found in Ref. 18 (especially Appendix A5-4), but to deal with the low-momentum singularities it is convenient to use those formulas as modified by the presence of the infinitesimal photon mass. The modified expressions are

$$\Sigma_f(k) = \Sigma_1(k) + mS_F(k)\Sigma_2(k), \quad \Sigma_1 = (\alpha/2\pi m)I_1, \quad \Sigma_2 = (\alpha/2\pi m)(I_2 + 2I_1/\rho + I_3) \tag{6}$$

and

$$I_1 = \int_0^1 dx\,(1+x)\ln\left(1+\frac{x(1-x)\rho}{x^2+\eta^2(1-x)}\right),$$

$$I_2=\int_0^1 dx\,(1-x)\,\ln\left(1+\frac{x(1-x)\rho}{x^2+\eta^2(1-x)}\right),$$

$$I_3=-2\int_0^1 \frac{dx\,(1-x^2)x}{x^2+\eta^2(1-x)}\,,$$

with $\rho=(k^2+m^2)/m^2$ and $\eta^2=\lambda^2/m^2$.

The amplitude (5) can be evaluated by means of Eqs. (2). Hence the integrand associated with the electron self-energy insertion is (notice that we have a factor of 4 since four diagrams contribute to this process)

$$-16\vec{k}^2 m\Sigma_2(p-k)/k^2(k-2p)^2(k^2-2pk)^2\,. \qquad (7)$$

Before integrating Eq. (7), it is convenient that we reexpress Σ_2 in the form

$$\Sigma_2(p-k)=\frac{\alpha}{2\pi m}(k^2-2pk)\left(\frac{2+\ln\eta^2}{k^2-2pk}+\frac{1}{2}\int_0^1\frac{dx\,x}{(k^2-2pk)(1-x)+m^2x}-2m^2\int_0^1\frac{dx\ln x}{[(k^2-2pk)(1-x)+m^2x]^2}\right)\,. \qquad (8)$$

[Equation (8) can be obtained from Eq. (6) by repeated integrations by parts and by isolating the cutoff-dependent terms. After isolating the singular term we set $\lambda=0$ in the remainder.]

Equation (7) is integrated by means of the standard Feynman parametrization. By multiplying the result by the appropriate factor, we obtain

$$\Delta E_{2\gamma}^{\text{el.self-en}}=-\frac{4\alpha^2|\phi(0)|^2}{m^2}\frac{\alpha}{\pi}\left[\frac{1}{2}+\frac{\pi^2}{6}-\ln 2-2\ln^2 2+i\pi\left(\frac{1}{2}+2\ln 2\right)-\ln\eta^2\left(\ln 2-1-\frac{i\pi}{2}\right)\right]. \qquad (9)$$

V. VERTEX INSERTION

The amplitude associated with diagram 1(c) is

$$\bar{u}(p)\gamma^\mu S_F(p-k)\gamma^\nu v(\bar{p})\bar{v}(\bar{p})\gamma_\nu S_F(p-k)\times\Gamma_\mu(p-k,p)u(p)\,, \qquad (10)$$

where

$$\Gamma_\mu(p',p)=\gamma_\mu(1+L)+\Lambda_{\mu f}(p',p)\,,$$
$$\Lambda_{\mu f}(p',p)=-\frac{\alpha}{2\pi}\int_0^1 dx\int_0^x dy\left(\frac{K_\mu}{a^2}+\gamma_\mu F\right)\,. \qquad (11)$$

In Eq. (11) we use the notation of Ref. 18, with the following modifications due to the introduction of the photon mass λ:

$$F=\int_0^1 dz\,\frac{a^2-a^2(0)}{a^2(0)+[a^2-a^2(0)]z}+2m^2\left(1-x-\frac{x^2}{2}\right)\left(\frac{1}{a^2}-\frac{1}{a^2(0)}\right)\,, \qquad (12)$$

$$a^2=m^2x^2+k^2y(x-y)+(p^2+m^2)(1-x)(x-y)+(p'^2+m^2)(1-x)y+\lambda^2(1-x)\,, \qquad (13)$$

$$a^2(0)=m^2x^2+\lambda^2(1-x)\,. \qquad (14)$$

For diagram 1(c), since $p^2=-m^2$ and $p'=p-k$, we have

$$a^2=k^2y(x-y)+(k^2-2pk)y(1-x)+m^2x^2+\lambda^2(1-x)\,. \qquad (13')$$

Using Eq. (2), we can see that the only contributing terms are $\gamma_\mu C$, $\gamma_\mu F$, and $\sigma_{\mu\nu}k^\nu$. For example, the term containing $\sigma_{\mu\lambda}$ is proportional to

$$\bar{u}(p)\sigma_{\mu\lambda}k^\lambda[i\gamma(p-k)-m]\gamma_\nu v(\bar{p})=-(4m/\sqrt{2})\epsilon_{0\mu\lambda\nu}k^\lambda\,,$$

where we used Eq. (2) and the equation of motion

$$(i\gamma\bar{p}-m)v(\bar{p})=0$$

(notice that at threshold $p=\bar{p}$). After some integrations by parts, followed by setting $\lambda=0$ in all terms which do not exhibit low-momentum singularities, we can write the integrand associated with diagram 1(c) in the form

$$\frac{2\alpha}{\pi}\frac{\vec{k}^2}{k^2(k-2p)^2(k^2-2pk)^2}\left[-\frac{1}{2}\int_0^1\frac{dy\,k^2y(1-2y)}{k^2y(1-y)+m^2}+\int_0^1 dx\int_0^x\frac{dy}{a^2}\left(a^2+\frac{k^2}{2}-\frac{(k-2p)^2}{2}(1-2x)+(k^2-2pk)\times(\tfrac{1}{2}y-x)-\frac{2m^2a^2}{a^2(0)}(1-x-\tfrac{1}{2}x^2)\right)\right], \qquad (15)$$

where a^2 is given by Eq. (13′), and for which we have decomposed the denominator in such a way as to reduce as much as possible the number of Feynman parameters to be introduced in each integral.

It is straightforward to show the equivalence of the eight diagrams with the vertex insertion. In particular, consider diagram 1(c) and the diagram with the insertion in the lower right-hand corner. To prove the equivalence of these two diagrams, we make the transformations $k\to 2p-k$ and $y\to x-y$ in the second diagram, and obtain the same contribution [Eq. (15)] as that associated with diagram 1(c). We now let $y\to xy$ and $x\leftrightarrow y$ inside the large parentheses in Eq. (15), so that $a^2=yA$, where

$$A=(k^2-2pk)x-(kx-p)^2y. \tag{16}$$

The integrand corresponding to the eight diagrams with the vertex insertion is therefore

$$16\frac{\alpha}{\pi}\frac{\vec{k}^2}{k^2(k-2p)^2(k^2-2pk)^2}\left[-\frac{1}{2}\int_0^1\frac{dy\,k^2y(1-2y)}{k^2y(1-y)+m^2}-\int_0^1 dx\int_0^1\frac{dy}{A}\left(Ay+\frac{k^2}{2}-\frac{(k-2p)^2}{2}(1-2y)+(k^2-2pk)\right.\right.$$
$$\left.\left.\times\, y(\tfrac{1}{2}x-1)-2m^2\frac{Ay(1-y-\frac{1}{2}y^2)}{m^2y^2+\lambda^2(1-y)}\right)\right]. \tag{17}$$

Notice that we set $\lambda=0$ in A. The only cutoff-dependent term in Eq. (17) is the last one.

Before proceeding further, we consider the box diagrams, since it is convenient, for computational purposes, to combine the integrand of these diagrams with Eq. (17).

VI. BOX DIAGRAMS

To evaluate the trace most simply it is advantageous to consider diagram 1(d) together with the corresponding diagram with the crossed photon configuration. Then the amplitude due to all "box" diagrams is

$$2\bar{u}(p)\gamma_\lambda S_F(p-q)\left[\gamma_\mu S_F(p-q-k)\gamma_\nu+\gamma_\nu S_F(p-q-k')\gamma_\mu\right]S_F(-p-q)\gamma^\lambda v(\bar{p})\bar{v}(\bar{p})\gamma^\nu S_F(p-k)\gamma^\mu u(p)$$
$$=-16\frac{\epsilon_{0\mu\nu\lambda}k^\lambda}{\mathfrak{D}}\left(\frac{\epsilon_{\mu\nu\alpha\beta}k^\alpha q^\beta}{m}2pq-q^2\epsilon_{0\mu\nu\alpha}k^\alpha-(q^2-2pq)\epsilon_{0\mu\nu\alpha}q^\alpha-2m^2\epsilon_{0\mu\nu\alpha}k^\alpha\right) \tag{18}$$
$$=-16\frac{\epsilon_{0\mu\nu\lambda}k^\lambda}{\mathfrak{D}}\left(\frac{\epsilon_{\mu\nu\alpha\beta}k^\alpha q^\beta}{m}2pq-q^2\epsilon_{0\mu\nu\alpha}k^\alpha-(q^2-2pq)\epsilon_{0\mu\nu\alpha}q^\alpha\right.$$
$$\left.-\frac{2m^2\epsilon_{0\mu\nu\alpha}k^\alpha}{k^2-2pk}\left[(q+k)(q+k-2p)+(2pq-q^2-2qk)\right]\right), \tag{19}$$

where

$$\mathfrak{D}=k^2(k-2p)^2(k^2-2pk)q^2(q^2+2pq)(q^2-2pq)(q+k)(q+k-2p).$$

In order to simplify the evaluation of the cutoff-dependent term we added and subtracted

$$2m^2\epsilon_{0\mu\nu\alpha}k^\alpha\,\frac{(q+k)(q+k-2p)}{k^2-2pk}$$

inside the large parentheses in Eq. (18).

After the integration over d^4q is performed, the contribution of all four box diagrams is proportional to

$$32i\pi^2\frac{\vec{k}^2}{k^2(k-2p)^2(k^2-2pk)^2}\left[-4m^2k_\lambda I^\lambda+2\left(\frac{\pi}{\eta}+\frac{\ln\eta^2}{2}-1\right)+\int_0^1 dx\int_0^1\frac{dy}{A}\left(\frac{k^2}{2}+\frac{(k-2p)^2}{2}-(k^2-2pk)y\right)\right], \tag{20}$$

where A is defined by Eq. (16), and I_λ is given by

$$I_\lambda=\frac{1}{2}\int_0^1 dx\int_0^1 dy\,(1-y)\left(\frac{(p-k)_\lambda}{[(k^2-2pk)x-(kx-p)^2y]^2}-\frac{[kx+p(1-2x)]_\lambda}{\{(k^2-2pk)x-[kx+p(1-2x)]^2y\}^2}\right). \tag{21}$$

The $1/\eta$ singularity that appears in Eq. (20) can be safely neglected, since it serves to reproduce the lowest-order $m\alpha^5$ result for two-photon annihilation.[3,4] A similar situation is found in the evaluation of the first-order radiative corrections to the one-photon virtual annihilation.[3,4] To verify this

TABLE I. Summary of the evaluation of the integrals in Eq. (22). 1, 2, ... represents the 1st, 2nd, ... term inside the large parentheses in Eq. (22). The entries are all multiplied by $2\alpha^3|\phi(0)|^2/\pi m^2$.

	Constant	π^2	ln2	π^2 ln2	$\ln^2 2$	$\zeta(3)$	$i\pi$ (nonsingular)	$\ln\eta^2$
1	2	$-\frac{17}{6}$	$-\frac{1}{4}$	−1	−2	$\frac{85}{4}$	$2\ln 2-\frac{1}{4}\pi^2$	
2	−6	$\frac{3}{4}$						
3	10		−10				5	$4-4\ln 2+2i\pi$
4	4		−4				2	$-2+2\ln 2-i\pi$
5	2		−2				1	
6	$-\frac{27}{2}$	$-\frac{3}{2}$						
7+8	$\frac{3}{2}$	$-\frac{1}{12}$	4		−2		$2\ln 2-2$	
Total	0	$-\frac{11}{3}$	$-\frac{49}{4}$	−1	−4	$\frac{85}{4}$	$6+4\ln 2-\frac{1}{4}\pi^2$	$2-2\ln 2+i\pi$

point, we also calculated the contribution of the box diagrams with the electron and positron off the mass shell by using the Schrödinger wave function and making all photons massless *ab initio*. The two methods of evaluation gave identical results.

The $m\alpha^6$ energy contribution from all vertex and box diagrams is given by

$$\Delta E_{2\gamma}^{\text{box+vertex}} = \frac{8\alpha^2|\phi(0)|^2}{m^2}\,\frac{\alpha}{\pi}\,\frac{m^2}{i\pi^2}\int\frac{d^4k\,\vec{k}^2}{k^2(k-2p)^2(k^2-2pk)^2}\Bigg(4m^2kI-\int_0^1 dy\,\frac{k^2y(1-2y)}{k^2y(1-y)+m^2}-4m^2\int_0^1 dy\,y \times\frac{1-y-\frac{1}{2}y^2}{m^2y^2+\lambda^2(1-y)}-2(\tfrac{1}{2}\ln\eta^2-1)+\int_0^1 dx\int_0^1\frac{dy}{A} \times[2Ay-(k-2p)^2(\tfrac{3}{2}-2y)+\tfrac{1}{2}k^2 -(k^2-2pk)y(1-x)]\Bigg)\,. \tag{22}$$

The integration of Eq. (22) is outlined in Table I. (Useful integrals are given in Tables II and III.) We obtain

$$\Delta E_{2\gamma}^{\text{box+vertex}} = \frac{4\alpha^2|\phi(0)|^2}{m^2}\,\frac{\alpha}{\pi} \times[-\tfrac{11}{6}\pi^2-\tfrac{49}{8}\ln 2-\tfrac{1}{2}\pi^2\ln 2-2\ln^2 2+\tfrac{85}{8}\zeta(3) +i\pi(3+2\ln 2-\tfrac{1}{8}\pi^2)+\ln\eta^2(1-\ln 2+\tfrac{1}{2}i\pi)]\,. \tag{23}$$

VI. CONCLUSION

Collecting the contributions from the different diagrams [Eqs. (4), (9), and (23)], we obtain[17]

$$\Delta E_{2\gamma}^{A}(\alpha^4R_\infty) = -(\alpha^4R_\infty/2\pi^2)[1+\tfrac{35}{9}\pi^2+(\tfrac{41}{4}+\pi^2)\ln 2 -\tfrac{85}{4}\zeta(3)-i\pi(5-\tfrac{1}{4}\pi^2)]\,. \tag{24}$$

TABLE II. Trilog integrals:

$$\int_0^1 dx\,f(x)=a_1\zeta(3)+a_2\pi^2\ln 2+a_3\ln^3 2.$$

A more complete list of trilog integrals can be found in Ref. 23. See also Table 5 of Ref. 16 for other useful integrals of this type.

$f(x)$	$\zeta(3)$	π^2 ln2	$\ln^3 2$
$\frac{\ln^2 x}{1+x}$	$\frac{3}{2}$		
$\frac{\ln(1-x)[\ln(1+x)-\ln 2]}{1-x}$	$\frac{7}{8}$	$-\frac{1}{12}$	$\frac{1}{6}$
$\frac{\ln^2(1+x)-\ln^2 2}{1-x}$	$\frac{1}{4}$	$-\frac{1}{6}$	$\frac{2}{3}$
$\frac{\ln x[\ln x-\ln(1-x)]}{1-2x}$	$\frac{7}{8}$	$\frac{1}{4}$	

TABLE III. Miscellaneous integrals involving logarithms: $\int_0^1 dx f(x) = b_1 + b_2\pi^2 + \cdots$. See also Table IV of Ref. 16 for other useful integrals of this type.

$f(x)$	Constant	π^2	ln2	$\ln^2 2$
$\ln x \ln(1-x)$	2	$-\frac{1}{6}$		
$\ln x \ln(1+x)$	2	$-\frac{1}{12}$	-2	
$\ln(1+x)\ln(1-x)$	2	$-\frac{1}{6}$	-2	1
$x \ln x \ln(1-x)$	1	$-\frac{1}{12}$		
$x \ln x \ln(1+x)$	$-\frac{1}{2}$	$\frac{1}{24}$		
$x \ln(1+x)\ln(1-x)$	$\frac{1}{4}$		-1	
$\frac{\ln(1-2x)^2}{x}$		$-\frac{1}{2}$		
$\frac{\ln(1+x)-\ln 2}{1-x}$		$-\frac{1}{12}$		$\frac{1}{2}$
$\frac{\ln^2(1+x)}{x^2}$		$\frac{1}{6}$		-2
$\frac{\ln x \ln(1+x)}{(1+x)^2}$		$\frac{1}{12}$	-1	$-\frac{1}{2}$
$\frac{\ln x \ln(1+x)}{(1+x)^3}$	$-\frac{3}{8}$	$\frac{1}{24}$		$-\frac{1}{4}$
$\frac{\ln(1-x)-\ln x}{1-2x}$		$\frac{1}{4}$		
$\frac{\ln(1-x)-\ln x}{(1-2x)^2}$	0	0	0	0
$\frac{\ln x(\ln(1-x)-\ln x)}{(1-2x)^2}$		$-\frac{1}{3}$		
$\frac{\ln x[\ln(1+x)+\ln(1-x)]}{x^2}$		$\frac{1}{4}$	-2	
$\frac{\ln(1+x)[\ln(1+x)-x]}{x^3}$	$\frac{1}{2}$	$-\frac{1}{12}$		
$\frac{\ln(1+x)\ln(1-x)}{x^2}$		$-\frac{1}{12}$		-1
$\frac{\ln^2 x}{(1-x)^3}+\frac{\ln x}{(1-x)^2}$	$\frac{1}{2}$	$\frac{1}{6}$		
$\frac{\ln x \ln(1-x)}{x^2}+\frac{\ln x}{x}$	-1	$\frac{1}{6}$		
$\frac{\ln^2 x}{(1-x)^3}-\frac{1}{1-x}$	$\frac{3}{2}$	$\frac{1}{6}$		
$\frac{\ln x[\ln(1+x)-\ln 2]}{(1-x)^2}$		$\frac{1}{8}$		$-\frac{1}{2}$
$\frac{2\ln x[\ln(1+x)-\ln 2]}{(1-x)^3}+\frac{\ln x}{(1-x)^2}$	$\frac{1}{2}$	$\frac{1}{16}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{\ln x[\ln(1-x)+x]}{x^3}+\frac{1}{2x}$	$-\frac{3}{8}$	$\frac{1}{12}$		
$\frac{\ln^2(1+x)-\ln^2(1-x)}{x^3}$		$-\frac{1}{4}$	-2	

The real part of Eq. (24) corresponds to the desired energy shift, while the imaginary part is related to the radiative correction to the lifetime of parapositronium by

$$\mathrm{Im}\Delta E_{2\gamma}^{A}(\alpha^4 R_\infty) = -\tfrac{1}{2}\Gamma_{2\gamma}(\alpha^4 R_\infty). \tag{25}$$

The agreement of Eq. (25) with the result of Harris and Brown[22] serves as a partial check on our calculation.

Evaluation of the real part of Eq. (24) leads to the frequency shift

$$\Delta\nu_{2\gamma}^{A}(\alpha^4 R_\infty) = -\pi^{-2}\alpha^4 R_\infty c(13.92) = -13.13 \text{ MHz}. \tag{26}$$

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